oretischer Eigenschaften von Banachverbanden mit Hilfe disjunkter Folgen, Arch. Math. 24 (1973), pp. 640–647.
[20] L. Tzafriri, *Remarks on contractive projections in L_p spaces*, Israel J. Math. 7 (1969), pp. 9–15.
[21] M. Zippin, *On bases in Banach spaces*, Thesis: Hebrew University, Jerusalem (1968).

Received August 10, 1974 (881)
Revised version May 27, 1975

STUDIA MATHEMATICA, T. LVIII. (1976)

Extension of real-valued α-additive set functions

by
R. B. KIRK and J. A. CRENSHAW (Carbondale, Ill.)

Abstract. The extension of real-valued α-additive finite finitely additive regula. real-valued set functions from an algebra of sets to larger σ-algebras of sets is givenr The extensions are then used to obtain results on $\sigma(A^*, A)$ convergence of τ-additive functionals on an algebra A of real-valued functions on a set X.

Introduction. Let A be a uniformly closed algebra of bounded real-valued functions on a set X which separates the points of X and contains the constants. Let X be equipped with the τ_A topology which is the weakest topology on X which makes each $f \in A$ continuous. In [4] the concept of α-additive set functions on a paving $\mathscr{W}$ of subsets of X was introduced to represent the α-additive functionals in A^*, and it was indicated that the α-additive set functions could be extended to α-additive elements on a larger paving (this includes the fact that τ-additive Baire measures in $C^b(X)$ can be extended to Borel measures on X). We shall establish this extension process which depends on which definition of outer measure is chosen. We then employ the extension to questions about weak, $\sigma(A^*, A)$, convergence of elements in A^*. We anticipate that working with a paving and that working with subalgebras of $C^b(X)$ will prove useful in probability theory, and in this direction we obtain a weakened form of Prochorovs' theorem. Also for subalgebras $A_2 \subset A_1$ we give sufficient conditions for weak convergence of τ-additive φ in A_1^* to be determined by the elements of A_2.

The authors wish to thank the referee for pointing out that our results in Section 1 should extend to exaustive functions with range a suitably endowed topological group. He also noted some of the rich literature on the subject such as done by Drewnowski [2], Sion [6] and Traynor [7]. The referee is of course correct and the authors intend to show this and that the weak additivity condition does yield the usual additivity condition in a different paper.

§ 1. Extension. We refer the reader to [4] for many of the basic definitions and results; however, we shall indicate here some of the essential definitions.

A *paving* on X is a family $\mathscr{W}$ of subsets which contains $\emptyset$, is closed under finite unions and intersections, and has $X = \bigcup \mathscr{W}$. The paving is *full* if $X \in \mathscr{W}$ and in this paper all pavings will be assumed to be full.

Let $\mathcal{F}(\mathcal{W})$ be the algebra of subsets of X generated by $\mathcal{W}$, then $M(\mathcal{W})$ will denote the set of all finite, finitely additive real-valued set functions on $\mathcal{F}$ which are regular in the sense that for each $F \in \mathcal{F}$ there is a $W \in \mathcal{W}$ such that $W \subset F$ and $|m(G)| \leqslant \varepsilon$ whenever $G \in \mathcal{F}(\mathcal{W})$ with $G \subset F - W$.

For an infinite cardinal α, we say that $m \in M(\mathcal{W})$ is *α-additive* if $\inf\{|m|(W_i)\} = 0$ for every collection $\{W_i \in \mathcal{W}\colon i \in I\}$ which is directed downward to $\emptyset$ with card $I \leqslant \alpha$. The set of α-additive elements will be denoted by M_α (or $M_\alpha(\mathcal{W})$) and τ will denote the least cardinal such that $M_\tau = M_\beta$ when $\tau \leqslant \beta$. Finally, $m \in M(\mathcal{W})$ is *α-singular* if there is a family $\{W_i \in \mathcal{W}\colon i \in I\}$ which is directed downward to $\emptyset$ with card $I \leqslant \alpha$ and such that $|m|(V) = |m|(V \cap W_i)$ for all $V \in \mathcal{W}$ and all $i \in I$.

For the extension process to develop adequately it is essential we choose the proper definition of outer measure; we now give this and remark that if $X = [0, 1]$, and if m is the Lebesgue measure on the Borel sets $\mathcal{W}$, then $m \in M_\sigma(\mathcal{W})$, and m^* agrees with the usual outer measure and the extension process yields the Lebesgue measure.

DEFINITION 1.1. Let $\mathcal{W}$ be a full paving on X and let $m \in M^+(\mathcal{W})$. For $A \subset X$,

$$m_\alpha^*(A) = \inf\Big\{\sup_{W \in I} m(W^c)\colon I \subset \mathcal{W},\ I \text{ directed downward},\ \operatorname{card}(I) \leqslant \alpha \text{ and } A \subset \bigcup\{W^c\colon W \in I\}\Big\}.$$

LEMMA 1.2. *Let $\mathcal{W}$ be a paving and let $m \in M^+(\mathcal{W})$. Then m_α^* is an outer measure on X.*

Proof. It is clear that m_α^* is monotone and non-negative. Let $A = \bigcup_{n=1}^{\infty} A_n$, and fix $\varepsilon > 0$. For each $n \in \mathbf{N}$, let $I_n \subset \mathcal{W}$ be downward directed with $\operatorname{card}(I_n) \leqslant \alpha$ and such that

$$A_n \subset \bigcup\{W^c\colon W \in I_n\} \quad \text{and} \quad m_\alpha^*(A_n) + \varepsilon/2^n > \sup\{m(W^c)\colon W \in I_n\}.$$

Let I denote the family of all finite intersections of members of $\bigcup\{I_n\colon n \in \mathbf{N}\}$. Then I is directed downward with $\operatorname{card}(I) \leqslant \alpha$ and $A \subset \bigcup\{W^c\colon W \in I\}$. Hence it follows that

$$\begin{aligned} m_\alpha^*(A) &\leqslant \sup\{m(W^c)\colon W \in I\} \\ &\leqslant \sup\{m(W_1^c \cup \ldots \cup W_m^c)\colon W_i \in \bigcup I_n,\ i = 1, \ldots, m\} \\ &\leqslant \sup\Big\{\sum_{i=1}^{m} (W_i^c)\colon W_i \in \bigcup I_n,\ i = 1, \ldots, m\Big\} \\ &\leqslant \sum_{i=1}^{\infty} m_\alpha^*(A_i) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete.

For $\alpha < \tau = \tau(\mathcal{W})$, let $\mathcal{W}_\alpha$ denote the family of all sets $W \subset X$ for which there is $J \subset \mathcal{W}$ with $\operatorname{card}(J) \leqslant \alpha$ and $W = \bigcap J$. (That is, $\mathcal{W}_\alpha$ is the family of subsets of X obtained as intersections over subsets of $\mathcal{W}$ of cardinal at most α.) For $\alpha = \tau$, $\mathcal{W}_\alpha$ will denote the family of all intersections over subsets of $\mathcal{W}$. It is clear that $\mathcal{W}_\alpha$ is a paving and that it is full if and only if $\mathcal{W}$ is full.

LEMMA 1.3. *Let $\mathcal{W}$ be a full paving and let $m \in M^+$. Then $\Sigma(\mathcal{W}_\alpha)$ is a subalgebra of the m^*-Carathéodory measurable sets.*

Proof. It is sufficient to show that if $W_0 \in \mathcal{W}_\alpha$, then W_0 is Carathéodory measurable. Fix $\varepsilon > 0$ and $A \subset X$. We must show that $m_\alpha^*(A) \geqslant m_\alpha^*(A \cap W_0) + m_\alpha^*(A \cap W_0^c)$. Let $I \subset \mathcal{W}$ be downward directed with $\operatorname{card}(I) \leqslant \alpha$ and $\bigcap I = W_0$. Let $J \subset \mathcal{W}$ be a downward directed net with $\operatorname{card}(J) \leqslant \alpha$, $A \subset \bigcup\{U^c\colon U \in J\}$ and $m_\alpha^*(A) + \varepsilon > \sup_{U \in J} m(U^c)$. Since m is regular, for each $W \in I$ there is $Z_W \in \mathcal{W}$ with $Z_W \subset W^c$ and $m(W^c) < m(Z_W) + \varepsilon$. Now fix $W_1 \in I$. Since the family $\{U^c\colon U \in J\}$ is upward directed,

$$\sup_{U \in J}[m(U^c \cap W_1) + m(U^c \cap W_1^c)] = \sup_{U \in J} m(U^c \cap W_1) + \sup_{U \in J} m(U^c \cap W_1^c).$$

Furthermore,

$$0 \leqslant m\big((U^c \cap Z_{W_1}^c) - (U^c \cap W_1)\big) = m(W_1^c \cap Z_{W_1}^c \cap U^c) \leqslant m(W_1^c - Z_{W_1}) \leqslant \varepsilon.$$

Hence it follows that,

$$\begin{aligned} m_\alpha^*(A) + \varepsilon &\geqslant \sup_{U \in J} m(U^c) \geqslant \sup_{U \in J} m(U^c \cap W_1) + \sup_{U \in J} m(U^c \cap W_1^c) \\ &\geqslant \sup_{U \in J} m(U^c \cap Z_{W_1}^c) - \varepsilon + \sup m(U^c \cap W_1^c) \\ &\geqslant m_\alpha^*(A \cap Z_{W_1}^c) - \varepsilon + \sup_{U \in J} m(U^c \cap W_1^c) \\ &\geqslant m_\alpha^*(A \cap W_0) - \varepsilon + \sup_{U \in J} m(U^c \cap W_1^c). \end{aligned}$$

Since $W_1 \in I$ was arbitrary, we thus have that for all $W \in I$,

(1) $$m_\alpha^*(A) + 2\varepsilon \geqslant m_\alpha^*(A \cap J_0) + \sup_{U \in J} m(U^c \cap W^c).$$

Let $K = \{U \cup W\colon U \in J,\ W \in I\}$. Then K is downward directed, $\operatorname{card}(K) \leqslant \alpha$ and $A \cap W_0^c \subset \bigcup\{V^c\colon V \in K\}$. Hence it follows from (1) that,

$$m_\alpha^*(A) + 2\varepsilon \geqslant m_\alpha^*(A \cap W_0) + \sup_{\substack{U \in J \\ W \in I}} m(U^c \cap W^c) \geqslant m_\alpha^*(A \cap W_0) + m_\alpha^*(A \cap W_0^c).$$

The proof is complete.

LEMMA 1.4. *Let $\mathcal{W}$ be a full paving and let $m \in M_\alpha^+$. Then for each $F \in \mathcal{F}(\mathcal{W})$, $m(F) = m_\alpha^*(F)$. Furthermore, the restriction of m_α^* to $\mathcal{F}(\mathcal{W}_\alpha)$ belongs to $M_\alpha^+(\mathcal{W}_\alpha)$.*

Proof. For $m \in M_\alpha^+(\mathcal{W})$, it is clear from the definition that $m(W^c) = m_\alpha^*(W^c)$ for all $W \in \mathcal{W}$. Since m_α^* is finitely additive on $\mathcal{F}(\mathcal{W})$ by Lemma

2.7, $m(W) = m_\alpha^*(W)$ for all $W \in \mathscr{W}$. That $m(F) = m_\alpha^*(F)$ for all $F \in \mathscr{F}(\mathscr{W})$ follows from Proposition [3, 1.2].

In order to show that $M_\alpha^*|_{\mathscr{F}(\mathscr{W}_\alpha)} \in M_\alpha^+(\mathscr{W}_\alpha)$, in view of Lemma 1.3, it is only necessary to show that m_α^* is α-additive and $\mathscr{W}_\alpha$-regular. We will begin with α-additivity which will be verified in two steps.

(1) Let $I \subset \mathscr{W}$ be downward directed with $\operatorname{card}(I) \leqslant \alpha$ and $W_0 = \bigcap I$. Then

$$m_\alpha^*(W_0) = \inf_{W \in I} m(W).$$

By definition, $m_\alpha^*(W^c) \leqslant \sup_{W \in I} m(W^c)$. On the other hand, $W_0 \subset W$ implies that $m_\alpha^*(W_0^c) \geqslant m_\alpha^*(W^c) = m(W^c)$. Thus, $m_\alpha^*(W_0^c) = \sup_{W \in I} m(W^c)$, and (1) follows.

(2) If $I \subset \mathscr{W}_\alpha$ is directed downward with $W_0 = \bigcap I$, then

$$m_\alpha^*(W_0) = \inf_{W \in I} m_\alpha^*(W).$$

First note that $W_0 \in \mathscr{W}_\alpha$ since $\alpha \cdot \alpha = \alpha$. For each $W \in I$, let $I_W \subset \mathscr{W}$ be downward directed with $\operatorname{card}(I_W) \leqslant \alpha$ and $W = \bigcap I_W$. Let J denote the family of all finite intersections of the elements of the set $\bigcup \{I_W : W \in I\}$. Then $J \subset \mathscr{W}$ is directed downward with $\operatorname{card}(J) \leqslant \alpha$ and $\bigcap J = W_0$. Hence, by (1), $m_\alpha^*(W_0) = \inf_{U \in J} m(U)$. Furthermore, if $\varepsilon > 0$ is given, there is $U_1 \in J$ with $\inf_{U \in J} m(U) + \varepsilon > m(U_1)$. Since I is directed downward, there is $W_1 \in I$ with $W_1 \subset U_1$. Hence it follows that,

$$m_\alpha^*(W_0) = \inf_{U \in J} m(U) > m(U_1) - \varepsilon \geqslant m_\alpha^*(W_1) - \varepsilon \geqslant \inf_{W \in I} m_\alpha^*(W) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof of (2) is complete.

We will now verify that m_α^* is $\mathscr{W}_\alpha$-regular on $\mathscr{F}(\mathscr{W}_\alpha)$. The proof will require three steps.

(3) Let $W_0 \in \mathscr{W}_\alpha$. Then

$$m_\alpha^*(W_0^c) = \sup\{m_\alpha^*(W) : W \in \mathscr{W}_\alpha \text{ and } W \subset W_0^c\}.$$

Let $I \subset \mathscr{W}$ be directed downward with $\operatorname{card}(I) \leqslant \alpha$ and $W_0 = \bigcap I$. Fix $\varepsilon > 0$. Since $m_\alpha^*(W_0) = \inf_{U \in I} m(U)$ by (1), there is $U_0 \in I$ with $m_\alpha^*(U_0) = m(U_0) < m_\alpha^*(W_0) + \varepsilon$. This is equivalent to

$$m_\alpha^*(W_0^c) < m_\alpha^*(U_0^c) + \varepsilon = m(U_0^c) + \varepsilon.$$

Since m is $\mathscr{W}$-regular, there is a $W \in \mathscr{W}$ with $W \subset W_0^c$ and $m(U_0^c) < m(W) + \varepsilon$. Then $W \in \mathscr{W}_\alpha$, $W \subset W_0^c$ and $m_\alpha^*(W_0^c) < m(U_0^c) + \varepsilon < m(W) + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the proof of (3) is complete.

(4) Let $W_0, W_1 \in \mathscr{W}_\alpha$. Then

$$m_\alpha^*(W_1 - W_0) = \sup\{m_\alpha^*(W) : W \in \mathscr{W}_\alpha \text{ and } W \subset W_1 - W_0\}.$$

Fix $\varepsilon > 0$. By (3) there is a $W_2 \in \mathscr{W}_\alpha$ with $W_2 \subset W_0^c$ and $m_\alpha^*(W_0^c) < m_\alpha^*(W_2) + \varepsilon$. Let $W = W_1 \cap W_2$. Then $W \in \mathscr{W}_\alpha$ and $W \subset W_1 - W_0$. Since $(W_1 - W_0) - W \subset W_0^c - W_2$, it follows that $0 \leqslant m_\alpha^*(W_1 - W_0) - m_\alpha^*(W) \leqslant m_\alpha^*(W_0^c - W_2) \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, (4) follows.

(5) m_α^* is $\mathscr{W}_\alpha$-regular on $\mathscr{F}(\mathscr{W}_\alpha)$.

This is immediate from (4) and Proposition [3, 1.2]. The proof of Lemma 1.4 is now complete.

LEMMA 1.5. *Let $\mathscr{W}$ be a full paving and let $m \in (M_\alpha(\mathscr{W})^\perp)^+$. Then $m_\alpha^* = 0$.*

Proof. By [4; 4.4] $M_\alpha(\mathscr{W})^\perp$ is a band so there is an increasing net (m_i) of α-singular elements of $M(\mathscr{W})$ with $0 \leqslant m_i \uparrow m$. (Since (m_i) is increasing, it is easy to verify that $m_i(F) \uparrow m(F)$ for all $F \in \mathscr{F}(\mathscr{W})$.) Fix $\varepsilon > 0$ and take i_0 with $m(X) < m_{i_0}(X) + \varepsilon$. Let $I \subset \mathscr{W}$ be an α-system with $m_{i_0}(X) = m_{i_0}(W)$ for all $W \in I$. Since $0 \leqslant m - m_{i_0}$, it follows that, for each $F \in \mathscr{F}(\mathscr{W})$, $0 \leqslant m(F) - m_{i_0}(F) = (m - m_{i_0})(F) \leqslant (m - m_{i_0})(X) < \varepsilon$. Hence

$$0 \leqslant m_\alpha^*(X) \leqslant \sup_{W \in I} m(W^c) \leqslant \varepsilon + \sup_{W \in I} m_{i_0}(W^c) = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $m_\alpha^*(X) = 0$, and so $m_\alpha^* = 0$.

PROPOSITION 1.6. *Let $\mathscr{W}$ be a full paving, and let $m \in M^+(\mathscr{W})$. Assume that $m = m_1 + m_2$ where $m_1 \in M_\alpha^+(\mathscr{W})$ and $m_2 \in (M_\alpha(\mathscr{W})^\perp)^+$. Then m_α^* restricted to $\mathscr{F}(\mathscr{W}_\alpha)$ belongs to $M_\alpha^+(\mathscr{W}_\alpha)$ and $m_\alpha^* = (m_1)_\alpha^*$. Hence, in particular, $m_1 = m_\alpha^*$ on $\mathscr{F}(\mathscr{W})$.*

Proof. It is easily shown that $m_\alpha^* = (m_1)_\alpha^* + (m_2)_\alpha^*$. The result is now immediate from Lemmas 1.4 and 1.5.

PROPOSITION 1.7. *Let $\mathscr{W}$ be a full paving, and let $m \in M_\alpha^+(\mathscr{W})$. Then there is a unique element $\mu \in M_\alpha^+(\mathscr{W}_\alpha)$ whose restriction to $\mathscr{F}(\mathscr{W})$ is m. In fact, if $\lambda \in M^+(\mathscr{W}_\alpha)$ is any element whose restriction to $\mathscr{F}(\mathscr{W})$ is m, then $\mu = \lambda$.*

Proof. Let μ denote the restriction of m_α^* to $\mathscr{F}(\mathscr{W}_\alpha)$. By Proposition 1.6, $\mu \in M_\alpha^+(\mathscr{W}_\alpha)$ and $\mu = m$ on $\mathscr{F}(\mathscr{W})$. Now let $\lambda \in M^+(\mathscr{W}_\alpha)$ and assume that $\lambda = m$ on $\mathscr{F}(\mathscr{W})$. Fix $W \in \mathscr{W}_\alpha$. Let $I \subset \mathscr{W}$ be downward directed with $\operatorname{card}(I) \leqslant \alpha$ and $W = \bigcap I$. Then $0 \leqslant \lambda(W) \leqslant \inf_{U \in I} \lambda(U) = \inf_{U \in I} \lambda(U) = \mu(W)$. Since $W \in \mathscr{W}_\alpha$ was arbitrary, the $\mathscr{W}_\alpha$-regularity of λ and μ guarantee that $0 \leqslant \lambda \leqslant \mu$. Since $M_\alpha(\mathscr{W}_\alpha)$ is an ideal, it follows that $\lambda \in M_\alpha^+(\mathscr{W}_\alpha)$. Hence $\lambda(W) = \inf_{U \in I} \lambda(U) = \inf_{U \in I} \mu(U) = \mu(W)$. Thus $\lambda(W) = \mu(W)$ for all $W \in \mathscr{W}_\alpha$. The $\mathscr{W}_\alpha$-regularity of λ and μ now imply that $\lambda = \mu$.

Define a map T_α from $M_\alpha(\mathscr{W})$ into $M_\alpha(\mathscr{W}_\alpha)$ as follows. (We continue to assume that $\mathscr{W}$ is a full paving.) For $m \in M_\alpha^+(\mathscr{W})$, let $T_\alpha(m)$ denote the restriction of m_α^* to $\mathscr{F}(\mathscr{W}_\alpha)$. Then $T_\alpha(m) \in M_\alpha^+(\mathscr{W}_\alpha)$ by Proposition 1.7. For arbitrary $m \in M_\alpha(\mathscr{W})$, define $T_\alpha(m) = T_\alpha(m^+) - T_\alpha(m^-)$.

THEOREM 1.8. *Let $\mathscr{W}$ be a full paving. The map T_α is a Banach lattice isomorphism of $M_\alpha(\mathscr{W})$ onto a band in $M_\alpha(\mathscr{W}_\alpha)$. Furthermore, for $\mu \in T_\alpha[M_\alpha(\mathscr{W}_\alpha)]$, $T_\alpha^{-1}(\mu)$ is the restriction of μ to $\mathscr{F}(\mathscr{W})$.*

Proof. It is easy to verify that T_α is linear on $M_\alpha^+(\mathscr{W})$. From this it is immediate that T_α is a positive linear transformation on $M_\alpha(\mathscr{W})$. If $T_\alpha(M) = 0$, then $T_\alpha(m^+) = T_\alpha(m^-)$ so that $m^+ = m^-$ by Proposition 1.7. Hence if $T_\alpha(m) = 0$, then it follows that $m = 0$. Finally, note that if $0 \leqslant T_\alpha(m)$, then $0 \leqslant m$ since m is the restriction of $T_\alpha(m)$ to $\mathscr{F}(\mathscr{W})$.

In order to verify that T_α is lattice preserving, it is sufficient to show that $|T_\alpha(m)| = T_\alpha(|m|)$ for all $m \in M_\alpha(\mathscr{W})$. Since T_α is a positive transformation, $|T_\alpha(m)| \leqslant T_\alpha(|m|)$. Let m' denote the restriction of $|T_\alpha(m)|$ to $\mathscr{F}(\mathscr{W})$. Then $0 \leqslant m' \leqslant |m|$. It is immediate from this that $m' \in M(\mathscr{W})$. Since $M_\alpha(\mathscr{W})$ is an ideal, it now follows that $m' \in M_\alpha(\mathscr{W})$. By Proposition 1.7, $T_\alpha(m') = |T_\alpha(m)|$. Thus $m \leqslant m'$ and $-m \leqslant m'$ so that $|T_\alpha(m)| = T_\alpha m' \geqslant T_\alpha |m|$. Hence $|T_\alpha(m)| = T_\alpha(|m|)$. Moreover, $\|T_\alpha(m)\| = |T_\alpha(m)|(X) = T_\alpha(|m|)(X) = |m|(X) = \|m\|$ so that T_α is norm-preserving.

We will now show that the image of $M_\alpha(\mathscr{W})$ under T_α is an ideal in $M_\alpha(\mathscr{W}_\alpha)$. Since T_α is lattice preserving, $T_\alpha[M_\alpha(\mathscr{W})]$ is a Riesz subspace of $M_\alpha(\mathscr{W}_\alpha)$. Now let $\lambda \in M_\alpha(\mathscr{W}_\alpha)$ satisfy $0 \leqslant \lambda \leqslant T_\alpha(m)$ for some $m \in M_\alpha(\mathscr{W})$. Let m' be the restriction of λ to $\mathscr{F}(\mathscr{W})$. Then $0 \leqslant m' \leqslant m$ so that $m' \in M_\alpha(\mathscr{W})$ since $M_\alpha(\mathscr{W})$ is an ideal. But then $T_\alpha(m') = \lambda$ by Proposition 1.7. Hence $\lambda \in T_\alpha[M_\alpha(\mathscr{W})]$.

Finally, in order to demonstrate that $T_\alpha[M_\alpha(\mathscr{W})]$ is a band, let $(T_\alpha(m_i))$ be an upward directed net in $(T_\alpha[M_\alpha(\mathscr{W})])^+$ with $T_\alpha(m_i) \uparrow \mu \in M_\alpha(\mathscr{W}_\alpha)$. Let m denote the restriction of μ to $\mathscr{F}(\mathscr{W})$. Since $m_i(X) = T_\alpha(m_i)(X) \uparrow \uparrow \mu(X) = m(X)$, if $\varepsilon > 0$ is fixed, there is an i_0 with $m(X) < m_{i_0}(X) + \varepsilon$. Thus $m(F) \leqslant m_{i_0}(F) + \varepsilon$ for all $F \in \mathscr{W}$. It now follows immediately that m is $\mathscr{W}$-regular so that $m \in M(\mathscr{W})$. Since $m_i(F) \uparrow m(F)$ for all $F \in \mathscr{F}(\mathscr{W})$ and since (m_i) is directed upward, $m_i \uparrow m \in M(\mathscr{W})$. Since $M_\alpha(\mathscr{W})$ is a band, $m \in M_\alpha(\mathscr{W})$. Finally, by Proposition 1.7, $T_\alpha(m) = \mu$. The proof is complete.

The map T_α is not onto $M_\alpha(\mathscr{W}_\alpha)$ in general as the following example shows.

EXAMPLE. Let $X = [0, 1]$. Define $W_0 = [0, 1)$ and $W_n = [1 - 1/n, 1]$ for $n \in \boldsymbol{N}$. Let $\mathscr{W}$ be the smallest paving on X containing $\{W_i\colon\ i = 0, 1, \ldots\}$. (Thus $W = \{X, W_0, \emptyset\} \cup \{W_n\colon\ n \in \boldsymbol{N}\} \cup \{W_0 \cap W_n\colon\ n \in \boldsymbol{N}\}$.) Note that $\mathscr{W}$ is a full paving.

For $F \in \mathscr{F}(\mathscr{W}_\sigma)$ $(\sigma = \aleph_0)$, define $\lambda(F) = 1$ if $1 \in F$ and $\lambda(F) = 0$ if $1 \notin F$. Then $\lambda \in M_\sigma(\mathscr{W}_\sigma)$ as is easily seen. If m denotes the restriction of λ to $\mathscr{F}(\mathscr{W})$, then $m \notin M(\mathscr{W})$ since m is not $\mathscr{W}$-regular. (Indeed, $m(W_0^c) = 1$, but $\sup\{m(W)\colon\ W \in \mathscr{W}$ and $W \subset W_0^c\} = 0$.)

The following gives a simple condition on $\mathscr{W}$ which guarantees that T_α is onto $M_\alpha(\mathscr{W}_\alpha)$. (Note that the family $\mathscr{Z}$ of all zero sets on a topological space satisfies the condition, and this accounts for the fact that every Baire measure on a topological space is $\mathscr{Z}$-regular.)

THEOREM 1.9. *Let $\mathscr{W}$ be a full paving. Assume that if $W_0 \in \mathscr{W}$, then there is a sequence (W_n) in $\mathscr{W}$ with $W_0^c = \bigcup_{n=1}^{\infty} W_n$. Then for every infinite cardinal α, T_α maps $M_\alpha(\mathscr{W})$ onto $M_\alpha(\mathscr{W}_\alpha)$.*

Proof. Let $\mu \in M_\alpha^+(\mathscr{W}_\alpha)$ and let m denote the restriction of μ to $\mathscr{F}(\mathscr{W})$. Then m is a non-negative, finite, finitely-additive function on $\mathscr{F}(\mathscr{W})$. All that need be verified is that m is $\mathscr{W}$-regular. Hence let $W_0 \subset \mathscr{W}$ and choose an increasing sequence (W_n) in $\mathscr{W}$ with $W_0^c = \bigcup\{W_n\colon\ n \in \boldsymbol{N}\}$. Since μ is α-additive, $m(W_0^c) = \lim m(W_n) \leqslant \sup\{m(W)\colon\ W \in \mathscr{W}$ and $W \subset W_0^c\}$. By Proposition [4; 1.4(3)], it follows that m is $\mathscr{W}$-regular.

§ 2. Applications of the extension.

In this section, we wish to apply the extension theorems to obtain certain results on weak convergence in A^* where A is a uniformly closed algebra of bounded real-valued functions on X which separates the points of X and contains the constants. We shall denote the paving of zero sets of A by $\mathscr{Z}(A)$. If $\mathscr{W}$ is a full paving on X, then a *standard representation* of A^* is an isometric isomorphism I of A^* onto $M(\mathscr{W})$ such that $I_\varphi(W) = \inf\{\varphi(f)\colon f \in A,\ \chi_W \leqslant f\}$ for all $W \in \mathscr{W}$.

LEMMA 2.1. *Let A_1, A_2 be algebras on X with $A_2 \subset A_1$ and let $\tau_{A_1} = \tau_{A_2}$. Let $\mathscr{W}_1$, $\mathscr{W}_2$ be two full pavings of closed sets in X which are bases for the τ_{A_i} closed sets. Let $\varphi \in (A_1^*)_\tau^+$. If $M(\mathscr{W}_i)$ represents A_i^*, $i = 1, 2$, and $m \in M_\tau^+(\mathscr{W}_1)$ represents φ while $\mu \in M_\tau^+(\mathscr{W}_2)$ represents $\varphi|A_2$, then $T_\tau m = T_\tau \mu$.*

Proof. Since $\mathscr{W}_1$ and $\mathscr{W}_2$ are bases for the closed sets for same topology, $(\mathscr{W}_1)_\tau = (\mathscr{W}_2)_\tau$ is the family of all closed sets for the topology. By Theorem 1.8, it is sufficient to prove that $m(W) = T_\tau \mu(W)$ for all $W \in \mathscr{W}_1$. Hence fix $W_0 \in \mathscr{W}_1$ and $\varepsilon > 0$. Choose $f_0 \in A_1$ with $\chi_{W_0} \leqslant f_0$ and $m(W_0) > \varphi(f_0) - \varepsilon$. Then $\varepsilon + m(W_0) > \varphi(f_0) = \int_X f_0 \, dT_\tau \mu \geqslant T_\tau \mu(W_0)$. Thus $T_\tau \mu(W_0) \leqslant m(W_0)$ for all $W_0 \in \mathscr{W}_1$.

Since $\mathscr{W}_2$ is base for the closed sets, $\mathscr{W}_1 \subset (\mathscr{W}_2)_\tau$. Hence, since $T_\tau \mu$ is τ-additive, there is $W_1 \in \mathscr{W}_2$ with $W_0 \subset W_1$ and $T_\tau \mu(W_0) + \varepsilon > \mu(W_1)$. Furthermore, there is $f_1 \in A_2$ with $\chi_{W_1} \leqslant f_1$ and $\varphi(f_1) < \mu(W_1) + \varepsilon$. Hence, $T_\tau \mu(W_0) > \mu(W_1) - \varepsilon > \varphi(f_1) - 2\varepsilon = \int_X f_1 \, dT_\tau m - 2\varepsilon \geqslant T_\tau m(W_1) - 2\varepsilon \geqslant T_\tau m(W_0) - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $T_\tau \mu(W_0) \geqslant T_\tau m(W_0) = m(W_0)$ for all $W_0 \in \mathscr{W}_1$. The proof is complete.

We note that $\mathscr{W}$ is a *normal base* if $\mathscr{W}$ is paving of closed sets on X which is a base for the closed sets of X and satisfies:

(i) If G is a closed set in X and if $x \in G^c$, then there are $W_1, W_2 \in \mathscr{W}$ with $W_1 \cap W_2 = \emptyset$ and $x \in W_1, G \subset W_2$.

(ii) If $W_1, W_2 \in \mathscr{W}$ with $W_1 \cap W_2 = \emptyset$, then there are $V_1, V_2 \in \mathscr{W}$ with $V_1 \cup V_2 = X$ and $W_i \subset V_i^c$ for $i = 1, 2$.

We remark that a normal base $\mathscr{W}$ gives rise to a compactification $X_{\mathscr{W}}$ and if this compactification is X_A (the compactification such that every element $f \in A$ can be extended to $C(X_A)$ and every element of $C(X_A)$ is such an extension), then $M(\mathscr{W})$ represents A^* by [4; 3.12].

THEOREM 2.2. *Let A_1, A_2 be algebras on X with $A_2 \subset A_1$ and $\tau_{A_1} = \tau_{A_2}$; and let $\mathscr{W}_1$ and $\mathscr{W}_2$ be normal bases with X_{A_j} the $\mathscr{W}_j$-compactification for $j = 1, 2$. Let (φ_i) be a net in $(A_1^*)^+$, let $\varphi \in (A_1^*)^+$ and assume that $\varphi_i(f) \to \varphi(f)$ for all $f \in A_2$. Then $\varphi_i(f) \to \varphi(f)$ for all $f \in A_1$ if either of the following two conditions hold.*

(1) $\mathscr{W}_2 \subset \mathscr{W}_1$.

(2) $\varphi_i \in (A_1^*)_\tau^+$ *for all i.*

Proof. First assume that $\mathscr{W}_2 \subset \mathscr{W}_1$. (That is, condition (1) holds.) Let $m_i, m \in M^+(\mathscr{W}_1)$ represent φ_i and φ, respectively, and let $\mu_i, \mu \in M^+(\mathscr{W}_2)$ represent the restrictions of φ_i and φ to A_2. By Proposition [4; 4.8] $m \in M_\tau^+(\mathscr{W}_1)$ and $\mu \in M_\tau^+(\mathscr{W}_2)$. Fix $W_0 \in \mathscr{W}_1$ and $\varepsilon > 0$. Since $\mathscr{W}_2$ is a normal base, there is a $W_1 \in \mathscr{W}_2$ with $W_0 \subset W_1$ and $\mu(W_1) < T_\tau \mu(W_0) + \varepsilon$. For all i, $m_i(W_1) = \inf\{\varphi(f)\colon f \in A_1, \chi_{W_1} \leqslant f\} \leqslant \inf\{\varphi_i(f)\colon f \in A_2, \ \chi_{W_1} \leqslant f\} = \mu_i(W_1)$. Using this together with Theorem [4; 3.13] and Lemma 2.1 we obtain that,

$$\limsup m_i(W_0) \leqslant \limsup m_i(W_1) \leqslant \limsup \mu_i(W_1) \leqslant \mu(W_1) < T_\tau \mu(W_0) + \varepsilon = m(W_0) + \varepsilon.$$

The result is now an immediate consequence of theorem [4; 6.3].

Now assume that condition (2) holds. Again let $m_i, m \in M_\tau^+(\mathscr{W}_1)$ represent φ_i and φ and let $\mu_i, \mu \in M_\tau^+(\mathscr{W}_2)$ represent the restrictions of φ_i and φ to A_2. By Lemma 2.1, $T_\tau \mu_i = T_\tau m$ and $T_\tau \mu = T_\tau m$. Fix $W_0 \in \mathscr{W}$ and $\varepsilon > 0$. Choose $W_1 \in \mathscr{W}_2$ such that $W_0 \subset W_1$ and $0 \leqslant \mu(W_1) - m(W_0) = T_\tau \mu(W_1 - W_0) \leqslant \varepsilon$. Then by theorem [4; 6.3] and Lemma 2.1

$$\limsup m_i(W_0) = \limsup T_\tau \mu_i(W_0) \leqslant \limsup \mu_i(W_1) \leqslant \mu(W_1) \leqslant m(W_0) + \varepsilon.$$

The result follows by another application of theorem [4; 6.3].

COROLLARY 2.3. *Let X be a compact Hausdorff space, (m_i) a net of Borel measures on X and $\mathscr{W}$ a base for the closed sets in X. Then (m_i) converges weakly to a Borel measure m if and only if $m_i(X) \to m(X)$ and $\limsup m_i(W) \leqslant m(W)$ for all $W \in \mathscr{W}$.*

We remark that in [3; appendix(d)] we showed that $M_\sigma(\mathscr{W})$ is not weakly sequentially complete even when $\mathscr{W}$ is a normal base. (It is well known that if $A = C^b(X)$, then $M_\sigma(\mathscr{Z})$ is weakly sequentially complete. See [8].) Thus we state the following

PROBLEM. Let (φ_n) be a sequence in A_σ^* and assume that $\varphi_n \to \varphi$ in the $\sigma(A^*, A)$-sense. Does it necessarily follow that $\varphi \in A_\sigma^*$? If not, what conditions on the algebra will guarantee that this is so?

LEMMA 2.4. *Let A be an algebra on X, and let C^b denote the algebra of bounded, real-valued τ_A-continuous functions on X. Then the following hold:*

(1) *Let $\mathscr{W}$ be a normal base for τ_A with X_A the $\mathscr{W}$-compactification. Then there is a unique Riesz space isomorphism T of $M_\tau(\mathscr{W})$ into $M_\tau(\mathscr{Z}(C^b))$ which is a homeomorphism for the weak topologies.*

(2) *Let $\mathscr{W} = \mathscr{Z}(A)$, then the restriction map S of $M_\tau(\mathscr{Z}(C^b))$ to $\mathscr{F}(\mathscr{Z}(A))$ is a Riesz space isomorphism of $M_\tau(\mathscr{Z}(C^b))$ onto $M_\tau(\mathscr{Z}(A))$ which is a homeomorphism for the weak topologies.*

(1). The uniqueness is immediate from the fact that $L(X)$ is weakly dense in $M_\tau(\mathscr{W})$ by Proposition [4; 6.4]. Since $\mathscr{W}$ is a normal base, $\mathscr{W}_\tau = (\mathscr{Z}(C^b))_\tau = \mathscr{I}$, the family of all τ_A-closed sets. Let T_1 be the map T_τ of $M_\tau(\mathscr{W})$ into $M_\tau(\mathscr{I})$ of Theorem 1.8, and let T_2 be the corresponding map of $M_\tau(\mathscr{Z}(C^b))$ onto $M_\tau(\mathscr{I})$. Let $T = T_2^{-1} \circ T_1$. Then T is a Riesz space isomorphism which keeps $L(X)$ pointwise fixed. It is clear that T^{-1} is continuous for the weak topologies. Let (m_i) be a net in $M_\tau^+(\mathscr{W})$ and $m \in M_\tau^+(\mathscr{W})$. Assume that $m_i \to m$ weakly. We must show that $Tm_i \to Tm$.

Hence fix $Z_0 \in \mathscr{Z}(C^b)$ and $\varepsilon > 0$. Take $W_0 \in \mathscr{W}$ with $Z_0 \subset W_0$ and $m(W_0) < T_1 m(Z_0) + \varepsilon$. Then by theorem [4; 6.3]

$$\begin{aligned}\limsup Tm_i(Z_0) &= \limsup T_1 m_i(Z_0) \leqslant \limsup T_1 m_i(W_0) \\ &\leqslant \limsup m_i(W_0) \leqslant m(W_0) = T_1 m(W_0) \\ &\leqslant T_1 m(Z_0) + \varepsilon = Tm(Z_0) + \varepsilon.\end{aligned}$$

It now follows by theorem [4; 6.1] that $Tm_i \to Tm$ weakly. The proof of (2) is similar to that of (1) except that we use [4; 6.1] in place of [4; 6.3]. See also [1, p.12] and [8].

Using Lemma 2.4, we can now obtain several facts about weak compactness as consequences of results which are known in $C^b(X)^*$. (See [5].)

THEOREM 2.5. *Let A be an algebra on X and assume that (X, τ_A) is metrizable as a separable metric space. Then a set $B \subset (A_\tau^*)^+$ is relatively weakly compact if and only if it is relatively weakly sequentially compact.*

Proof. Let $B' \subset M_\tau^+(\mathscr{Z}(A))$ be the set of measures which represent the elements of B according to theorem [4; 4.8]. By (1) in Lemma 2.4, B' is relatively weakly compact if and only if $S[B']$ is relatively weakly compact in $M_\tau^+(\mathscr{Z}(C^b))$. But by theorem 27 of [5, p. 76], $S[B']$ is relatively weakly compact if and only if $S[B']$ is relatively weakly sequentially compact. Again by Lemma 2.4, this is equivalent to B' being weakly sequentially compact. The proof is complete.

THEOREM 2.6. *Let A be an algebra on X, and let $B \subset (A_\tau^*)^+$. If B is relatively weakly countably compact in $(A_\tau^*)^+$, then B is relatively weakly compact in $(A_\tau^*)^+$.*

Proof. It is enough to prove the theorem in the special case, $A = C^b(X)$. The general result then follows from Lemma 2.4 as in the proof of Theorem 2.5 above. But if $A = C^b$, then $M_\sigma(\mathscr{Z})$ is complete for the Mackey topology $m(M_\sigma(\mathscr{Z}), C^b)$. (Indeed, it is shown in [3] that M_σ is complete for a topology e^b for which the dual of M_σ is C^b. Hence it is complete for the Mackey topology.) It then follows from Eberlein's theorem that if $B \subset M_\sigma(\mathscr{Z})$ is relatively weakly countably compact, then it is relatively weakly compact. The proof is complete.

Remark. We have shown above that any relatively weakly sequentially compact subset of $M_\sigma(\mathscr{Z}(C^b))$ is necessarily relatively weakly compact.

We note from [4] that for a paving $\mathscr{W}$, a set $S \subset X$ is $\mathscr{W}$-compact if for every filter $\mathscr{U} \subset \mathscr{W}$ with $S \cap U \neq \emptyset$ for all $U \in \mathscr{U}$, then $\bigcap\{S \cap U: U \in \mathscr{U}\} \neq \emptyset$.

DEFINITION 2.7. Let $\mathscr{W}$ be a paving on X. A set $B \subset M(\mathscr{W})$ is *tight* if $\sup\{\|m\|: m \in B\} < \infty$ and if for every $\varepsilon > 0$, there is a $\mathscr{W}$-compact set $W_0 \in \mathscr{W}_\tau$ such that $|m|(W) \leqslant \varepsilon$ for all $W \in \mathscr{W}$ with $W \cap W_0 = \emptyset$ and all $m \in B$.

It is clear that the mapping S of Lemma 2.4 preserves tight sets. Hence using Lemma 2.4 and Theorem 31 of [8, p. 66], we obtain the following weakened version of Prochorov's theorem.

THEOREM 2.8. *Let A be an algebra and assume that (X, τ_A) is locally compact or that (X, τ_A) is metrizable with a complete metric. Then $B \subset M_\tau^+(\mathscr{Z}(A))$ is relatively $\sigma(A^*, A)$-compact if and only if it is tight.*

We conclude this paper with a last application to obtain a generalization of a known result (see for example [5; 5.1(d)]). We denote the set of tight elements of $M(\mathscr{W})$ by $M_t(\mathscr{W})$.

THEOREM 2.9. *Let $\mathscr{W}$ be a normal base with compactification X_A or let $\mathscr{W} = \mathscr{Z}(A)$ for A an algebra on X. If $m \in M_t(\mathscr{W})$, then there is a unique compact regular Borel measure μ on X such that $\mu|\mathscr{F}(\mathscr{W}) = m$ and $\mu^+, \mu^- \in M_\tau(\mathscr{W}_\tau)$.*

Proof. We note that the hypothesis implies the τ_A-compact sets are $\mathscr{W}$-compact and conversely. By [4; 5.6] $m \in M_t(\mathscr{W})$ is in $M_\tau(\mathscr{W})$. Therefore there exist unique $\mu^+, \mu^- \in M_\tau(\mathscr{W}_\tau)$ such that $\mu^+|\mathscr{F}(\mathscr{W}) = m^+$ and $\mu^-|\mathscr{F}(\mathscr{W}) = m^-$ by 1.7. Since $\mathscr{W}$ is a base for the τ_A closed sets of X, $\mathscr{W}_\tau$ is the paving of all closed sets so that μ^+, μ^- are Borel measures on X.

Finally, since m is tight, for any $\varepsilon > 0$ there is a $\mathscr{W}$-compact set W_0 such that $|m|(W) < \varepsilon$ if $W \cap W_0 = \emptyset$. Since W_0 is τ_A-compact and $\mathscr{W}$ is a basis for the τ_A closed sets, it follows that for any closed set F with $F \cap W_0 = \emptyset$, there is a $W \in \mathscr{W}$ such that $F \subset W$ and $W \cap W_0 = \emptyset$. Consequently, $|\mu|(F) \leqslant |\mu|(W) = |m|(W) < \varepsilon$ so that $|\mu|$ is compact regular.

References

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.
[2] L. Drewnowski, *Decompositions of set functions*, Studia Math. 68 (1973), pp. 23–48.
[3] R. B. Kirk, *Algebras of bounded real-valued functions I, II*, Nederl. Akad. Wetensch Proc. Sev. A 75, Indg. Math. 34 (1972), pp. 443–463.
[4] — and J. A. Crenshaw, *A generalized topological measure theory*, Trans. Amer. Math. Soc. 207 (1975), pp. 189–217.
[5] F. D. Sentilles, *Bounded continuous functions*, Trans. Amer. Math. Soc. 168 (1972), pp. 311–336.
[6] M. Sion, *Outer measures with values in a topological group*, Proc. London Math. Soc. 19 (1969), pp. 89–106.
[7] Tim Traynor, *A General Hewit–Yosida Decomposition*, Canad. J. Math. 26 (1972), pp. 1164–1169.
[8] V. S. Varadarajan, *Measures in topological spaces* (Russian), Mat. Sbornik N. S. 55 (1961), pp. 33–100. Translated in Amer. Math. Soc. Transl. (2) 72 (1965), pp. 161–228.

Received October 23, 1974
Revised version May 30, 1975 (907)