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Use of the Power inequality in connection of coefficient regions for bounded univalent functions

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Abstract. Consider bounded and normalized univalent functions

$$f(z) = bz + b_2 z^2 + \ldots,$$
$$|z| < 1, \quad |f(z)| < 1,$$
$$b \in (0, 1], \quad a_n = \frac{b_n}{b}.$$

For them coefficient regions have been studied mainly by aid of Löwner – and variational methods. It appears, that also the Power inequality

$$\sum_{-N}^{\infty} k|y_k|^2 + 2\operatorname{Re}(\bar{x}_0 y_0) \leqslant \sum_{-N}^{\infty} k|x_k|^2 \quad (N = 1, 2, \ldots),$$
$$y_k = \sum_{\nu=-N}^{k} x_\nu c_{\nu k} \quad (k \geqslant -N),$$
$$f(z)^\nu = \sum_{k=\nu}^{\infty} c_{\nu k} z^k, \quad \log\frac{f(z)}{z} = \sum_{k=0}^{\infty} c_{0k} z^k \psi,$$

implies information of the coefficient regions and characterizes the boundary functions in cases where they satisfy the necessary equality condition

$$x_0 \log f + \sum_{-N}^{N}{}' x_\nu f^\nu = \sum_{-N}^{N} y_k z^k.$$

In the lowest case $N = 1$ results concerning (a_3, a_2) can be described in detail and they are comparable to those of Schaeffer and Spencer. $N = 3$ gives analogous but more complicated information of the next coefficient region (a_4, a_3, a_2).

1. Power inequality. We consider bounded and univalent functions f defined in the unit disc $U = \{z \in C \mid |z| < 1\}$ and normalized according to the following notations:

$$f(z) = bz + b_2 z^2 + \ldots,$$
$$|f(z)| < 1,$$
$$b \in (0, 1]; \quad \frac{b_n}{b} = a_n \quad (b_1 = b).$$

The class of functions, for which the leading coefficient b is constant, is denoted by $S(b)$.

For these functions one can derive an effective inequality by starting from Green's identity and from the inequality implied in it:

$$0 \leqslant \iint_D |g'|^2 d\sigma = \frac{1}{i} \int_{\partial D} \mathrm{Re}\, g(w) \cdot g'(w) dw .$$

Here the domain of integration D is determined as follows. Let ∂K_r be a circle in U, center at the origin and radius r (<1). Its image under f is $C = f(\partial K_r)$. The ring which is bounded by C and the unit circumference ∂K_1 in the w-plane is cut open by a properly chosen slit and the result is the domain D.

The function g, which we call the *generating function*, is defined to be

$$g(w) = x_0 \log w + \sum_{-N}^{N}{}' x_\nu w^\nu \qquad (\nu \neq 0 \text{ in } \Sigma';\ N = 1, 2, \ldots).$$

Here the numbers x_ν are free complex parameters with the assumption that $x_0 \in R$. This limitation is unessential for the inequality resulting; the inequality can easily be extended to concern complex x_0 too.

By applying Green's identity to the pair (D, g) we obtain an inequality which, further, allows the limit process $r \to 1$. The result is called the *Power inequality* or shortly P_N*-inequality* and reads as follows:

$$\sum_{-N}^{\infty} k |y_k|^2 + 2\, \mathrm{Re}(\bar{x}_0 y_0) \leqslant \sum_{-N}^{N} k |x_k|^2 \qquad (N = 1, 2, \ldots).$$

Here the numbers y_k mean the combinations

$$y_k = \sum_{\nu = -N}^{k} c_{\nu k} x_\nu \qquad (k \geqslant -N),$$

which are got from the expansion

$$g\bigl(f(z)\bigr) = x_0 \log z + \sum_{-N}^{\infty} y_k z^k .$$

The numbers $c_{\nu k}$ are called the *Power-coefficients* and are defined as coeficients of the following expansions

$$f(z)^\nu = \sum_{k=\nu}^{\infty} c_{\nu k} z^k, \qquad \log \frac{f(z)}{z} = \sum_{k=0}^{\infty} c_{0k} z^k .$$

In applications we utilize the truncated form of the Power-inequality

$$\sum_{-N}^{N} k |y_k|^2 + 2\, \mathrm{Re}(\bar{x}_0 y_0) \leqslant \sum_{-N}^{N} k |x_k|^2 .$$

This can further be transformed by aid of Schwarz inequality, under the assumption that $\mathrm{Re}(\bar{x}_0 y_0) = 0$, to the following bilinear form

$$\left|\sum_1^N k(y_{-k}y_k + x_{-k}x_k)\right| \leqslant \sum_1^N k(|y_{-k}|^2 + |x_k|^2).$$

Equality in these transformations is preserved provided that $y_{N+1} = \ldots = 0$. Therefore, we get for the possible extremum function a necessary condition, by comparing the generating function and the corresponding expansion:

$$x_0 \log f + \sum_{-N}^{N}{}' x_\nu f^\nu = x_0 \log z + \sum_{-N}^{N} y_\nu z^\nu.$$

This means, that only such problems can be sharply solved by aid of the Power inequality, which have an extremal function satisfying a condition of the above type. There appears, that many interesting coefficient problems are of this nice character. We are going to illustrate this by considering two first coefficient bodies in $S(b)$ by aid of this Power inequality method.

2. The optimized P_1-inequality for $f(z)$. The quadratic P_1-inequality reads

$$2\,\mathrm{Re}(\bar{x}_0 y_0) + |y_1|^2 \leqslant |x_1|^2 + (1-b^2)|u_1|^2,$$

where

$$\begin{aligned} y_{-1} &= -u_1 \text{ is a parameter;} \\ y_0 &= x_0 \log b + u_1 a_2, \\ y_1 &= u_1(a_3 - a_2^2) + x_0 a_2 + x_1 b. \end{aligned}$$

If we suppose that $x_0 = 0$, the ratio x_1/u_1 is left as a parameter in the inequality. Denoting $u_1 = 1$ we thus get for the combination

$$\delta = a_3 - a_2^2$$

the condition

$$|\delta + b x_1|^2 - |x_1|^2 - (1-b^2) \leqslant 0.$$

If $x_0 \neq 0$ we similarly way take $x_0 = 1$ and obtain

$$|u_1 \delta + a_2 + x_1 b|^2 - |x_1|^2 - (1-b^2)|u_1|^2 - 2\,\mathrm{Re}(u_1 a_2) - 2\log b \leqslant 0.$$

In both the cases, the left-hand side is quadratic in some complex variables and can, of course, be explained quadratic in real variables too by splitting the complex variables in real and imaginary parts. It is a very natural procedure to optimize the inequality by choosing variables so that the left-hand side is maximized. In the case $x_0 \neq 0$, the optimal choice appears to be the following:

$$u_1 = -\frac{(1-b^2)\bar{a}_2+\bar{\delta}a_2}{|\delta|^2-(1-b^2)^2}; \quad x_1 = b\bar{u}_1.$$

This gives to the inequality the form

$$|\delta-\delta_0| \leqslant R; \quad \delta_0 = \frac{a_2^2}{2\log b}, \quad R = 1-b^2+\frac{|a_2|^2}{2\log b} > 0.$$

Geometrically this means that δ lies in a disc having center at δ_0 and radius R. The parameter values found show, that

$$x_{-1} = -\bar{x}_1, \quad y_{-1} = -\bar{y}_1.$$

Thus, equality is reached provided that the following condition defines a $S(b)$-function:

$$\log f + b(\bar{u}_1 f - u_1 f^{-1}) = \log z + \bar{u}_1 z - u_1 z^{-1} + y_0.$$

If $x_0 = 0$, we obtain similar equality condition, without the logarithmic terms.

The functions defined by the above condition can be completely determined by aid of boundary correspondence and the monodromy theorem. There appears, that if $|a_2| \leqslant 2b\,|\log b|$ the whole boundary of the δ-disc is reached by extremal functions. From this limit upwards only a shrinking part of the boundary is connected to $S(b)$-functions.

In the case where $a_2 \in R$, we are led to a 3-dimensional coefficient-region by letting the δ-disc move in the direction of the a_2-axis.

Especially, if all the coefficients $a_2, a_3, \ldots$ are real, we get an intersection of the previous figure. In this case the coefficient body can be determined completely by filling the caps left by aid of Löwner's method.

The family of extremal functions is an interesting one, consisting of symmetric and non-symmetric two-slit domains.

The reason for the cap left is obvious. The extremal domains belonging to the cap are expected to be of one-slit or forked slit type. The Power inequality is not constructed to fit with such externmal cases.

3. The optimized P_3-inequality for $\sqrt{f(z^2)}$. It is a very natural and interesting question to check possibilities of generalizing the preceding results for higher coefficients. The next step is the coefficient region (a_4, a_3, a_2) which can be studied by applying the P_3-inequality to the function $F(z) = \sqrt{f(z^2)}$. The expressions involved are much more complicated and almost too difficult to handle if we use the quadratic inequality. The bilinear form is more reasonable. When studying preservation of equality in the use of Schwarz inequality we end up to the conditions

$$x_{-\nu} = -\bar{x}_\nu, \quad y_{-\nu} = -\bar{y}_\nu.$$

Because these were true in the preceding extremum case, we have all reasons to expect sharp results from the bilinear P_3-inequality.

As before, use as parameters

$$u_\nu = -\nu y_{-\nu} \quad (\nu = 1, 2, 3)$$

and apply the symmetric choice

$$x_{-\nu} = -\bar{x}_\nu \quad (\nu = 1, 2, 3)$$

together with

$$x_0 = x_2 = 0.$$

This implies

$$u_2 = 0; \quad y_0 = y_2 = 0$$

and the bilinear P_3-inequality assumes the form

$$\mathrm{Re}(u_1 y_1 + u_3 y_3) - |u_1|^2 - \frac{|u_3|^2}{3} \leqslant 0.$$

Here is

$$y_1 = a_1 u_1 + b\bar{u}_1 + d_1 u_3 + e_1 \bar{u}_3,$$

$$y_3 = d_1 u_1 + \bar{e}_1 \bar{u}_1 + d_3 u_3 + e_3 \bar{u}_3;$$

$$a_1 = \frac{a_2}{2},$$

$$d_1 = \frac{a_3}{2} - \frac{3}{8} a_2^2,$$

$$e_1 = \frac{b}{2} \bar{a}_2,$$

$$d_3 = \frac{a_4}{2} - a_2 a_3 + \frac{13}{24} a_2^3,$$

$$e_3 = \frac{b}{4} |a_2|^2 + \frac{b^3}{3}.$$

Again, we optimize the inequality by choosing u_1 and u_3 so that the left-hand side is maximized. The result is a complete generalization of the preciding one and reads as follows. Let a_2 and a_3 be given. Then d_3 lies in a disc

$$|d_3 + d_1\lambda + \bar{e}_1\mu| \leqslant 1/3 - e_3 - d_1\mu - \bar{e}_1\bar{\lambda},$$

where

$$\lambda = \frac{\bar{a}_2\left(a_3 - \frac{3}{4}a_2^2\right) + 2b(1-b)a_2}{[2(1-b)]^2 - |a_2|^2},$$

$$\mu = \frac{b\bar{a}_2^2 + 2(1-b)\overline{\left(a_3 - \frac{3}{4}a_2^2\right)}}{[2(1-b)]^2 - |a_2|^2}.$$

The boundary points of the d_3-disc can be located by aid of the parameter $u_3 = e^{i\omega}$ so that at the boundary

$$d_3 + d_1\lambda + \bar{e}_1\bar{\mu} = (1/3 - e_3 - d_1\mu - \bar{e}_1\bar{\lambda})u_3^{-2}.$$

Equality holds if $F(z) = \sqrt{f(z^2)}$ defines a $S(b)$-function f, when F is determined by the condition

$$\frac{b^{3/2}}{3}(\bar{u}_3F^3 - u_3F^{-3}) + b^{1/2}(\bar{s}F - sF^{-1}) = \tfrac{1}{3}(\bar{u}_3z^3 - u_3z^{-3}) + \bar{u}_1z - u_1z^{-1};$$

$$u_1 = \lambda u_3 + \mu\bar{u}_3, \qquad s = u_1 + \frac{a_2}{2}u_3.$$

In the extremum case the conditions

$$y_{-1} = -\bar{y}_1, \qquad y_{-3} = -\bar{y}_3$$

appear to be true, as was to be expected.

The extremum condition is much more complicated than that for (a_3, a_2). In general, we can not interprete it without numerical calculations. However, the special case $a_2, a_3, a_4 \epsilon R$, can be completely solved in closed form. a_4 is maximized at the point $u_3 = 1$:

$$d_3 \leqslant \frac{1}{3}(1-b^3) - \frac{b}{4}a_2^2 - \frac{1}{2}\frac{(a_3 - \frac{3}{4}a_2^2 + ba_2)^2}{2(1-b) - a_2}.$$

The domain which gives boundary functions, i.e. for which equality is attained, lies inside the coefficient body (a_3, a_2) and is bounded by two parabolas. This domain is divided by a parabola in two parts connected to 3-slit or forked slit extremum domains.

I have verified that also in the complex case several boundary points of the d_3-disc give sharp results. However, complete analysis in the complex case seems to me to involve too many parameters.

The above results are included in two joint papers with R. Kortram, published 1974 in the Annales of the Finnish Academy of Sciences.

References

[1] R. Kortram and O. Tammi, *On the first coefficient regions of bounded univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I no. 591 (1974).

[2] — — *On the coefficient region* (a_4, a_3, a_2) *of bounded univalent functions*, to appear, 1974.