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A convolution approximation property for $L^1(G)$

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Abstract. In this paper we define a certain convolution approximation property and give some examples of subspaces which have this property with respect to $L^1(G)$. We exhibit a connection between subspaces with the approximation property and multipliers of Fourier transforms. A result on (p, p) multipliers over noncompact LCA groups is also obtained.

Throughout this paper G is a LCA group, Γ the group dual to G, $M(G)$ the convolution algebra of bounded Borel measures on G and $\hat{}$ the Fourier-Stieltjes transformation. Denote by $L^1(G)$ the space of Haar integrable functions on G and $D(G)$ any subspace of $L^1(G)$ with

$$D(G) \supset \{f \in L^1(G)\colon \operatorname{supp}\hat{f} \text{ is a compact set}\}.$$

Suppose that whenever $\sigma_n \in L^1(G)$ $(n = 1, 2, \ldots)$ with $\limsup\limits_{n\to\infty} \|\sigma_n\|_1 = \infty$ there exists a $k \in D(G)$ such that $\limsup\limits_{n\to\infty} \|\sigma_n * k\|_1 = \infty$. Then $D(G)$ is said to have the *convolution approximation property with respect to* $L^1(G)$. It is not difficult to see that $L^1(G)$ itself has the convolution approximation property with respect to $L^1(G)$; see Example I.

By adapting and generalizing some methods in [1] and [3] we give an example of a rather small space $D(G)$ with the convolution approximation property whenever G is a noncompact LCA group. It turns out that spaces with the convolution approximation property admit a solution to a wide class of multiplier problems. A function φ on Γ is said to be a *multiplier of type* (D, L^1) if whenever $f \in D(G)$ there corresponds a $g \in L^1(G)$ such that $\varphi\hat{f} = \hat{g}$.

THEOREM 1. *If $D(G)$ has the convolution approximation property, then $\varphi \in (D, L^1)$ if and only if $\varphi = \hat{\mu}$ for some $\mu \in M(G)$.*

Proof. One half of the Theorem is obvious. For the converse let $\varphi \in (D, L^1)$. For any $k \in D(G)$ with $\operatorname{supp}\hat{k}$ a compact subset of Γ put

$$\sigma(x) = \int_\Gamma \varphi(\alpha)\hat{k}(\alpha)\alpha(x)\,d\alpha \qquad (x \in G).$$

Then $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ if and only if no matter what be k (as defined above) there exists an $M \in \boldsymbol{N}$ (the natural numbers) such that

$$\|\sigma\|_1 \leqslant M$$

whenever $\|k\|_1 \leqslant 2$. To prove sufficiency let $p(x) = \sum_{i=1}^{n} c_{a_i} a_i(x)$ be any trigonometric polynomial on G with $\|p\|_\infty \leqslant 1$. Now by [4], p. 53, there is a $k \in D(G)$ such that $\hat{k}$ has compact support,

$$\|k\|_1 \leqslant 2 \quad \text{and} \quad \hat{k}(a_i) = 1 \quad (i = 1, 2, \ldots, n).$$

So,

$$\Big|\sum_{i=1}^{n} c_{a_i} \varphi(a_i)\Big| = \Big|\sum_{i=1}^{n} c_{a_i} \hat{k}(a_i)\varphi(a_i)\Big|$$

and we obtain

$$\Big|\sum_{i=1}^{n} c_{a_i} \varphi(a_i)\Big| = \Big|\sum_{i=1}^{n} c_{a_i} \hat{\sigma}(a_i)\Big|.$$

Thus

$$\Big|\sum_{i=1}^{n} c_{a_i} \varphi(a_i)\Big| \leqslant \|p\|_\infty \|\sigma\|_1 \leqslant \|\sigma\|_1.$$

So

$$\Big|\sum_{i=1}^{n} c_{a_i} \varphi(a_i)\Big| \leqslant M$$

for all trigonometric polynomials $p(x) = \sum_{i=1}^{n} c_{a_i} a_i(x)$ with $\|p\|_\infty \leqslant 1$ and this in turn implies by the Bochner–Eberlein Theorem ([4], p. 32) that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ since φ is continuous.

Let us now suppose that $\varphi \notin M(G)^\wedge$. We shall force a contradiction: By the preceding argument there exists a sequence $k_n \in D(G)$ such that for each $n \in \boldsymbol{N}$, $\|k_n\|_1 \leqslant 2$, $\hat{k}_n$ has compact support and

$$\sigma_n(x) = \int_\Gamma \hat{k}_n(a)\varphi(a)a(x)\,da \quad (x \in G)$$

satisfies

(1) $$\limsup_{n\to\infty} \|\sigma_n\|_1 = \infty.$$

Since $D(G)$ has the convolution approximation property, there is a $k \in D(G)$ such that

(2) $$\limsup_{n\to\infty} \|\sigma_n * k\|_1 = \infty.$$

Observe that

(i) $$(\sigma_n * k)(x) = \int_\Gamma \hat{h}(a)\hat{k}_n(a)a(x)\,da$$

for some $h \in L^1(G)$ since $\varphi \in (D, L^1)$. Thus by the Inversion Theorem ([4], p. 22)

$$\sigma_n * k = h * k_n.$$

Inasmuch as $\|h * k_n\|_1 \leqslant 2\|h\|_1$ we have

(ii) $$\limsup_{n\to\infty} \|\sigma_n * k\|_1 \leqslant 2\|h\|_1.$$

Since (ii) says, in particular, that $\limsup_{n\to\infty} \|\sigma_n * k\|_1 < \infty$ this contradicts (2) and completes the proof.

We next present some examples.

EXAMPLE I. Let G be a LCA group. Then $L^1(G)$ has the convolution approximation property with respect to $L^1(G)$. This follows immediately from the Uniform Boundedness Principle and the fact that $L^1(G)$ has a bounded approximate identity.

As a consequence of Theorem 1 we obtain that $(L^1, L^1) = M(G)^\wedge$; this is, of course, well known. Our next example is more interesting.

EXAMPLE II. For G a noncompact LCA group put

$$A^0(G) = \{f \in L^1(G)\colon \hat{f} \in \bigcap_{0<p<\infty} L^p(\Gamma)\}.$$

Then $A^0(G)$ has the convolution approximation property with respect to $L^1(G)$. To see this fix $f \in L^1(G)$. Now given $\varepsilon > 0$ there is a $k \in L^1(G)$ such that: $\hat{k}$ has compact support;

(1) $$\|k * f - f\|_1 \leqslant \varepsilon/2 \quad \text{and} \quad \|k\|_1 \leqslant 1.$$

For $K * f \in L^1(G)$ there is a $g \in L^1(G)$ such that g has compact support K and $\|g - k * f\|_1 \leqslant \varepsilon/2$.

Then, if we assume that G contains a copy of $\boldsymbol{Z}$, it follows by the noncompactness of G that there is an $a \in G$ such that the sets

$$na + K \quad (n \in \boldsymbol{Z})$$

are pairwise disjoint. Following [3], p. 129, put

(2) $$k_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} k_{ia}(x) \quad (x \in G).$$

Then for some θ and all $n \in \boldsymbol{N}$

(3) $$\|f\|_1 = \|k_n * f\|_1 + \theta\varepsilon \quad (|\theta| \leqslant 1).$$

Now

$$\hat{k}_n(\gamma) = \begin{cases} \dfrac{1}{n}\left(\dfrac{1-\overline{\gamma(a)}^n}{1-\overline{\gamma(a)}}\right)\hat{k}(\gamma), & \gamma(a) \neq 1, \\ \hat{k}(\gamma), & \gamma(a) = 1. \end{cases}$$

Since the annihilator of the copy of $\boldsymbol{Z}$ has Haar measure zero in Γ, the expression above shows $\hat{k}_n(\gamma)\to 0$ a.e. Notice that $|\hat{k}_n(\gamma)| \leqslant |\hat{k}(\gamma)|$, hence it follows by the Lebesgue Dominated Convergence Theorem that

$$\|\hat{k}_n\|_p \to 0 \quad (0 < p < \infty). \tag{4}$$

Thus when G contains a copy of $\boldsymbol{Z}$ we have that given $\sigma_n \in L^1(G)$ and $\varepsilon_n > 0$ $(n \in \boldsymbol{N})$ there correspond $g_n \in A^0(G)$ such that

$$\|\sigma_n\|_1 = \|\sigma_n * g_n\|_1 + \theta_n \varepsilon_n \quad (|\theta_n| \leqslant 1). \tag{5}$$

Furthermore, the g's in (5) may be chosen to satisfy

$$\|\hat{g}_n\|_{1/n} \leqslant 1 \quad \text{and} \quad \|g_n\|_1 \leqslant 1. \tag{6}$$

Using structure theory for LCA groups it can be shown that (5) and (6) hold for every noncompact LCA group.

Observe that since p may lie in $(0, 1)$, Banach space techniques are not available to us (cf. Theorem 3.1 of [2], p. 188). To show that $A^0(G)$ has the convolution approximation property we modify and generalize an argument of Doss ([1], pp. 184–185) given for $G = \boldsymbol{R}$. We reproduce some details from [1] for the readers' convenience.

Suppose that $\sigma_n \in L^1(G)$ and

$$\limsup_{n\to\infty} \|\sigma_n\|_1 = \infty. \tag{7}$$

In (5) set $\varepsilon_n = \frac{1}{4}\|\sigma_n\|_1$ for all $n \in \boldsymbol{N}$.

Let $\langle g_n\rangle_1^\infty$ satisfy (5) and (6). For any subsequence $\langle g_{n_q}\rangle_1^\infty$ define

$$g(x) = \sum_{q=1}^{\infty} g_{n_q}(x)/3^q \quad (x \in G). \tag{8}$$

Since by (6) $\|g_{n_q}\|_1 \leqslant 1$ for all n_q, we conclude that $g \in L^1(G)$ and $\|g\|_1 \leqslant 1$. We now prove that $\hat{g} \in \bigcap_{0<p<\infty} L^p(\Gamma)$.

Given $p \in (0, \infty)$ choose $n \in \boldsymbol{N}$ such that $1/n < p$. Then $\|\hat{g}_m\|_{1/n}^{1/n} \leqslant 1$ for all $m \geqslant n$ since $\|\hat{g}_m\|_\infty \leqslant 1$. Put

$$\hat{s}_N(\gamma) = \sum_{q=1}^{N} \hat{g}_{n_q}(\gamma)/3^q.$$

Recall that for f_1 and $f_2 \in L^{1/n}(\Gamma)$

$$\|f_1+f_2\|_{1/n}^{1/n} \leqslant \|f_1\|_{1/n}^{1/n} + \|f_2\|_{1/n}^{1/n}.$$

Thus for all $N > n$

$$\|\hat{s}_N\|_{1/n}^{1/n} < \sum_{j=1}^{n} (\tfrac{1}{3})^{j/n} \|\hat{g}_{n_j}\|_{1/n}^{1/n} + 1/3(1/3^{1/n}-1).$$

Inasmuch as $\hat{s}_N$ converges uniformly to $\hat{g}$ we have by the Fatou Lemma that $\hat{g} \in L^{1/n}(\Gamma)$ and this implies since $\|\hat{g}\|_\infty \leqslant 1$ that $\hat{g} \in L^p(\Gamma)$.

We shall show there is a subsequence of $\langle g_n\rangle_1^\infty$ such that $\limsup_{n\to\infty} \|g * \sigma_n\|_1 = \infty$ where g is defined by (8): Observe that if, for some $k \in \boldsymbol{N}$, g_k satisfies

$$\limsup_{n\to\infty} \|g_k * \sigma_n\|_1 = \infty,$$

then we are done. So suppose that given $k \in \boldsymbol{N}$ there corresponds an $M_k \in \boldsymbol{N}$ such that for all $n \in \boldsymbol{N}$

$$\|g_k * \sigma_n\|_1 \leqslant M_k. \tag{9}$$

For any sequence $\langle g_{n_q}\rangle_1^\infty$ and corresponding g defined by (8) we have

$$\|g * \sigma_{n_q}\|_1 \geqslant \frac{1}{3^q}\|\sigma_{n_q} * g_{n_q}\|_1 - \sum_{i<q} \frac{1}{3^i}\|\sigma_{n_q} * g_{n_i}\|_1 - \sum_{i>q} \frac{1}{3^i}\|\sigma_{n_q} * g_{n_i}\|_1.$$

From (5) and our choice of ε_n

$$\|g * \sigma_{n_q}\|_1 \geqslant \frac{3}{4}\frac{\|\sigma_{n_q}\|_1}{3^q} - \sum_{i<q} \frac{\|\sigma_{n_q} * g_{n_i}\|_1}{3^i} - \sum_{i>q} \frac{1}{3^i}\|\sigma_{n_q} * g_{n_i}\|_1$$

and by (9)

$$\|g * \sigma_{n_q}\|_1 \geqslant \frac{3}{4}\frac{\|\sigma_{n_q}\|_1}{3^q} - \sum_{i<q} \frac{M_{n_i}}{3^i} - \sum_{i>q} \frac{1}{3^i}\|\sigma_{n_q} * g_{n_i}\|_1.$$

Finally, since

$$\|\sigma_{n_q} * g_{n_i}\|_1 \leqslant \|g_{n_i}\|_1 \|\sigma_{n_q}\|_1,$$

we have *via* (6) the inequality

$$\|g * \sigma_{n_q}\|_1 \geqslant \frac{3}{4}\frac{1}{3^q}\|\sigma_{n_q}\|_1 - \sum_{i<q} \frac{M_{n_i}}{3^i} - \|\sigma_{n_q}\|_1 \sum_{i>q} \frac{1}{3^i}. \tag{10}$$

Thus,

$$\|g * \sigma_{n_q}\|_1 \geqslant \frac{1}{4}\frac{\|\sigma_{n_q}\|_1}{3^q} - \sum_{i<q} \frac{M_{n_i}}{3^i}; \tag{11}$$

and this inequality holds for all subsequences $\langle g_{n_q}\rangle_1^\infty$. Inasmuch as $\limsup_{n\to\infty} \|\sigma_n\|_1 = \infty$, (11) permits the construction of a subsequence $\langle g_{n_q}\rangle_1^\infty$ such that g satisfies $\limsup_{n\to\infty} \|g * \sigma_n\|_1 = \infty$. Our discussion is complete.

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As a consequence of our Theorem it follows that when G is a noncompact LCA group $\varphi \in (A^0, L^1)$ if and only if $\varphi = \hat{\mu}$ for some $\mu \in M(G)$, cf. Theorem 3.1 of [2], p. 188. This result has also been obtained by S. Saeki (private communication). Notice that $A^0(G)$ does not have the convolution approximation property when G is compact since $(L^2, L^2) = L^\infty(\Gamma)$.

For G a noncompact LCA group and $1 \leqslant p \leqslant 2$ put

$$A_1^p(G) = \{f \in L^p(G) \colon \hat{f} \in L^1(\Gamma)\}.$$

Recall that $A_1^p(G)$ is a Banach space when equipped with the norm $\|f\|_p + \|\hat{f}\|_1$, $f \in A_1^p(G)$. Denote by (A_1^p, L^p) the Banach space of all bounded linear maps from $A_1^p(G)$ to $L^p(G)$ which commute with translation.

By $M_p^p(G)$ we mean the space of all continuous linear operators from $L^p(G)$ to $L^p(G)$ which commute with translation. The proof of the following result may be of some interest and does not seem to be widely known:

THEOREM 2. *For $G = \boldsymbol{R}$ the operator $T \in (A_1^p, L^p)$ if and only if T extends to an operator in $M_p^p(\boldsymbol{R})$.*

Proof. One half of the Theorem is obvious. For the converse we have by the continuity of T a constant $K > 0$ such that

$$\|T(f)\|_p \leqslant K\{\|f\|_p + \|\hat{f}\|_1\} \qquad \big(f \in A_1^p(\boldsymbol{R})\big).$$

Let $f \in A_1^p(\boldsymbol{R})$, then

$$\lim_{a\to\infty} \|f + \tfrac{1}{2}f_a + \tfrac{1}{2}f_{-a}\|_p = (1 + 1/2^{p-1})^{1/p}\|f\|_p$$

and since T is linear and commutes with translation

$$\lim_{a\to\infty} \|T(f + \tfrac{1}{2}f_a + \tfrac{1}{2}f_{-a})\|_p = (1 + 1/2^{p-1})^{1/p}\|T(f)\|_p.$$

Now

$$(f + \tfrac{1}{2}f_a + \tfrac{1}{2}f_{-a})\hat{\ }(\alpha) = \left(1 + \frac{\alpha(a)}{2} + \frac{\overline{\alpha(a)}}{2}\right)\hat{f}(\alpha)$$

and since the kernel $\left(1 + \frac{\alpha(a)}{2} + \frac{\overline{\alpha(a)}}{2}\right)$ is non-negative, we obtain by the Riemann–Lebesgue Lemma that

$$\lim_{a\to\infty} \int_{\boldsymbol{R}} |\hat{f}(\alpha)| \left|1 + \frac{\alpha(a)}{2} + \frac{\overline{\alpha(a)}}{2}\right| d\alpha = \int_{\boldsymbol{R}} |\hat{f}(\alpha)|\, d\alpha.$$

We conclude therefore that

$$\|T(f)\|_p \leqslant K\{\|f\|_p + (1 + 1/2^{p-1})^{-1/p}\|\hat{f}\|_1\}.$$

Iterating the process n times we have

$$\|T(f)\|_p \leqslant K\{\|f\|_p + (1 + 1/2^{p-1})^{-n/p}\|\hat{f}\|_1\}.$$

Well, this means since $n \in \boldsymbol{N}$ is arbitrary that

$$\|T(f)\|_p \leqslant K\{\|f\|_p\} \qquad (f \in A_1^p(\boldsymbol{R}).$$

Since $A_1^p(\boldsymbol{R})$ is dense in $L^p(\boldsymbol{R})$ we may extend T in the desired fashion. This completes the proof.

It is easy to see that the proof of Theorem 2 generalizes to a wide class of noncompact LCA groups.

References

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