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q-variate minimal stationary processes

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Abstract. A complete description of non full rank in general q-variate minimal stationary processes over discrete Abelian groups are given. This result subsumes the minimality theorems of various authors in special cases.

1. Introduction. In his fundamental paper [1] A. N. Kolmogorov introduced the important concept of minimal processes. Next the concept have been extended to the q-variate case (cf. [2] and [6], Section 10). The interpolation problem for q-variate stationary processes over groups was studied by H. Salehi and J. K. Scheidt [8] and by A. Weron [9], [10]. Furthermore in those papers characterizations of q-variate minimal processes are also given. In [8] a generalization of Masani's minimality theorem for full rank processes is obtained. Two characterizations of non-full rank processes are given in [10], but unfortunately one of which ([10], Theorem 5.7) contains an error. In this paper a counter example for this (see Example 5.3) and a correct statement of this theorem (see Theorem 4.6(d)) is given. Moreover, we will get a general theorem on characterizations of q-variate minimal (not necessary full rank) processes.

Section 2 is devoted to the preliminary results on the spaces $L_{2,F}$ — of square integrable matrix-valued functions and $H_{2,F}$ — of Hellinger square integrable matrix-valued measures. Section 3 treats on q-variate stationary processes over a discrete Abelian group. Using methods of the earlier work [10] on stationary processes over locally compact Abelian (LCA) groups, we obtain an analytical characterization of a subspace N_e which is important in the minimality problem. In Section 4 we discuss the minimality problem and give some characterizations of minimal processes. As a corollary we then deduce Kolmogorov's and Masani's minimality theorems. Finally in Section 5 we give several examples to show that conditions in the presented theorems are essential ones as well as to illustrate them.

2. $L_{2,F}$ and $H_{2,F}$ spaces. Let $\mathfrak{B}$ be a σ-algebra of subsets of a space Ω and let $\Phi = [\varphi_{ij}]$, $1 \leqslant i, j \leqslant q$, be a matrix-valued function on Ω. Troughout this paper all matrices have complex entries and C denotes the set of complex numbers. A function Φ is $\mathfrak{B}$-measurable if each function φ_{ij} is $\mathfrak{B}$-measurable. If m is a non-negative real-valued measure on $\mathfrak{B}$, then by $L_{1,m}$ we denote the class of all $\mathfrak{B}$-measurable functions Φ such that each φ_{ij} is integrable with respect to (abbreviated to "w.r.t.") m. For $\Phi \in L_{1,m}$ we put

$$\int_\Omega \Phi dm = \Big[\int_\Omega \varphi_{ij} dm\Big].$$

By a matrix-valued measure on a σ-algebra $\mathfrak{B}$ we shall mean a function M from $\mathfrak{B}$ into the set of all $q\times q$-matrices over C, the complex numbers, such that for every disjoint sequence of sets $A_1, A_2, \ldots$ in $\mathfrak{B}$ with union A, $M(A) = \sum_{k=1}^{\infty} M(A_k)$. Obviously, $M = [M_{ij}]$ is a matrix-valued measure if and only if each of its entries M_{ij} is a complex-valued measure on $\mathfrak{B}$. If m is a non-negative σ-finite measure on $\mathfrak{B}$, we say that M is *absolutely continuous* w.r.t. m ($M \ll m$) if each entry M_{ij} is absolutely continuous w.r.t. m.

Let now Φ and Ψ be $\mathfrak{B}$-measurable matrix-valued functions on Ω and let M be a matrix-valued measure on $\mathfrak{B}$. If m is non-negative σ-finite measure on $\mathfrak{B}$ such that $M \ll m$ and if $\Phi(dM/dm)\Psi^* \in L_{1,m}$, then we define

$$\int_\Omega \Phi dM \Psi^* = \int_\Omega \Phi(dM/dm)\Psi^* dm,$$

where A^* denotes the conjugate transpose of A. It is known ([5], 3.1) that $\int_\Omega \Phi dM\Psi^*$ is independent of the choise of m.

(2.1) PROPOSITION ([3], p. 406). *The four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$ have a unique solution for any matrix A.*

The unique solution of these equations is called the *generalized inverse* of A and written $X = A^{\#}$.

Let $\mathscr{R}(A) = \{y\colon y = xA\}$ will denote the range of A and $\mathscr{N}(A) = \{x\colon xA = 0\}$ will denote the null space of A. $P_{\mathscr{M}}$ will denote the orthogonal projection matrix onto the subspace $\mathscr{M} \subset C^q$. If $A^{\#}$ is the generalized inverse of A, then from [4], p. 355, we have

(2.2) $AA^{\#} = P_{\mathscr{R}(A^*)} = P_{\mathscr{N}(A)^\perp}$,

(2.3) $A^{\#}A = P_{\mathscr{R}(A)} = P_{\mathscr{N}(A^*)^\perp}$,

(2.4) $(A^{\#})^{\#} = A$,

(2.5) $(A^*)^{\#} = (A^{\#})^*$,

(2.6) Let H be Hermitian and $G = AHA^*$. Then $\mathscr{R}(H) \subset \mathscr{R}(A)$ if and only if $A^{\#}G(A^{\#})^* = H$.

We note that (2.1)–(2.5) implies

(2.7) *If A is non-negative and Hermitian, then $A^{\#}$ is also non-negative and Hermitian and $\mathscr{R}(A) = \mathscr{R}(A^{\#})$.*

Let M and N be matrix-valued measures on $\mathfrak{B}$. Following [4], p. 361, we say that N is *strongly absolutely continuous* w.r.t. M ($N \lll M$) if exists a non-negative σ-finite measure m such that $M \ll m$, $N \ll m$ and $\mathscr{R}(dN/dm) \subset \mathscr{R}(dM/dm)$ m-a.e. We say that the matrix-valued functions Φ and Ψ are *equivalent* w.r.t. $M\big(\Phi = \Psi(\mathrm{mod}\, M)\big)$ if $\int_E (\Phi-\Psi)\, dM = 0$ for all $E \in \mathfrak{B}$.

(2.8) PROPOSITION (cf. [4], 5.4). *Let M and N be matrix-valued measures on $\mathfrak{B}$. Then $N \lll M$ if and only if exists a matrix-valued function Φ such that $\Phi \in L_{1,M}$ and $N(E) = \int_E \Phi dM$ for all $E \in \mathfrak{B}$. This Φ is unique* (mod M): *in fact, $\Phi = (dN/dm)(dM/dm)^{\#}$, where m is any non-negative measure such that $M \ll m$.*

The next Cramér's decomposition is an easy consequence of [4], 6.14.

(2.9) PROPOSITION. *Let N be a non-negative Hermitian matrix-valued measure and let m be a non-negative σ-finite measure on $\mathfrak{B}$. Then there exist unique matrix-valued measures N^a (the absolutely continuous part) and N^s (the singular part) such that $N = N^a + N^s$, $N^a \ll m$, $N^s \perp m$ and N^a, N^s are non-negative Hermitian matrix-valued measures.*

Let M, N and F be matrix-valued measures on $\mathfrak{B}$. If m is a non-negative σ-finite measure on $\mathfrak{B}$ such that $M \ll m$, $N \ll m$, $F \ll m$ and if $(dM/dm)(dF/dm)^{\#}(dN/dm)^* \in L_{1,m}$, then we define the Hellinger integral

$$\int_\Omega dM\, dN^*/dF = \int_\Omega (dM/dm)(dF/dm)^{\#}(dN/dm)^* dm.$$

It is well known ([7]) that $\int_\Omega dM\,dN^*/dF$ is independent of the choise of m.

(2.10) DEFINITION (cf. [5], [7]). Let F be a non-negative Hermitian matrix-valued measure on $\mathfrak{B}$.

(a) By $H_{2,F}$ we denote the class of all matrix-valued measures M on $\mathfrak{B}$ for which the integral $(M, M)_F = \int_\Omega dM\,dM^*/dF$ exists and M is strongly absolutely continuous w.r.t. F.

(b) By $L_{2,F}$ we denote the class of all $\mathfrak{B}$-measurable matrix-valued functions Φ on Ω for which the integral $(\Phi, \Phi)_F = \int_\Omega \Phi dF \Phi^*$ exists.

We remark that our definition of $H_{2,F}$ space is different from original one, given by H. Salehi ([7]). Example (5.1) shows that Theorem 2(b) in [7] is not true without the assumption $M \lll F$ and therefore we add it in the definition.

(2.11) THEOREM (cf. [7], Theorem 2). *Let F be a non-negative Hermitian matrix-valued measure on $\mathfrak{B}$. Then $M \in H_{2,F}$ if and only if there exists a function Φ in $L_{2,F}$ such that $M(E) = \int_E \Phi dF$ for each $E \in \mathfrak{B}$.*

Proof. Let $M \in H_{2,F}$. Since $M \langle\langle\langle F$, then by (2.8) there exists a function $\Phi \in L_{1,F}$ such that $M(E) = \int_E \Phi dF$ for each $E \in \mathfrak{B}$. In fact, $\Phi = (dM/dm)(dF/dm)^{\#}(\mathrm{mod} F)$, where m is any non-negative σ-finite measure on $\mathfrak{B}$ such that $F \ll m$ (for example, $m = \mathrm{tr} F$). Obviously,

(2.12) $$\int_\Omega dM\,dM^*/dF = \int_\Omega (dM/dm)(dF/dm)^{\#}(dM/dm)^* dm = \int_\Omega \Phi dF \Phi^*.$$

Hence $\Phi \in L_{2,F}$.

Conversely, let $M(E) = \int_E \Phi dF$ for all $E \in \mathfrak{B}$ and $\Phi \in L_{2,F}$. Then $\Phi \in L_{1,F}$ (see [5]) and, by (2.8), $M \langle\langle\langle F$. According to (2.12), the integral $\int_\Omega dM\,dM^*/dF$ exists and therefore $M \in H_{2,F}$. ∎

(2.13) Corollary. *Let F be as before. Then M is zero in $H_{2,F}$ if and only if $M(E) = 0$ for each $E \in \mathfrak{B}$.*

Proof. The sufficiency is trivial. For necessity let us consider $M \in H_{2,F}$ such that $\int_\Omega dM\,dM^*/dF = 0$. By (2.11), there exists a function $\Phi \in L_{2,F}$ such that $M(E) = \int_E \Phi dF$, $E \in \mathfrak{B}$. Since $(\Phi, \Phi)_F = (M, M)_F$, from ([5]) $\Phi = 0$ $(\mathrm{mod} F)$. Hence $M(E) = \int_E \Phi dF = 0$ for each $E \in \mathfrak{B}$. ∎

3. Stationary processes. Let G be any discrete Abelian group with multiplication. Then Γ, the dual group of G, is a compact Abelian group under compact-open topology. We will denote the elements of G by g and those of Γ by γ. The value of $\gamma \in \Gamma$ at $g \in G$ will be denoted by $\langle g, \gamma \rangle$. The Borel field of a topological group is the minimal σ-field generated by closed subsets. Throughout this paper the letter $\mathfrak{B}$ will denote the Borel field of Γ. On every locally compact Abelian group there exists a non-negative measure, finite on compact sets and positive on non-empty open sets, the so-called Haar measure of the group, which is translation-invariant. We denote by dg and $d\gamma$ the Haar measures on G and Γ. Without loss of generality, we will assume $d\gamma(\Gamma) = 1$.

(3.1) Definition. A *q-variate stationary process over G* is a function $(X_g)_{g \in G}$ such that:

(a) $X_g \in H^q$ for all $g \in G$ (H is a fixed complex Hilbert space) and

(b) the Grammian matrix $(X_g, X_h) = (X_{gh^{-1}}, X_e) = K(gh^{-1})$ depends only on gh^{-1} for all $g, h \in G$.

$K(g)$ is positive-definite and, in view of Bochner's theorem, can be written in the form

(3.2) $K(g) = (X_g, X_e) = \int_\Gamma \overline{\langle g, \gamma \rangle} dF$, where F is the non-negative Hermitian matrix-valued measure on $\mathfrak{B}$, the so-called *spectral measure* of $(X_g)_{g \in G}$. This F is unique.

Let $\mathfrak{M}$ denote the time domain of the stationary process $(X_g)_{g \in G}$,

i.e., the closed subspace of H^q spanned over the elements X_g, $g \in G$, with $q \times q$-matrix coefficients.

(3.3) Theorem (cf. [10], Theorem 3.7). *If $(X_g)_{g \in G}$ is a q-variate stationary process over G, with the spectral measure F, then the spaces $\mathfrak{M}$, $L_{2,F}$ and $H_{2,F}$ are isomorphic, where*

(a) *the mapping $V_1\colon X_g \mapsto \overline{\langle g, \gamma \rangle} I$, I denoting the unit matrix, induces an isomorphism between $\mathfrak{M}$ and $L_{2,F}$,*

(b) *the mapping $V_2\colon \Phi \mapsto M_\Phi$, for any matrix-valued function $\Phi \in L_{2,F}$ with values on the set of measures M on $\mathfrak{B}$ given by $M_\Phi(E) = \int_E \Phi dF$, is an isomorphism between $L_{2,F}$ and $H_{2,F}$.*

Let g be a fixed element of G. By $\mathfrak{M}_g$ we will denote the closed subspace of H^q spanned by X_h, $h \neq g$, and

$$\mathfrak{N}_g = \mathfrak{M} \ominus \mathfrak{M}_g.$$

The following theorem in the general case of locally compact Abelian groups is proved in [9], 2.8.

(3.4) Theorem. *Let $(X_g)_{g \in G}$ be a q-variate stationary process over G, and F its spectral measure. Then $V_2 V_1 \mathfrak{N}_e$ consists of all matrix-valued measures N_A from the space $H_{2,F}$, where $N_A(E) = A d\gamma(E)$ for each $E \in \mathfrak{B}$, and A is any $q \times q$-matrix.*

Remark. Let $X \in \mathfrak{N}_e$. From the diagram which is presented in [10], p. 175, it follows that, for each $E \in \mathfrak{B}$,

(3.5) $$M_X(E) = N_{(X, X_e)}(E) = (X, X_e)\, d\gamma(E), \quad \text{where } M_X = V_2 V_1 X.$$

Let F^a and F^s denote the absolutely continuous and singular parts of F in the Craimér's decomposition (2.9) w.r.t. the Haar measure $d\gamma$ and let F' denote the derivative of F^a. If we put $\Phi_X = V_1 X$, then, by (3.3) and (3.5), we have

$$\int_E (X, X_e)\, d\gamma = M_X(E) = \int_E \Phi_X dF = \int_E \Phi_X dF^a + \int_E \Phi_X dF^s.$$

Since $F^s \perp d\gamma$, we conclude that

(3.6) $$\begin{cases} \Phi_X F' = (X, X_e)\, d\gamma\text{-a.e.,} \\ \Phi_X = 0 \text{ on } S, \text{ where } S \text{ denotes the support of } F^s. \end{cases}$$

(3.7) Lemma. *Let F be the spectral measure of a q-variate stationary process $(X_g)_{g \in G}$ over G and let $N_A = A d\gamma$, where A is any $q \times q$-matrix. Then $N_A \in H_{2,F}$ if and only if $\mathscr{R}(A) \subset \mathscr{R}(F')\, d\gamma$-a.e. and the integral $\int_\Gamma A(F')^{\#} A^* d\gamma$ exists.*

Proof. *Necessity.* Let $N_A \in H_{2,F}$. Then $N_A \langle\langle\langle F$ and by (2.11) there exists a function $\Phi_A \in L_{2,F}$ such that for each $E \in \mathfrak{B}$

$$N_A(E) = \int_E \Phi_A dF.$$

Since, for all $E \in \mathfrak{B}$,

$$A d\gamma(E) = N_A(E) = \int_E \Phi_A dF = \int_E \Phi_A F' d\gamma + \int_E \Phi_A dF^s,$$

we conclude that $\Phi_A F' = A$ $d\gamma$-a.e. and $\Phi_A = 0$ on S, where $S = \operatorname{supp} F^s$. Thus $\mathscr{R}(A) = \mathscr{R}(\Phi_A F') \subset \mathscr{R}(F')$ $d\gamma$-a.e. Furthermore

$$\int_\Gamma dN_A dN_A^* / dF = \int_\Gamma \Phi_A dF \Phi_A^* = \int_\Gamma \Phi_A F' \Phi_A^* d\gamma = \int_\Gamma A(F')^{\#} A^* d\gamma$$

and consequently the integral $\int_\Gamma A(F')^{\#} A^* d\gamma$ exists. The sufficiency is trivial. ■

4. The minimality theorem. Let $(X_g)_{g \in G}$ be a q-variate stationary process over discrete Abelian group G and F its spectral measure. The derivative of the absolutely continuous part of F in the Cramér's decomposition w.r.t. the Haar measure $d\gamma$ will be called the *spectral density* of $(X_g)_{g \in G}$ and will be denoted by F'.

(4.1) Definition. We say that the stationary process $(X_g)_{g \in G}$ is *minimal* if $X_e \notin \mathfrak{N}_e$.

For the proof of the minimality theorem, we need the following lemmas.

(4.2) Lemma ([4], 3.2). *Let E be a measurable set and let Φ be a non-negative Hermitian matrix-valued function on Ω such that $\int_E \Phi dm$ exists. Then*

(a) *for almost all ω in E $\mathscr{N}\left(\int_E \Phi dm\right) \subset \mathscr{N}(\Phi(\omega))$,*

(b) *for almost all ω in E $\mathscr{R}(\Phi(\omega)) \subset \mathscr{R}\left(\int_E \Phi dm\right)$,*

(c) *if for almost all ω in E, $\mathscr{R}(\Phi(\omega)) \subset \mathscr{M}$, where $\mathscr{M}$ is a subspace of C^q, then $\mathscr{R}\left(\int_E \Phi dm\right) \subset \mathscr{M}$.*

The following lemma is easy to prove.

(4.3) Lemma. *Let Φ be a matrix-valued function on Ω and let A, B be any matrices. Then*

(a) *the integral $\int_\Omega A \Phi dm$ exists if and only if the integral $\int_\Omega P_{\mathscr{R}(A)} \Phi dm$ exists; if these integrals exist, then*

$$\int_\Omega A \Phi dm = A \int_\Omega P_{\mathscr{R}(A)} \Phi dm,$$

(b) *the integral $\int_\Omega \Phi B dm$ exists if and only if the integral $\int_\Omega \Phi P_{\mathscr{R}(B^*)} dm$ exists; if these integrals exist, then*

$$\int_\Omega \Phi B dm = \int_\Omega \Phi P_{\mathscr{R}(B^*)} dm \cdot B.$$

(4.4) Lemma (cf. [10], p. 181). *If Φ is a non-negative Hermitian matrix-valued function, then the integral $\int_\Omega \Phi dm$ exists if and only if the integral $\int_\Omega \operatorname{tr} \Phi dm$ exists.*

Let Y_e be the orthogonal projection of X_e onto $\mathfrak{N}_e$. The letter J will denote the orthogonal projection matrix onto the range of (Y_e, Y_e). Now, we prove the following theorem which gives a characterization of the space $\mathscr{R}((Y_e, Y_e))$.

(4.5) Theorem. *Let $(X_g)_{g \in G}$ be a q-variate stationary process over discrete Abelian group G with the spectral measure F. The range $\mathscr{R}((Y_e, Y_e))$ is a maximal closed linear subspace $\mathscr{M}$ of C^q satisfying the following conditions:*

(a) *the integral $\int_\Gamma P_{\mathscr{M}} (F')^{\#} P_{\mathscr{M}} d\gamma$ exists,*

(b) *$\mathscr{M} \subset \mathscr{R}(F')$ $d\gamma$-a.e.,*

(c) *$\mathscr{R}\left(\int_\Gamma P_{\mathscr{M}} (F')^{\#} P_{\mathscr{M}} d\gamma\right) = \mathscr{M}$.*

Proof. Let $M = V_2 V_1 Y_e$. By (3.5), $M(E) = (X, Y_e) d\gamma(E) = (Y_e, Y_e) d\gamma(E)$ for all $E \in \mathfrak{B}$. Furthermore, by (3.7), $\mathscr{R}((Y_e, Y_e)) \subset \mathscr{R}(F')$ $d\gamma$-a.e. and the integral $\int_\Gamma J (F')^{\#} J d\gamma$ exists, $J = P_{\mathscr{R}((Y_e, Y_e))}$. These facts in combination with (2.1)–(2.3) imply

$$\begin{aligned} (Y_e, Y_e)^{\#} &= (Y_e, Y_e)^{\#} (M, M)_F (Y_e, Y_e)^{\#} \\ &= (Y_e, Y_e)^{\#} \int_\Gamma (Y_e, Y_e)(F')^{\#} (Y_e, Y_e) d\gamma (Y_e, Y_e)^{\#} \\ &= \int_\Gamma J (F')^{\#} J d\gamma. \end{aligned}$$

Consequently, by 2.7, $\mathscr{R}((Y_e, Y_e)) = \mathscr{R}\left(\int_\Gamma J (F')^{\#} J d\gamma\right)$. Thus, conditions (a), (b), (c) hold for $\mathscr{M} = \mathscr{R}((Y_e, Y_e))$.

Let $\mathscr{M}$ be a closed linear subspace of C^q satisfying conditions (a), (b), (c). We shall show that $\mathscr{M} \subset \mathscr{R}((Y_e, Y_e))$. Put $B = \int_\Gamma P_{\mathscr{M}} (F')^{\#} P_{\mathscr{M}} d\gamma$. By (3.4) and (3.7), $N_B \in V_2 V_1 \mathfrak{N}_e$ where $N_B(E) = B d\gamma(E)$ for all $E \in \mathfrak{B}$. Let Y be an element of $\mathfrak{N}_e$ such that $N_B = V_2 V_1 Y$; then, by (3.5), $B = (Y, Y_e)$. Since $Y \in \mathfrak{N}_e$, $Y = Y_e D$ for any matrix D. Hence

$$\mathscr{R}((Y_e, Y_e)) \supset \mathscr{R}((Y_e D, Y_e)) = \mathscr{R}(B) = \mathscr{M}. \ ■$$

Now we state the main result of this paper.

(4.6) Theorem. *The following properties of q-variate stationary process $(X_g)_{g \in G}$ over a discrete Abelian group G are equivalent:*

(a) *the process $(X_g)_{g \in G}$ is minimal,*

(b) *the Hellinger integral $\int_\Gamma dN_J dN_J / dF$ exists and is non-zero, where for each $E \in \mathfrak{B}$, $N_J(E) = J d\gamma(E)$,*

(c) *there exists a linear closed subspace* $\mathscr{M} \subset \mathscr{R}(F')$ $d\gamma$*-a.e. such that the integral* $\int_\Gamma P_{\mathscr{M}}(F')^{\#} P_{\mathscr{M}} d\gamma$ *exists and is non-zero,*

(d) *there exists a matrix* A *with* $\mathscr{R}(A) \subset \mathscr{R}(F')$ $d\gamma$*-a.e. such that the integral* $\int_\Gamma \operatorname{tr}\{A(F')^{\#}A^*\} d\gamma$ *exists and is non-zero.*

Proof. First of all we remark that in view of (4.4) we may use also the trace in the integrals in conditions (b) and (c).

(a) $\Rightarrow$ (b). Let $(X_g)_{g\in G}$ be minimal. If we denote $M = V_2 V_1 Y_e$, then by assumption and by (3.3), $(M, M)_F = (Y_e, Y_e) \neq 0$. Using the same argument as that in the proof of (4.5), we obtain

$$(4.7) \qquad (Y_e, Y_e)^{\#} = \int_\Gamma J(F')^{\#} J d\gamma = \int_\Gamma dN_J dN_J / dF.$$

Hence, the integral $\int_\Gamma dN_J dN_J/dF$ exists and is non-zero.

(b) $\Rightarrow$ (c). Suppose that $\int_\Gamma dN_J dN_J/dF$ exists and is non-zero. Put $\mathscr{M} = \mathscr{R}((Y_e, Y_e))$. From (4.5), $\mathscr{M} = \mathscr{R}((Y_e, Y_e)) \subset \mathscr{R}(F')$ $d\gamma$-a.e. Since $\int_\Gamma P_{\mathscr{M}}(F')^{\#} P_{\mathscr{M}} d\gamma = \int_\Gamma dN_J dN_J/dF$, the first integral exists and is non-zero.

(c) $\Rightarrow$ (d). Let us suppose that (c) holds. If we put $A = P_{\mathscr{M}}$, then it is obvious that the integral $\int_\Gamma \operatorname{tr}\{A(F')^{\#}A^*\} d\gamma$ exists and $\mathscr{R}(A) = \mathscr{M} \subset \mathscr{R}(F')$ $d\gamma$-a.e. Since F is the non-negative Hermitian function, then, by (2.7), $A(F')^{\#}A^*$ is also non-negative and Hermitian. By the inequality $0 \leqslant \Psi \leqslant (\operatorname{tr}\Psi) I$ which is satisfied for any non-negative and Hermitian matrix, we deduce that the integral $\int_\Gamma \operatorname{tr}\{A(F')^{\#}A^*\} d\gamma$ is non-zero.

(d) $\Rightarrow$ (a). Let now A be any matrix such that (d) is satisfied. We put $\Psi = A(F')^{\#}A^*$. By (4.4), follows that $\Psi \in L_{1,d\gamma}$. Put $B = \int_\Gamma P_{\mathscr{R}(A)}(F')^{\#} P_{\mathscr{R}(A)} d\gamma$ (by (4.3) this integral exists). Since $0 \neq \int_\Gamma \operatorname{tr}\{A(F')^{\#}A^*\} d\gamma$ and

$$\int_\Gamma \operatorname{tr}\{A(F')^{\#}A^*\} d\gamma = \operatorname{tr}\Big(\int_\Gamma A(F')^{\#}A^* d\gamma\Big) = \operatorname{tr}(ABA^*),$$

B is non-zero. Clearly, by (4.2)(c), $\mathscr{R}(B) \subset \mathscr{R}(A) \subset \mathscr{R}(F')$ $d\gamma$-a.e. If we put $N_B(E) = \int_E B d\gamma$ for all $E \in \mathfrak{B}$, then from (3.7) and (3.4) it follows that $N_B \in V_2 V_1 \mathfrak{N}_e$. We note that (2.13) in combination with the fact $B \neq 0$ implies that N_B is non-zero in $H_{2,F}$. In view of the isomorphism theorem (3.3), the space $\mathfrak{N}_e$ is non-zero. Concluding, the process $(X_g)_{g\in G}$ is minimal. ■

This result implies the univariate minimality theorem (cf. [1]) and the theorem by H. Salehi and J. K. Scheidt ([8], 3.6) for q-variate full rank processes over groups.

(4.8) Corollary (Kolomogorov's minimality theorem). *Let* $(X_n)_{n\in Z}$ *be a univariate stationary process over the group* Z *of integers, and* F' *its spectral density. Then* $(X_n)_{n\in Z}$ *is minimal if and only if* $\int_0^{2\pi} d\gamma / F'(\gamma) < \infty$.

Proof. This assertion is an easy consequence of (4.6)(c) and the fact that $\mathscr{M}$ is a closed linear subspace of C if and only if $\mathscr{M} = \{0\}$ or $\mathscr{M} = C$. ■

(4.9) Corollary (Masani's minimality theorem). *Let* $(X_g)_{g\in G}$ *be a q-variate stationary process over discrete Abelian group* G *and* F *its spectral density. Then* $(X_g)_{g\in G}$ *is minimal and* $\operatorname{rank}(Y_e, Y_e) = q$ *if and only if* $(F')^{-1}$ *exists* $d\gamma$*-a.e. and* $(F')^{-1} \in L_{1,d\gamma}$.

Proof. Let $(X_g)_{g\in G}$ be minimal and $\operatorname{rank}(Y_e, Y_e) = q$. Then by the proof of (4.5), $\mathscr{R}((Y_e, Y_e)) \subset \mathscr{R}(F')$ $d\gamma$-a.e. and the integral $\int_\Gamma J(F')^{\#} J\, d\gamma$ exists, $J = P_{\mathscr{R}(Y_e, Y_e)}$. Since $\mathscr{R}((Y_e, Y_e)) = C^q$, it follows that $(F')^{-1}$ exists $d\gamma$-a.e. and $(F')^{-1} \in L_{1,d\gamma}$.

Conversely, let $(F')^{-1}$ exist $d\gamma$-a.e. and $(F')^{-1} \in L_{1,d\gamma}$. By (4.5) and (4.2)(b), $\mathscr{R}((Y_e, Y_e)) = C^q$. Hence the process $(X_g)_{g\in G}$ is minimal and $\operatorname{rank}(Y_e, Y_e) = q$. ■

Finally we give the formulas for the linear interpolation. Let $(X_g)_{g\in G}$ be a q-variate stationary process over discrete Abelian group G. Suppose that all X_h for $h \neq g$ are known, g is a fixed element of G. We say that X_g^0 is a prediction of X_g based on observations from the complement of the element g if $X_g^0 \in \mathfrak{M}_g$ and

$$\|X_g - X_g^0\| = \min_{X \in \mathfrak{M}_g} \|X_g - X\|.$$

It follows that X_g^0 is the projection of X_g onto $\mathfrak{M}_g$. We note that the closed subspace of $\mathfrak{M}$ spanned by $X_g - X_g^0$ is exactly the space $\mathfrak{N}_g$ which was defined in Section 3. Thus, $X_g^0 = X_g - Y_g$, where, as before, Y_g denotes the orthogonal projection of X_g onto $\mathfrak{N}_g$. If we put $\Phi_g^0 = V_1 X_g^0$, $M_g^0 = V_2 \Phi_g^0$ then, from (4.7) and (2.6),

$$(Y_e, Y_e) = \Big(\int_\Gamma J(F')^{\#} J d\gamma\Big)^{\#}.$$

Consequently, by (3.3), (3.5), (3.6), we have the following formulas for the prediction:

$$(4.10) \qquad M_e^0(E) = F(E) - \Big(\int_\Gamma J(F')^{\#} J d\gamma\Big)^{\#} d\gamma(E), \qquad E \in \mathfrak{B},$$

$$(4.11) \qquad \Phi_e^0(\gamma) = I - \Big(\int_\Gamma J(F')^{\#} J d\gamma\Big)^{\#} \big(F'(\gamma)\big)^{\#} \pmod F,$$

where J denotes the orthogonal projection matrix onto the range of (Y_e, Y_e).

It is well known that if $\{U_g\}_{g\in G}$ is a shift groups of unitary operators on $\mathfrak{M}$ defined by equality $U_g X_h = X_{hg}$, $g, h \in G$, then for each $g \in G$ and

$Y \in \mathfrak{M}$, $V_1 U_g Y = \overline{\langle g, \gamma\rangle} V_1 Y$. Consequently, from the obviously equality $X_g^0 = U_g X_e^0$, we obtain

$$\Phi_g^0(\gamma) = \overline{\langle g, \gamma\rangle}\, \Phi_e^0(\gamma) \pmod F, \tag{4.12}$$

$$M_g^0(E) = \int_E \overline{\langle g, \gamma\rangle}\, dM_e^0. \tag{4.13}$$

Let us note that the space $\mathscr{R}\big((Y_e, Y_e)\big)$ used in these formulas has the characterization given in (4.5).

5. Examples. In this section we will give several examples illustrating the above results.

(5.1) Example. Let M, N be the real-valued measures on the Borel field of $[0, 2\pi)$, absolutely continuous w.r.t. the Haar measure dt, and let theirs derivatives be given by

$$M'(t) = 1_{[0,\pi)}(t), \quad N'(t) = 1_{[\pi, 2\pi)}(t), \quad t \in [0, 2\pi),$$

where $1_A(t)$ denotes the indicator of A. Obviously, the integral $\int_0^{2\pi} dM dM^*/dN$ exists. But does not exist a function Φ such that for each measurable set E, $M(E) = \int_0^{2\pi} \Phi(t)\, dN(t)$. Therefore the assumption M is strongly absolutely continuous w.r.t. N in (2.11) is essential one.

(5.2) Example. Let $(X_n)_{n\in Z}$ and $(Y_n)_{n\in Z}$ be mutually orthogonal stationary processes over the group Z, of the integers, with the absolutely continuous spectral measures F_1 and F_2. Let $F_1'(t) = 1_{[0,\pi)}(t)$, $F_2'(t) = 1_{[\pi,2\pi)}(t)$, $t \in [0, 2\pi)$. Then $W_n = [X_n, Y_n]$, $n \in Z$, is a bivariate stationary process over Z with the spectral density given by

$$F'(t) = \begin{bmatrix} 1_{[0,\pi)}(t) & 0 \\ 0 & 1_{[\pi,2\pi)}(t) \end{bmatrix}, \quad t \in [0, 2\pi).$$

From the univariate Kolmogorov's minimality theorem we have that the processes $(X_n)_{n\in Z}$ and $(Y_n)_{n\in Z}$ are not minimal, and therefore the process $(W_n)_{n\in Z}$ is not minimal. We note that

$$(F')^{\#}(t) = \begin{bmatrix} 1_{[0,\pi)}(t) & 0 \\ 0 & 1_{[\pi,2\pi)}(t) \end{bmatrix}$$

is integrable w.r.t. dt, but does not exist a non-zero closed linear subspace $\mathscr{M} \subset C^2$ such that $\mathscr{M} \subset \mathscr{R}(F')$ dt-a.e. It shows that the assumption $\mathscr{M} \subset \mathscr{R}(F')$ $d\gamma$-a.e. in (4.6)(c) is essential.

(5.3) Example. Let $(X_n)_{n\in Z}$ and $(Y_n)_{n\in Z}$ be mutually orthogonal univariate stationary processes over Z, with the absolutely continuous spectral measures F_1 and F_2. Let $F_1'(t) = t$, $F_2'(t) = 1$, $t \in [0, 2\pi)$. Then $W_n = [X_n, Y_n]$, $n \in Z$, is a bivariate stationary process over Z with the spectral density

$$F'(t) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}, \quad t \in [0, 2\pi).$$

Since $V = [0, X_0] \notin \mathfrak{M}_0$ and $V \neq 0$, it follows that the process $(W_n)_{n\in Z}$ is minimal. But the integral $\int_0^{2\pi} (F')^{\#}\, dt$ does not exist. Hence, the theorems [6], 10.2, and [10], 5.7, are not true, but their correct version is given in (4.6)(d). It is clear that if we put $\mathscr{M} = C_2$, where the number at the bottom is numbering of the axes in the product $C^2 = C_1 \times C_2$, then condition (c) of (4.6) is satisfied.

In the last example we show how to obtain the prediction formulas in some special case.

(5.4) Example. Let $(W_n)_{n\in Z}$ be as in (5.3). Since $(F')^{\#}(t) = (F')^{-1}(t)$ and $(F')^{-1} \notin L_{1,dt}$, we have, by (4.5),

$$J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

According to formulas (4.10) and (4.11) we obtain:

$$(dM_0^0/dt)(t) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}, \quad t \in [0, 2\pi),$$

$$\Phi_0^0(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad t \in [0, 2\pi).$$

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The range of vector measures into Orlicz spaces

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Abstract. It is shown that the range of a σ-additive vector measure having values in an Orlicz space $L_\varphi(X, A, \mu)$, where φ is unbounded and satisfies condition Δ_2, is bounded. This implies that every scalar-valued, bounded measurable function can be integrated with respect to any vector measure taking values in such a space $L_\varphi(X, A, \mu)$. In the special case of the sequence spaces l^p, $0 < p < 1$, the range is relatively compact, and the closure is even convex and compact if the measure is nonatomic.

1. It is known that the range of every σ-additive vector measure with values in a locally pseudoconvex vector space is bounded (cf. [1]). On the other hand P. Turpin has shown in [11] that there exists a non-locally pseudoconvex F-space and a vector measure having unbounded range in that space. With regard to integration theory it would be important to know whether a vector measure has always bounded range in an Orlicz space $L_\varphi(X, A, \mu)$ (cf. [8]). P. Turpin states this question in [9] and [11].

In this note we answer the question positively for the class of Orlicz spaces $L_\varphi(X, A, \mu)$, where φ is unbounded and satisfies condition Δ_2. It is done by showing that every normbounded, convex set in $L_\varphi(X, A, \mu)$ is bounded and then using the fact that the convex hull of the range of such a vector measure is normbounded. The latter follows from an inequality for Orlicz spaces, which is essential for the proof that in these spaces unconditional convergence is equivalent to bounded multiplier convergence ([4], [10]).

As a consequence every scalar-valued, bounded measurable function can be integrated with respect to any vector measure taking values in such a space $L_\varphi(X, A, \mu)$.

In the special case of the sequence spaces l^p, $0 < p < 1$, the range is even relatively compact. When such a vector measure is also nonatomic, the closure of its range is compact and convex.

2. Throughout the paper, Ω will denote a set and Σ a σ-algebra of subsets. Let Y be an F-space (i.e. a complete metric topological linear