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Дифференциальные формы, ортогональные голоморфным формам и их свойства

Л. А. Айзенберг (Красноярск)

1. Ряд результатов теории функций одного комплексного переменного опирается на известное утверждение: если $\varphi(z)$ непрерывна на гладкой границе ∂D ограниченной области D, то

$$\int_{\partial D} f(z)\varphi(z)\,dz = 0$$

для всех голоморфных на $\bar{D}$ функций $f(z)$ тогда и только тогда, когда $\varphi(z)$ продолжается в D как голоморфная.

Аналогичная задача интересна и в теории функций многих комплексных переменных: описать внешние дифференциальные формы a на ∂D ортогональные голоморфным формам типа $(n-p, 0)$, при $0 \leqslant p \leqslant n$ (в частности, при $p = n$ — голоморфным функциям) в том смысле, что

$$\int_{\partial D} a \wedge \mu = 0$$

для всех μ, голоморфных на $\bar{D} \subset C^n$.

Кон и Росси описали формы, ортогональные $\bar{\partial}$-замкнутым формам типа $(n-p, q)$ при $q > 0$. Их работа не охватывает наш случай $q = 0$.

2. Некоторые шаги в решении поставленной задачи сделали Вайнсток и Айзенберг. Затем Ш. А. Даутов показал для строго псевдовыпуклой D, что формы, ортогональные голоморфным, продолжаются внутрь $\bar{D}$ как $\bar{\partial}$-замкнутые формы типа $(p, n-1)$. При доказательстве использовались интегральные представления Мартинелли–Бохнера–Компельмана и „барьерная" функция Хенкина. В работах Дуатова даны интересные приложения.

Опираясь на этот результат, Л. А. Айзенберг установил, что в случае произвольной ограниченной D всякая форма, ортогональная голоморфным, представляется как сумма сужений на ∂D двух форм типа $(p, n-1)$, первая $\bar{\partial}$-замкнута в D, а вторая — вне $\bar{D}$ и $\bar{\partial}$-

точна вне оболочки голоморфности компакта $\bar{D}$ (которая предполагается однолистной). Второе слагаемое, вообще говоря, по существу уже сделано (Даутов).

3. $\bar{\partial}$-замкнутые формы типа $(p, n-1)$ являются аналогами голоморфных функций одного комплексного переменного, т.е. сохраняют ряд важных свойств этих функций, которых нет у голоморфных функций многих комплексных переменных. Для таких форм доказаны теоремы об апроксимации и разделении особенностей, аналоги теорем Кузена и Морера, рассмотрена аддитивная краевая задача о скачке; указана связь этих форм с распределениями из $\boldsymbol{R}^{2n-1}$ и предположено новое определение умножения распределений (Вайнсток, Айзенберг, Даутов, Андреотти, Кытманов).

Compact holomorphic mappings on Banach spaces

by R. M. Aron and M. Schottenloher (Lexington)

Let E and F be complex Banach spaces. A holomorphic mapping $f\colon E \to F$ is said to be *compact* if for each point x of E, there is a neighbourhood V_x of x such that $f(V_x)$ is relatively compact in F. The space of compact holomorphic mappings from E to F is denoted $H_K(E; F)$.

The space $H_K(E; F)$ shares many properties with the space of compact linear mappings from E to F. For example, $f \in H_K(E; F)$ if and only if the linear mapping $\varphi \to \varphi \circ f$ is compact from F'_β to $(H(E; C), \mathscr{T}_\omega)$, where $\mathscr{T}_\omega$ is the Nachbin ported topology. Various other necessary and sufficient conditions for a holomorphic mapping to be compact are discussed, as well as the relation of compact holomorphic mappings to the aproximation property.

Метод граничных вариаций в задачах о неналегающих областях

Г. П. Бахтина (Киев)

С помощью метода граничных вариаций М. Шиффера решены задачи о максимуме функционала $\prod_{i=1}^{n} |f_i'(0)|$ для системы $\{f_i\}_1^n$ взаимно неналегающих однолистных конформных отображений единичного круга в следующих классах:

1. В классе систем мероморфных отображений, удовлетворяющих условию

$$\prod_{1 \leqslant i < k \leqslant n} |f_i(0) - f_k(0)|^{3/n(n-1)} = 1.$$

При $n \geqslant 4$ автором получен качественный результат,

при $n = 2$ — это известный результат М. А. Лаврентьева,

при $n = 3$ — полученный результат эквивалентен одной теореме Г. М. Голузина,

при $n = 4$ показывается, что система областей, дополнительная до двух ортогональных прямых, не является экстремальной.

2. В классе систем регулярных отображений в единичный круг.

При $n \geqslant 3$ получен качественный результат,

при $n = 2$ экстремальной являестя пара областей, каждая из которых есть полукруг.

Аналогичные задачи решены для многосвязных областей.

Non-orientable partial regular Riemannian coverings

by I. Bârzâ (Bucharest)

It presents the Hurwitz–Kerékjártó–Andrian formula for the ramification index of a covering map between surfaces, in the non-orientable case.

Sur les coefficients de fonctions étoilées symétriques

par O. Bereśniewicz-Rajca (Gliwice)

Soit $\hat{S}_G^{(k)}$, $k \geqslant 2$, une famille de fonctions étoilées et univalentes dans le circle $|z| < 1$, de la forme

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots,$$

avec k prémiers coefficiénts réels.

On montre:

Théorème I.

$$\inf_{\hat{S}_G^{(3)}} \{a_3\} = -1.$$

La fonction extrémale est de la forme

$$f(z) = \frac{z}{1+z^2} = z - z^3 + z^5 - \ldots$$

THÉORÈME II.

$$-3,2 < \inf_{\hat{S}_G^{(5)}} a_5 < -0,79.$$

Les démonstrations sont basées sur la relation entre les fonctions étoilées et les fonctions de Carathéodory, ainsi que sur un théorème caracterisant les fonctions de Carathéodory.

La note sera publiée dans les Zeszyty Naukowe Politechniki Śląskiej.

Über ein Problem von F. Leja

von F. BIERSKI (Kraków)

Zu Konferenz in Lublin im Jahre 1970 der Herr Prof. F. Leja hat einem Vorschlag über eigene Verallgemeinerungen der Kapazität und Extremalfunktionen geben. Ich lege daher einige Ergebnise vor.

Es sei im metrischen Raum T, dessen Punkte wir mit $p_1, p_2, \ldots, u$ bezeichnen, eine nicht negative, stetige und symmetrische Funktion $\omega(p_1, \ldots, p_a)$ von $a > 2$ Punkte definiert. Es sei in diesem Raume T eine kompakte und beschränkte Menge E gegeben.

Für ein beliebiges System

$$p^{(n)} = (p_1, \ldots, p_n, \ldots, p_{n+a-1}) \tag{1}$$

von $(n+a-1)$ Punkten der Menge E bezeichnen wir durch

$$V_n^{(i_1,\ldots,i_k)}(p^{(n)}) = \prod_{\substack{1 \leqslant j_1 < \ldots < j_{a-k} \leqslant n+a-1 \\ i_1,\ldots,i_k \neq j_1}} \omega(p_{i_1}, \ldots, p_{i_k}, p_{j_1}, \ldots, p_{j_{a-k}})$$

das Produkt von $\binom{n+a-k-1}{a-k}$ Faktoren, wo k $(=0, 1, \ldots, a-1)$ Punkte $p_{i_1}, \ldots, p_{i_k}$ des Systems (1) sind beliebig festgestellt.

Es sei nun

$$V_n^{(k)}(E, \omega) = \sup_{p^{(n)}} \big(\min_{i_1,\ldots,i_k} V_n^{(i_1,\ldots,i_k)}(p^{(n)}) \big)$$

und

$$v_n^{(k)}(E, \omega) - V_n^{(k)}(E, \omega)^{\beta_n}, \qquad \text{wo } \beta_n = 1 \Big/ \binom{n+a-k-1}{a-k}.$$

I. Die Folgen $\{V_n^{(k)}(E, \omega)\}$ erfüllen folgende Ungleichungen:

a) $V_{\mu+\nu}^{(a-1)}(E, \omega) \leqslant V_{\mu}^{(a-1)}(E, \omega)\, V_{\nu}^{(a-1)}(E, \omega)$,

b) $V_n^{(k)}(E, \omega)^{\frac{n+a-k-2}{a-k}} \leqslant V_n^{(k-1)}(E, \omega) \leqslant V_1^{(k)}(E, \omega)\, V_2^{(k)}(E, \omega) \ldots V_n^{(k)}(E, \omega)$

für $k = 1, \ldots, a-1$.

II. Die Folgen $\{v_n^{(k)}(E, \omega)\}$ konvergieren nach einem endlichen Grenzen:

$$\lim_{n\to\infty} v_n^{(a-1)}(E, \omega) = \lim_{n\to\infty} v_n^{(a-2)}(E, \omega) = \ldots = \lim_{n\to\infty} v_n^{(0)}(E, \omega) = v(E, \omega).$$

Wir nennen den Greznwert $v(E, \omega)$ verallgemeinerte Kapazität der Menge E in besug auf die Funktion ω.

Es sei u ein beliebiger Punkt der Raum T und (1) ein System der Punkten der Menge E, für welche $\omega(p_{j_1}, \ldots, p_{j_a}) \neq 0$, $1 \leqslant j_1 < j_2 < \ldots < j_a \leqslant n+a-1$. Führen wir folgende Funtionen des Punktes u ein:

$$\Phi_n^{(k)}(u, E) = \inf_{p^{(n)}} \Bigg(\max_{i_1, \ldots, i_k} \prod_{\substack{1 \leqslant j_1 < \ldots < j_{a-k} \leqslant n+a-1 \\ i_1, \ldots, i_k \neq j_1}} \frac{\omega(u, p_{i_2}, \ldots, p_{i_k}, p_{j_1}, \ldots, p_{j_{a-k}})}{\omega(p_{i_1}, p_{i_2}, \ldots, p_{i_k}, p_{j_1}, \ldots, p_{j_{a-k}})} \Bigg),$$

$$k = 1, \ldots, a \dot{-} 1.$$

III. Die Folgen $\{\Phi_n^{(k)}(u, E)\}$ erfüllen volgende Ungleichungen

a) $\Phi_{\mu+\nu}^{(a-1)}(u, E) \dot{\Phi}_\mu^{(a-1)}(u, E) \cdot \Phi_\nu^{(a-1)}(u, E)$,

b) $\Phi_n^{(k)}(u, E)^{\frac{n+a-k}{a-k}} \geqslant \Phi_n^{(k-1)}(u, E) \geqslant \Phi_1^{(k)}(u, E) \cdot \Phi_2^{(k)}(u, E) \ldots \Phi_n^{(k)}(u, E)$,
$$k = 1, \ldots, a-1.$$

IV. Wenn $v(E, \omega) > 0$, dann konvergieren die Folgen $\{[\Phi_n^{(k)}(u, E)]^{\beta_n}\}$ nach endlichen Grenzen $\varphi^{(k)}(u, E)$, $k = 1, \ldots, a-1$. Die Grenz-Funktionen $\varphi^{(k)}(u, E)$ nennen wir Extremal-Funktionen.

Sur la factorialité des anneaux de fonctions analytiques

par J. Bochnak (Kraków)

Théorème 1. *Soit M une variété analytique réelle, compacte, connexe avec $H^1(M, Z_2) = 0$. L'anneau $\mathcal{O}(M)$ de fonctions analytiques réelles sur M est factorial.*

Il est fort probable que la réciproque du Théorème 1 est vraie. En tout cas, on a

Théorème 2. *Pour une variété analytique réelle M, de dimension $\leqslant 2$ les condition suivantes sont équivalentes:*

a) *$\mathcal{O}(M)$ est factoriel;*

b) *M est compacte, connexe et $H^1(M, Z_2) = 0$;*

c) *$M = S^2$ (sphère de dimension 2).*

Pour un ensemble $A \subset R^n$ (resp. $A \subset C^n$) notons par $\mathcal{O}_A$ l'anneau des germes de fonctions analytiques (resp. holomorphes) sur A.

THÉORÈME 3. *Soit $A \subset \boldsymbol{R}^n$ (resp. $A \subset \boldsymbol{C}^n$) un ensemble semi-analytique compacte et connexe avec $H^1(A, \boldsymbol{Z}_2) = 0$ (resp. $H^2(A, \boldsymbol{Z}) = 0$ et A admet un système fondamental de voisinages de Stein). L'anneau $\mathcal{O}_A$ est alors factoriel.*

Sur les ensembles analytiques à singularités données à priori

par J. BOCHNAK (Kraków)

Soient M une variété analytique réelle ou complexe et V un ensemble analytique (réel ou complexe, selon le cas) d'un ouvert de M. Notons par $I_x(V)$ l'idéal de V en $x \in V$. Un germe d'un ensemble analytique cohérent est appelé une *intersection complète* à singularité isolée an x, si l'ensemble pointe $V \setminus \{x\}$ est une sous-variété analytique lisse de codimension p dans un voisinage de x dans M, et que l'idéal $I_x(V)$ est engendré par exactément p éléments.

Notons par $\boldsymbol{K}$ le corps des nombres reels $\boldsymbol{R}$ ou complexes $\boldsymbol{C}$ et posons $Z(\varphi) = \varphi^{-1}(0)$, où $\varphi\colon M \to \boldsymbol{K}^s$. Désignons par $\mathrm{Diff}_x^\omega(M)$ (resp. $\mathrm{Diff}_x^k(M)$) le groupe des homeomorphismes biholomorphe (resp. difféomorphismes de classe C^k) locaux $\tau\colon (M, x) \to (M, x)$ au voisinage x de M; $\tau(x) = x$. Par V nous désignons le germe d'ensemble V en point x.

TÉÉORÈME. *Soient M une variété analytique complexe ou réelle et D un sous-ensemble discret de M. Supposons donnés, en chaque point $x \in D$, un germe d'ensemble analytique V_x de codimension p, qui soit une intersection complète à singularité isolée en x. Alors*

a) (cas complexe). *Si M est de Stein, il existe une application holomorphe $\varphi\colon M \to \boldsymbol{C}^p$, telle que l'ensemble $Z(\varphi) \setminus D$ soit une sous-variété analytique complexe de M de* codim p, *et pour chaque point $x \in D$ il existe $\tau \in \mathrm{Diff}_x^\omega(M)$, avec $\tau(V_x) = Z(\varphi)_x$.*

b) (cas reel). *Il existe, pour tout $k \in \boldsymbol{N}$, une application analytique réelle $\varphi\colon M \to \boldsymbol{R}^p$, telle que $Z(\varphi) \setminus D$ soit une sous-variété analytique réelle de M de* codim p, *et pour chaque $x \in D$, il existe $\tau \in \mathrm{Diff}_x^k(M)$ tel que $\tau(V_x) = Z(\varphi)_x$.*

c) *Si M est un espace affine et si D est fini l'application φ peut être choisie polynômiale.*

Le Théorème (a) a été demonstre par Strehl [2] dans le cas des hypersurfaces ($p = 1$).

PROBLÈME. Le theoreme (a) et (b) reste-t-il valable si les singularites ne sont pas des intersections complets?

Références

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On modular and domain majorizations in the classes of bounded functions

by F. Bogowski (Lublin)

Let S denote the class of functions F holomorphic and univalent in the unit disc K_1, where $K_r = \{z\colon |z| < r\}$, such that $f(0) = 0$, $F'(0) = 1$. Let S_M, $M > 1$, denote the subclass of the class S of the functions satisfying the condition: $|F(z)| < M$ in K_1 and let H be the class of the functions $f\colon f(0) = 0$, $f'(0) \geqslant 0$, holomorphic in the unit disc K_1. The author proves the following

THEOREM. *If $f \in H$, $F \in S_M$, $M > M_0 = 3.259\ldots$ and $|f(z)| \leqslant |F(z)|$ for $z \in K_1$, then*

$$f(K_r) \subset F(K_r)$$

for every $r < r_0(M) = r_0(H, S_M)$, where $r_0(M)$ is the unique, positive root of the equation:

$$\frac{\sqrt{(1-r)^2+4r\tau}+1-r}{\sqrt{(1+r)^2-4r\tau}+1+r} = \sqrt{r}, \qquad \tau = \frac{1}{M},$$

and where

$$M_0 = \frac{p_0(1+p_0^2)^2}{4p_0^3-(1+p_0^2)(1-p_0^2)^2},$$

$$\log\frac{(1+p_0)^2}{1+p_0^2} = \arctan\frac{1-p_0^2}{\sqrt{2}p_0}, \qquad p_0 > 0.$$

In the case $M \to \infty$ we get a result of F. F. Jabłoński [2].

References

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Analytic functions and transcendental numbers

by E. BOMBIERI (Pisa)

This is a survey lecture on recent methods in the theory of analytic functions and their applications to the theory of transcendental numbers.

A conjecture of Lelong on plurisubharmonic functions is also discussed, and some generalizations are proposed.

Periodic solutions of holomorphic differential equations

by M. L. CARTWRIGHT and N. G. LLOYD (Cambridge)

Let $z \in \boldsymbol{C}^N$, $t \in \boldsymbol{R}$, and let $\mathscr{F}$ be the class of continuous functions f: $\boldsymbol{R} \times \boldsymbol{C}^N \to \boldsymbol{C}^N$ that for fixed t are holomorphic for all z, and that satisfy, for a fixed ω, $f(t+\omega, z) = f(t, z)$. The differential equations

$$\frac{dz}{dt} = f(t, z), \qquad f \in \mathscr{F}, \tag{$*$}$$

are considered, and in particular the number of their periodic solutions. Equation ($*$) is identified with $f \in \mathscr{F}$ and is given a locally convex topology in a natural way. The concept of singular periodic solutions is defined and such solutions characterized. After examining the structure of that subset $\mathscr{A}$ of $\mathscr{F}$ consisting of equations without singular periodic solutions, results are proved concerning the number of periodic solutions of equations of type ($*$) in terms of the components of $\mathscr{A}$.

Propriété de Runge et enveloppe d'holomorphie de certaines variétés analytiques de dimension infinie

par G. COEURÉ (Nancy)

Dans un espace du type $X \times E$ où X est une variété de Stein et E est un espace localement convexé, séparé et complexe, on étudie quelques propriétés des fonctions analytiques sur un ouvert Ω de $X \times E$ dont les sections suivant E sont disquées. Si X est de dimension finie, on montre que $(\omega \times E) \cap \Omega$ et Ω forment une paire de Runge pour tout ouvert de Stein ω dans X, dont on déduit que l'enveloppe d'holomorphie de Ω est univalente au-dessus de l'enveloppe d'holomorphie de X. Sans hypothèse de finitude sur X, on obtient le même résultant en supposant que X est enveloppe d'holomorphie convenable.

Isolated critical points of analytic functions of three variables

by I. DOLGAČEV (Moscow)

It is known since F. Klein that there exists the close relation between regular polytops and some singularities of algebraic surfaces. It was shown by V. Arnold that the classification of analytic functions near its isolated critical points is related with the Klein's singularities. In his further investigations he introduced 14 new types of singularities which describe unimodal germs of analytic functions.

In the lecture will be shown that all these Arnold's singularities can be obtained with help of the construction generalizing one of Klein and which connects the theory of linear representation of automorphism groups of compact Riemann surfaces and singularities of analytic surfaces.

Moduli problems in several complex variables

by O. FORSTER (Regensburg)

The moduli problem goes back to Riemann, who stated in his memoir *Theorie der Abel'schen Functionen* that a closed Riemann surface of genus p depends on $3p-3$ moduli, i.e. $3p-3$ complex parameters. This statement was made more precise by Teichmüller, Ahlfors and Bers. Kodaira and Spencer were the first to consider moduli problems in higher dimensions and created a deformation theory of complex manifolds. They use almost complex structures and potential theory. Therefore these methods cannot be applied to the deformation theory of complex spaces with singularities. Recently new methods have been developped by Douady, Grauert, Knorr/Forster to attack the deformation problem of compact complex spaces. In the talk we want to explain some of these methods.

Exceptional points on real submanifolds

by M. FREEMAN (Lexington)

A real submanifold $M^n = \{p_1 = \ldots = p_n = 0\}$ defined in C^n by smooth independent real-valued functions ϱ_j has an *exceptional point* at $p \in M^n$ if $\partial_p\varrho_1 \wedge \ldots \wedge \partial_p\varrho_1 = 0$. This means that the complex tangent space $H_pM = \bigcap_1^n \ker \partial_p \varrho_j$ has positive dimension. We consider exceptional points p, where $\dim_C H_pM = 1$, and assume that M^n is *Levi-flat* at every exceptional point. This is the well-known second-order condition that the restricted Levi forms $L_p\varrho_j | H_pM = 0$, all j. Levi-flatness is invariant in the sense that it is independent of a choice of defining functions ϱ_j and of local holomorphic coordinates for C^n.

We also assume that M^n is *non-degenerate* near ϱ. This is a new second-order condition on the "purely holomorphic" part of the Taylor expansion of the ϱ_j's near p which is invariant in the same sense. The main results support the idea that near such an exceptional point M^n has the same "good" function – theoretic behavior as a totally real submanifold (one whose complex tangent space has everywhere dimension zero): It is shown that near a non-degenerate exceptional point a Levi-flat submanifold M^n enjoys the following properties. A) M^n is holomorphically convex near p in the strong sense that there exists a smooth non-negative function σ such that $M^n = \{\sigma = 0\}$ near p and σ is strictly plurisubharmonic at every non-exceptional point of C^n near p. B) There exists a compact M^n-neighborhood N of p and a strictly pseudoconvex domain $D \supset N$ such that every continuous function on N is a uniform limit on N of functions holomorphic on D. This setting subsums certain special cases of recent interest. It also suggests several directions of possible generalization.

Determination of the conformal module of a quadrilateral

by D. Gaier (Giessen)

The conformal module of a quadrilateral V is a conformal invariant important for theoretical and practical purposes. Here, a survey is given on the methods to obtain numerical approximations for $m(V)$.

1. Difference methods. Here an extremal characterization of $m(V)$ is used: $m(V) = \inf\{D[f]: f \in K\}$, where K denotes the class of functions with $f = 0$ and $f = 1$ on the opposite sides Γ_0 and Γ_1 of V and with f_x, f_y piecewise continuous in G, and where $D[f] = \iint_G (f_x^2 + f_y^2)\, db$. Every $f \in K$ gives an upper bound for $m(V)$. To obtain suitable such f, bilinear spline functions are constructed and their determination leads to large systems of linear equations which can be solved by iteration. Convergence estimates are given if the mesh carrying V is refined. For details and several numerical experiments see Gaier [3].

In order to reduce the discretisation error, it is suggested that a refinement could be used similar to the method of Birkhoff and Fix ([1], p. 133).

2. Integral equation methods. If V is mapped conformally onto $\{w: |w| < 1\}$ and the corners of V correspond to Q_1, Q_2, Q_3, Q_4, we have

$$m(V) = K'(k)/K(k) \quad \text{with } k = 1/\sqrt{d},$$

where d is the cross ratio of the Q_i and K, K' are complete elliptic integrals. So any of the mapping methods via integral equations gives $m(V)$. Experiments by Knierim [5] show that this method is faster than the difference method and can also be applied to infinite quadrilaterals, but it does not give bounds for $m(V)$.

Further it is suggested that the mixed boundary value problem

$$(*) \qquad \Delta u = 0, \quad u = \begin{matrix} 0 & \text{on } \Gamma_0 \\ 1 & \text{on } \Gamma_1 \end{matrix}, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial G \setminus (\Gamma_0 \cup \Gamma_1)$$

is solved by integral equation methods due to Hayes and Kellner [4]. Thereafter, $m(V) = \int_\gamma \frac{\partial u}{\partial n} ds$ for any arc γ connecting Γ_0 to Γ_1.

3. Analog methods. Even if no computer is available, $m(V)$ can be determined by simple analog methods with an error of about 1%. Experiments due to the author ([3], p. 193) were done with resistance paper. In recent experiments by Fenyi and Michael [2] the discrete analog of $(*)$ has been simulated by a grid of resistors and $m(V) = \int_\gamma \frac{\partial u}{\partial n} ds$ evaluated numerically.

References

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On analytic functions with values in l^p-spaces

by J. Globevnik (Ljubljana)

Throughout, Δ is the open unit disc in C. We denote by N (resp. $\bar{N}_0$) the set of all positive (resp. non-negative) integers.

Let $1 \leqslant p < \infty$ and let $\{e_i\}$ be the canonical basis in l^p. Suppose that

$$\zeta \mapsto f(\zeta) = \sum_{i=1}^{\infty} \varphi_i(\zeta) e_i = \{\varphi_i(\zeta)\}$$

is a function from Δ to l^p whose coordinate functions φ_i are analytic on Δ. R. Aron and J. A. Cima [1] proved that such an f is analytic if and only if the function

$$\zeta \mapsto \sum_{i=1}^{\infty} |\varphi_i(\zeta)|^p$$

is locally bounded on Δ. They have shown that in general only the analyticity of all φ_i is not sufficient for f to be analytic. Here we present some additional conditions on the coordinate functions φ_i which assure the analyticity of f.

THEOREM 1. *Let φ_i above be polynomials of uniformly bounded degree. Then f is analytic.*

THEOREM 2. *Let $1 \leqslant p < \infty$. Let $\{\psi_i\}$ be a sequence of univalent complex-valued analytic functions on Δ. Suppose that a point $\zeta_0 \in \Delta$ exists such that*

(i) $\{\psi_i(\zeta_0)\} \in l^p$,

(ii) $\{\psi_i'(\zeta_0)\} \in l^p$.

Then $\zeta \mapsto \{\psi_i(\zeta)\}$ is an analytic function from Δ to l^p.

THEOREM 3. *Let $1 \leqslant p < \infty$. Let $\{\psi_i\}$ be a sequence of complex-valued functions, analytic and r-valent in mean with respect to area in Δ* (see [3], p. 587, [2], p. 314) *for some fixed r. Suppose that $\psi_i(\zeta) \neq 0$ $(i \in N, \zeta \in \Delta)$. Let a point $\zeta_0 \in \Delta$ exist such that $\{\psi_i(\zeta_0)\} \in l^p$. Then $\zeta \mapsto \{\psi_i(\zeta)\}$ is an analytic function from Δ to l^p.*

THEOREM 4. *Let $1 \leqslant p < \infty$. Let $\{\psi_i\}$ be a sequence of complex-valued analytic functions on Δ. Suppose that for each $i \in N$ ψ_i does not assume any value on a ray from the origin. Suppose that a point $\zeta_0 \in \Delta$ exists such that $\{\psi_i(\zeta_0)\} \in l^p$. Then $\zeta \mapsto \{\psi_i(\zeta)\}$ is an analytic function from Δ to l^p.*

THEOREM 5. *Let $1 \leqslant p' < p < \infty$. Let $\{\psi_i\}$ be a sequence of complex-valued analytic functions on a disc $|\zeta| < R$ which are uniformly bounded there. Suppose that $\psi_i(\zeta) \neq 0$ $(i \in N, |\zeta| < R)$. Let $\{\psi_i(0)\} \in l^{p'}$. Then $\zeta \mapsto \{\psi_i(\zeta)\}$ is an analytic function from the disc $|\zeta| < R(p-p')/(p+p')$ to l^p.*

The proof of Theorem 1 is elementary. The proofs of Theorems 2–5 are simple and use some inequalities from the complex function theory ([4], p. 217, [2], p. 314, [2], p. 296, [5], p. 140) to show that in all cases the function

$$\zeta \mapsto \sum_{i=1}^{\infty} |\psi_i(\zeta)|^p$$

is locally bounded and then apply the Aron–Cima theorem.

Finally, we present the following counterexample.

THEOREM 6. *Let $1 \leqslant p < \infty$. There exists a function from Δ to l^p with the following properties:*

(i) *it is not continuous,*

(ii) *it is analytic on Δ as a function to $l^{p'}$ for any $p' > p$,*

(iii) *its derivatives (to $l^{p'}$) of all orders are functions from Δ to l^p.*

In particular, Theorem 6 gives the negative answer to the question in [1] whether every function from Δ to l^p whose coordinate functions are analytic and uniformly bounded on Δ is analytic on Δ.

The proof of Theorem 6 is long. Let us give the idea for the case when $p = 1$. One takes first the suitably modified counterexample of Aron–Cima [1]: there exists a sequence $\{\varphi_i\}$ of complex-valued polynomials such that the function

$$\zeta \mapsto \sum_{i=1}^{\infty} |\varphi_i(\zeta)|$$

is not locally bounded on Δ while $\{\varphi_i^{(n)}(\zeta)\} \in l^p$ $(n \in N_0;\ \zeta \in \Delta)$. One observes that each polynomial together with all its derivatives is uniformly bounded on Δ and then, taking a suitable sequence $\{p_i\}$ of positive integers one defines for $\zeta \in \Delta$

$$\{\psi_j(\zeta)\} = \{\underbrace{\varphi_1(\zeta)/p_1, \ldots, \varphi_1(\zeta)/p_1}_{p_1 \text{ times}}, \ldots, \underbrace{\varphi_n(\zeta)/p_n, \ldots, \varphi_n(\zeta)/p_n}_{p_n \text{ times}}, \ldots\}.$$

which shows that in the Aron–Cima counterexample one may assume that all polynomials ψ_i together with all their derivatives are uniformly bounded by 1 on Δ. Then, using such a sequence of polynomials $\{\psi_i\}$ one defines for $\zeta \in \Delta$

$$f(\zeta) = \{\psi_1(\zeta)/2, \psi_1(\zeta)/2, \ldots, \underbrace{\psi_n(\zeta)/2^n, \ldots, \psi_n(\zeta)/2^n}_{2^n \text{ times}}, \ldots\}.$$

The detailed proofs, especially the proof of Theorem 6 are too long to be presented here and will appear elsewhere.

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The additive Cousin-problem for partial differential equations

by B. GOLDSCHMIDT (Halle)

The additive and the multiplicative Cousin-Problem for meromorphic functions is well known. The aim of this paper is to solve corresponding problems in the case of linear partial differential equations or of systems of such equations:

$$L[u] = h.$$

PROBLEM. *Given is an open covering $\{U_i\}$ of a domain G and in every neighbourhood U_i a solution u_i of $L[u_i] = h$. In $U_i \cap U_k \neq 0$ the difference of the corresponding functions $g_{ik} = u_i - u_k$ must be Hölder-continuous (Condition of compatibility).*

The problem is to find a global solution of the differential equation, which is compatible with the local solutions.

I will restrict the proof here on the Vekua-equation:

$$L[w] = \frac{\partial w}{\partial z^*} + A(z)w(z) + B(z)w(z)^* = F(z).$$

As in the general case the proof will be made in two steps:

a. construction in compact subsets of G,

b. construction in the whole domain G with an exhaust-method.

For the Vekua-equation exists a Cauchy-integral formula

$$\frac{1}{2\pi i} \int_{\partial G} \Omega_1(z,t)w(t)\,dt - \Omega_2(z,t)w(t)^*\,dt^* = w(z) \qquad \text{for } z \in G$$

for every solution $w(z)$. In the holomorphic case is $\Omega_1(z,t) = \dfrac{1}{t-z}$ and $\Omega_2(z,t) = 0$. Let γ be a smooth curve. Then the function

$$d(z) = \frac{1}{2\pi i} \int_{\gamma} \Omega_1(z,t)g(t)\,dt - \Omega_2(z,t)g(t)^*\,dt^*$$

has the following properties:

1. $L[d] = 0$.
2. By crossing over γ, $d(z)$ has a jump $g(z)$.

3. On the boundary points of γ, $d(z)$ has the representation:

$$d(z) = \frac{w(z)}{2\pi i} \log(z - z_0) + a \text{ continuous function.}$$

For subdomains $G_n \Subset G$ exists a finite covering of neighbourhoods and it is possible to construct a net of small squares, such that every square lies completely in any U_i. Next form the jump-integrals:

$$d_{ik}(z) = \frac{1}{2\pi i} \int_{\gamma_{ik}} \Omega_1(z, t) g_{ik}(t) dt - \Omega_2(z, t) g_{ik}(t)^* dt^*,$$

where γ_{ik} is the common part of the boundaries of the squares Q_i and Q_k, and take the (finite) sum about all this functions. The sum has the same jump-behavior as the functions in the squares. The difference

$$u(z) = u_m(z) - \sum_{i,k} d_{ik}(z) \quad \text{in } Q_m$$

has no jumps at the boundaries of the squares and is compatible with the given local solutions.

In order to construct a solution in the whole domain G, one must construct a solution $u^{(n)}$ in every subdomain G_n, where the sequence $\{G_n\}$ exhausts the domain G. For the construction one needs approximation theorems in the kind of the Runge-theorem. For generalized analytic functions such a theorem is proved in [6].

By this considerations the problem for the Vekua-equation is solved. It is also possible to construct solutions in compact subsets by using of the partition of unity (see [2]).

The corresponding problem can be solved also for other partial differential equations, for instance for general elliptic equations, for equations with constant coefficients and for principal normal differential equations.

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Holomorphic continuation of separately analytic functions on a 3-dimensional set in C^2

by J. GÓRSKI (Katowice)

Let D_j, $j = 1, 2$ be a simply connected domain on z_j-plane and $E_j \subset D_j$ a segment on the real axis x_j. Let F be the family of all functions f which satisfy the following conditions: (i) f is bounded on the set $Z = E_1 \times \times D_2 \cup D_1 \times E_2$, $|f| < M_f$ on Z, (ii) f is separately analytic on Z. The first problem: Is $f \in F$ analytic continuable in a 4-dimensional domain? The second problem is the construction of the envelope of holomorphy of the set Z. Both problems were solved in general case without the condition of boundedness of f for arbitrary finite dimension n and general form of the set E_j by J. Siciak [Ann. Polon. Math. 22 (1969), p. 145–171]. The aim of this announcement is to give an other proof for both problems in the particular case $n = 2$ under conditions (i), (ii). Application to the edge of the wedge theorem are given.

Direct decomposition of the space $L^\infty(\Omega(\Gamma))$

by D. IVAŞCU (Bucuresti)

In this note we construct a continuous and linear projection of the space $L^\infty(\Omega)$ on the subspace $L^\infty(\Gamma, \Omega)$; Γ being a discontinuous group of Möbius transformations (Kleinian group), $\Omega = \Omega(\Gamma)$ the set of discontinuity of Γ and the space $L^\infty(\Gamma, \Omega)$ called the *space of Beltrami coefficient of Γ*, being defined by:

$$L^\infty(\Gamma, \Omega) = \{\mu \mid \mu \in L^\infty(\Omega), (\mu \circ T)\bar{T}'/T' = \mu \text{ for } T \in \Gamma\},$$

$L^\infty(\Gamma, \Omega)$, as closed subspace of $L^\infty(\Omega)$ is a Banach space.

With the view of an effective construction of a continuous linear projection of the space $L^\infty(\Omega)$ on $L^\infty(\Gamma, \Omega)$, it is necessary to remind two well-known properties of the Kleinian groups:

a) Every Kleinian group is a countable one.

b) For each Kleinian group there exists a fundamental domain $N = N(\Gamma) \subset \Omega(\Gamma)$ such that $m(\partial N) = 0$ (where $m(\partial N)$ denotes the two-dimensional measure of the boundary of N). Now, for a Kleinian group Γ, we fix a fundamental domain N with the property above, and denote by $L_N^\infty(\Omega)$ the following Banach space:

$$L_N^\infty(\Omega) = \Big\{\mu \mid \mu \in L^\infty(\Omega), \sum_{T \in \Gamma} \|\mu_{ITN}\|_\infty < \infty\Big\}$$

with the norm:

$$\|\mu\|_{N,\infty} = \sum_{T\in\Gamma} \|\mu_{ITN}\|_\infty .$$

First of all we define a bilinear continuous mapping:

$$U\colon l'(\Gamma) \times L^\infty(\Omega) \to L^\infty_N(\Omega)$$

by the formula:

$$U(\gamma, \mu) = \sum_{T\in\Gamma} \gamma(T)\chi_{TN}\mu$$

χ_{TN} being the characteristic function of the set TN).

Obviously the bilinear mapping U verifies the relations:

$$\|U(\gamma, \mu)\|_{N,\infty} = \sum_{\tilde{T}\in\Gamma} \Big\| \Big(\sum_{T\in\Gamma} \gamma(T)\chi_{TN}\mu\Big)_{|\tilde{T}N}\Big\|_\infty$$
$$= \sum_{T\in\Gamma} |\gamma(T)|\, \|\mu_{ITN}\|_\infty \leqslant \|\mu\|_\infty \sum_{T\in\Gamma} |\gamma(T)| = \|\mu\|_\infty \|\gamma\|_1 .$$

Now by means of U we can define the mapping:

$$\tilde{U}\colon l'(\Gamma) \times \big(L^\infty(\Omega)\big)^2 \to L^\infty(\Gamma, \Omega).$$

Namely:

$$\tilde{U}(\gamma, \nu, \mu) = \sum_{T\in\Gamma} \big([U(\gamma, \nu)\mu]\circ T\big)\overline{T}'/T'.$$

Obviously $\|\tilde{U}(\gamma, \nu, \mu)\|_\infty \leqslant \|\mu\|_\infty \|U(\gamma, \nu)\|_{N;\infty}$.

Moreover, if $\mu \in L^\infty(\Gamma, \Omega)$, then $\tilde{U}(\gamma, \nu, \mu) = \mu\cdot\big(\sum_{T\in\Gamma} U(\gamma, \nu)\circ T\big)$, therefore, if $1/\big(\sum_{T\in\Gamma} U(\gamma, \nu)\circ T\big) \in L^\infty(\Omega)$, the mapping:

$$P_{\gamma,\nu}\colon L^\infty(\Omega) \to L^\infty(\Gamma\Omega)$$

defined by the formula: $P_{\gamma,\nu}(\mu) = \tilde{U}(\gamma, \nu, \mu)/\big(\sum_{T\in\Gamma} U(\gamma, \nu)\circ T\big)$ is a linear continuous projection of $L^\infty(\Omega)$ on $L^\infty(\Gamma, \Omega)$. Consequently, $L^\infty(\Omega) \simeq L^\infty(\Gamma, \Omega) \oplus \ker P_{\gamma,\nu}$. Denoting by γ_0 the element of $l^1(\Gamma)$ defined by the relations: $\gamma_0(I) = 1$, $\gamma_0(T) = 0$ for $T\in\Gamma - \{I\}$ we obtain:

$$\sum_{T\in\Gamma} U(\gamma_0, 1)\circ T = \sum_{T\in\Gamma} \chi_N \circ T$$

and evidently

$$1/\sum_{T\in\Gamma} U(\gamma_0, 1) \in L^\infty(\Omega).$$

Therefore $L^\infty(\Omega) = L^\infty(\Gamma, \Omega) \oplus \ker P_{\gamma_0,1}$.

The above construction generalizes the Poincaré series.

Prolongement analytique des fonctions harmoniques

par M. JARNICKI (Kraków)

Nous posons le problème suivant: supposons que pour le domaine D dans $\boldsymbol{R}^n$ ($n \geqslant 2$) il existe un domaine G dans $\boldsymbol{C}^n$ ($D \subset G$) tel que chaque fonction harmonique dans D ait un prolongement holomorphique (ou bien un prolongement analytique non univoque) dans G; existe-t-il un domaine maximal ayant la même propriété? S'il existe nous le désignons par $\tilde{D}_H$ (ou $\tilde{D}_A$).

On sait que, pour chaque domaine D, $\tilde{D}_A$ existe et qu'il existe aussi un domaine $G \subset \tilde{D}_A$ ($D \subset G$) tel que chaque fonction harmonique dans D se prolonge à une fonction holomorphe dans G (P. Lelong).

La construction effective du domaine $\tilde{D}_A$ (ou bien $\tilde{D}_H$) constitue aussi un problème. Par exemple, on peut prouver (C. O. Kiselman et indépendamment, J. Siciak) que pour la boule $B = \{x \in \boldsymbol{R}^n\colon |x| < 1\}$ on a $\tilde{B}_H = \tilde{B}_A = \{z = x+iy \in \boldsymbol{C}^n\colon [|x|^2+|y|^2+2(|x|^2\cdot|y|^2-\langle x, y\rangle^2)^{1/2}]^{1/2} < 1\}$. Pareillement, pour la bague $P = \{x \in \boldsymbol{R}^n\colon r_1 < |x| < r_2\}$ on a

$$\begin{aligned} P_A = \{z = x+iy \in \boldsymbol{C}^n\colon r_1 < [|x|^2+|y|^2-2(|x|^2\cdot|y|^2-\langle x, y\rangle^2)^{1/2}]^{1/2} \\ \leqslant [x^2+y^2+2(|x|^2\cdot|y|^2-\langle x, y\rangle^2)^{1/2}]^{1/2} < r_2\} \end{aligned}$$

(notez qu'ils existent des fonctions harmoniques dans P qui ne se prolongent pas aux fonctions holomorphiques dans P_A).

Si $n = 2$, les problèmes se simplifient. On prouve que pour $D \subset C$ on a $\tilde{D}_A = \{(z_1, z_2) \in C^2\colon z_1+iz_2, \bar{z}_1+i\bar{z}_2 \in D\}$. De plus, si pour deux domaines D et E dans C la transformation $f = u+iv\colon D \to E$ est biholomorphe, les fonctions harmoniques u et v peuvent être prolongées à des fonctions holomorphes $\tilde{u}, \tilde{v}$ dans $\tilde{D}_A$ et $\tilde{f} = (\tilde{u}, \tilde{v})\colon \tilde{D}_A \to \tilde{f}(\tilde{D}_A) = \tilde{E}_A$ est une transformation biholomorphe.

A variational characterization of the capacity of a condenser in $\boldsymbol{R}^n$ and $\boldsymbol{C}^n$

by J. KALINA (Łódź)

A variational characterization of the capacity of an inhomogeneous condenser in $\boldsymbol{R}^n$, expressed by means of a hypersurface integral, is obtained by the author. This is a generalization of the corresponding result in $\boldsymbol{R}^2$ due to J. Ławrynowicz (Math. Nachr., 1973). The main theorem of the paper gives connections between stationary points of a functional whose value at these points is a capacity and solutions of the equation div$(p \operatorname{grad} u) = 0$, where p is a function describing inhomogeneity. An analogous result is obtained in the case of C^n, for some counterparts of the above capacities (J. Ławrynowicz, same Proceedings), where the admissible class of functions consists of suitably normalized plurisubharmonic functions.

K_{ml_i}-функции многих параметров и некоторые вопросы оценок коэффициентов однолистных полиномов

Ф. Кравченко (Каменец Подольский)

Показана роль K_{ml_i}-функций многих параметров в решении проблемы оценок коэффициентов однолистных полиномов.

Пользуясь представлениями нулей полиномов с помощью K_{ml_i}-функций доказывается предположение, высказанное Ю. С. Лаврыновичем, что в классе P_m однолистных полиномов имеет место точная оценка

$$|c_m| \leqslant \frac{(m-1)^{m-1}}{m^m}.$$

Кроме того, устанавливается точная оценка коэффициентов для полинома $P_m(w) = w + c_l w^l + c_m w^m$, однолистного в наибольшей области D, содержащей точку $w = 0$ и такой, что $P_m(w)| < 1$ для $w \in D$.

Некоторые свойства клейновых групп и их деформаций

С. Л. Крушкаль (Новосыбирск)

В докладе будет изложено решение ряда задач, стоящих на стыке теории квазиконформных отображений, клейновых групп и римановых поверхностей.

Пусть G-клейнова (разрывная неэлементарная) группа мебиусовых автоморфизмов расширенной комплексной плоскости $\overline{C}$, $\Omega(G)$ — множество разрывности G. Рассматриваются группы конечного типа, для которых $\Omega(G)/G$ — конечное объединение римановых поверхностей конечного типа.

Для клейновых групп и римановых поверхностей доказываются локальные теоремы существования квазиконформных деформаций с определёнными свойствами. В простейшем случае для квазиконформных автоморфизмов плоских областей эти теоремы показывают, что за счёт замены конформности квазиконформностью даже на скольугодно малом множестве положительной двумерной лебеговой меры можно произвольным образом задавать (в определённых пределах) например, значения отображений вместе с их производными любых конечных порядков в конечном числе заданных точек. Здесь же получаются теоремы о возможности продолжения различных классов деформаций группы G с множества $\Omega(G)$ или его некоторой части на всю плоскость $\overline{C}$ с сохранением определенных свойств.

На основании этих разультатов дается решение экстремальных задач для квазиконформных деформаций римановых поверхностей и клейновых групп, а также устанавливаются некоторые свойства пространств квазиконформных деформаций.

Исследуются вопросы квазиконформной стабильности клейновых групп и устанавливаются классы групп, удовлетворяющих определённым условиям стабильности.

Symmetrization and harmonic measure

by J. G. Krzyż (Lublin)

Following Ahlfors [1] we shall call a configuration a domain Ω of finite connectivity n with $l \geqslant 0$ distinguished boundary elements (or prime ends) and $m \geqslant 0$ distinguished interior points. All boundary components of the carrier Ω are supposed to be hyperbolic and can be taken without loss on generality as piecewise analytic arcs. It is well known that each class of conformally equivalent configurations can be described by a finite number N of real parameters. In particular $N = 3n-6$ for $l = m = 0$ and $n \geqslant 3$. Hence it easily follows that one-dimensional configurations ($N = 1$) correspond to carriers with $n \leqslant 2$ and thus can be easily found. These are:

(i) ring domain ($n = 2$, $l = 0, 1$, $m = 0$);

(ii) quadrilateral ($n = 1$, $l = 4$, $m = 0$);

(iii) twice punctured disk ($n = 1$, $l = 0$, $m = 2$);

(iv) punctured disk with a distinguished boundary arc ($n = 1$, $l = 2$, $m = 1$).

The class of all conformally equivalent one-dimensional configurations can be described by one real parameter — its characteristic conformal invariant. In the above described cases characteristic conformal invariants are:

(i) the modulus of the ring domain;

(ii) the modulus of the quadrilateral;

(iii) the Green function (or hyperbolic distance);

(iv) harmonic measure of the distinguished boundary arc.

According to the classical result of Pólya, cf. [4] circular and Pólya symmetrization both increase the modulus of a ring domain.

A quite natural question arises, how do the remaining characteristic conformal invariants behave under symmetrization. Since symmetrization does not give rise to any explicite correspondence between the distinguished points of the original and symmetrized domain in the general case, we must assume that the configurations have a special form.

In fact, if e.g. the distinguished sides a_1, a_2 of a quadrilateral Q lie on a straight line l and Q lies to the right of l, then a reflection in l and a subsequent sewing along a_1 and a_2 of symmetric quadrilaterals yield a ring domain B. Steiner symmetrization of B w.r. to the line l_1 perpendicular to l gives a ring domain B^* whose right-hand half can be considered as the symmetrized quadrilateral Q. A similar procedure can be applied to quadrilaterals whose sides lie on a circle in case of circular symmetrization. In both cases the modulus of Q is increased by symmetrization, cf. [5], p. 135. For the configurations of type (iii) a very natural correspondence between the distinguished interior points can be established by means of perpendicular projection of distinguished points on the axis of Steiner symmetrization, or by circular projection on the polar axis in case of circular symmetrization. As shown by the author [6], symmetrization increases the Green function and decreases the hyperbolic distance with the above described correspondence of distinguished interior points.

The remaining configuration (iv) has been treated six years later by K. Haliste [3]. She assumed that the distinguished boundary arc a is situated on a straight line l and the carrier Ω lies to the right of l. Then the Steiner symmetrization of the configuration $\{\Omega, z_0, a\}$ w.r.t. the line l_1 perpendicular to l can be defined in a natural way as $\{\Omega^*, x_0, a^*\}$, where x_0 is the projection of z_0 on l_1. Note that symmetrization of Ω and a makes sense.

Obviously circular symmetrization can be defined in an analogous manner. As shown by Haliste, harmonic measure is increased by symmetrization.

However, this result is a simple consequence of the symmetrization result for the configuration (iii) and the following observation which we quote as

LEMMA. *Let Ω be a simply connected domain of hyperbolic type with two distinguished interior points z_0, z_1. Then Ω has two lines of symmetry that are image lines of circular arcs of symmetry in the unit disk Δ (with distinguished interior points $0, r$) under the conformal mapping $f\colon \Delta \to \Omega$, $f(0) = z_0$, $f(r) = z_1$. Let a be the line of symmetry in Ω not containing z_0, z_1 and let Ω_0 be the component of $\Omega - a$ containing z_0. Then*

$$e^{-g} = \cos \tfrac{1}{2}\pi\omega,$$

where $g = g(z_0, z_1; \Omega)$ *is the Green function and* $\omega = \omega(z_0, a; \Omega_0)$ *denotes the harmonic measure.*

The proof follows by conformal invariance of g, ω and direct calculation in case of the unit disk.

Recently more general symmetrization results for the Green function and harmonic measure have been obtained, cf. e.g. [2].

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Über Identitäten und eine Kernfunktion in der Theorie der quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung

von R. KÜHNAU (Halle)

Von P. R. Garabedian und M. Schiffer, Trans. Amer. Math. Soc. 65 (1949), p. 187–328, wurde durch umfassende Anwendung der Methode der Randintegration ein umfangreiches System von Identitäten zwischen einigen konformen Normalabbildungen (Parallel-, Radial- und Kreisbogenschlitzabbildungen usw.) aufgestellt. Es werden hierzu Analoga, d.h. Identitäten zwischen gewissen quasikonformen Normalabbildungen angegeben, die durch Extremalprobleme bei ortsabhängiger Dilatationsbeschränkung auftreten. Diese Normalabbildungen bzw. einfache Hilfsausdrücke in diesen erfüllen dabei des lineare elliptische System

$$f_{\bar{z}} = \nu \cdot f_z \quad (0 \leqslant \nu < 1). \tag{1}$$

Dazu wird für Lösungen von (1) eine Theorie der Kernfunktion aufgebaut, die in diesem Zusammenhang gewisse Vorteile gegenüber bekannten Entwicklungen von St. Bergman und M. Schiffer (vgl. z.B. deren Buch *Kernel functions and elliptic differential equations in mathematical physics*, New York 1953) besitzt, z.B. Darstellungs und Konstruktionsmöglichkeit für die gennannten quasikomfortmen Normalabbildungen in unmittelbarerer Weise liefert.

Conformal mappings with quasi-conformal extensions

by O. Lehto (Helsinki)

Considerable interest has been devoted in recent years to quasi-conformal mappings of the plane which are conformal in a subdomain of the plane. Such mappings play an important role in the modern theory of Riemann surfaces and lead to many interesting problems in the theory of schlicht analytic function.

Properties of such mappings can be studied by means of variational methods, developed by Belinski and Schiffer and applied by Kühnau, Kruškal and others. Another starting point is the Bojarski representation formula for normalized quasi-conformal mappings whose complex dilatations have bounded support. By methods of the classical function theory, a general inequality can be derived which yields a number of various special results.

On some extremal functions

by F. Leja (Kraków)

Let T be a metric space, $\omega(p_1, \ldots, p_a)$ — a positive symmetric function defined in T^a, and let $p^{(n)}$ be a system of $n \geqslant a$ point $p_1, \ldots, p_n$ of T. Let $j_1, \ldots, j_\nu$ be a system of $\nu < a$ distinct integers $\leqslant n$, and let $z_1, \ldots, z_\nu$ be points of T. Consider two following products, each having $\binom{n-\nu}{a-\nu}$ factors,

$$\Delta^{(j_1,\ldots,j_\nu)}(p^{(n)}) = \prod_{1 \leqslant i_{\nu+1} < \cdots < i_a \leqslant n} (p_{j_1}, \ldots, p_{j_\nu}, p_{i_{\nu+1}}, \ldots, p_{i_a}),$$

$$\Phi^{(j_1,\ldots,j_\nu)}(z_1, \ldots, z_\nu;\ p^{(\nu)}) = \prod_{1 \leqslant i_{\nu+1} < \cdots < i_a \leqslant n} \frac{\omega(z_1, \ldots, z_\nu, p_{i_{\nu+1}}, \ldots, p_{i_a})}{\omega(p_{j_1}, \ldots, p_{j_\nu}, p_{i_{\nu+1}}, \ldots, p_{i_a})},$$

where the set $\{i_{\nu+1}, \ldots, i_a\}$ does not intersect the set $\{j_1, \ldots, j_\nu\}$.

Let K be a compact subset of T such that its "*transfinite diameter*" (l'ecart) with respect to ω is positive. Put

$$\Delta_n^{(\nu)}(K) = \sup_{p^{(n)} \subset K} \{ \min_{(j_1,\ldots,j_\nu)} \Delta^{(j_1,\ldots,j_\nu)}(p^{(n)})\},$$

$$\Phi_n^{(\nu)}(z_1, \ldots, z_\nu;\ K) = \inf_{p^{(n)} \subset K} \{ \max_{(j_1,\ldots,j_\nu)} \Phi^{(j_1,\ldots,j_\nu)}(z_1, \ldots, z_\nu;\ p^{(n)})\}.$$

The subject of this communication is the problem of convergence of the sequences

$$\delta_n^{(\nu)}(K) = [\Delta_n^{(\nu)}(K)]^{1/\binom{n-\nu}{a-\nu}},$$

$$\varphi_n^{(\nu)}(z_1, \ldots, z_\nu; K) = [\Phi_n^{(\nu)}(z_1, \ldots, z_\nu; K)]^{1/\binom{n-\nu}{a-\nu}},$$

$\nu - 1, \ldots, a-1$, as $n \geqslant a$ tends to ∞.

Un théorème de fonctions implicites pour les fonctions plurisousharmoniques et ses application à l'étude des fonctions analytiques

par P. LELONG (Paris)

Un théorème de fonctions a déja été donné dans le cours de Montréal (*Fonctionnelles analytiques et fonctions entières n variables*, Sém. de Math. Sup., été 1967); il permet d'obtenir des résultats variés en dimension finie et aussi en dimension infinie pour les fonctions analytiques; certains théorèmes obtenus récemment par SIU sur les ensembles analytiques complexes sont des conséquence de ce théorème qui ne fait appel qu'à des propriétés des fonctions plurisousharmoniques. On l'utilise pour étudier une notion nouvelle: les zéros semi-réels d'une fonctions entière F de n variables complexes, c'est-à-dire les points $z = (z_1, \ldots, z_n)$ tels que $z_k = ux_k$ où u est un nombre complexe et $x = (x_k) \in \boldsymbol{R}^n$ a des coordonnées réelles.

Modular majorization and inclusion of domains

by Z. LEWANDOWSKI and J. STANKIEWICZ (Lublin)

Let T denote a compact subclass of the class S of functions

$$F(z) = z + A_2 z^2 + \ldots$$

which are regular and univalent in K_1 ($K_r = \{z: |z| < r\}$).

Let H_n ($n = 1, 2, \ldots$) denote the class of functions

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$$

which are regular in K_1. Furthermore, if $n = 1$, then we suppose $a_1 \geqslant 0$.

We say that f is *modular subordinate* to F in K_r, or that F is *modular majorant* of f in K_r if $|f(z)| \leqslant |F(z)|$ for $z \in K_r$. We write then

$$|f(z)| \leqslant_r |F(z)|.$$

It is known that if $f \in H_n$, $F \in T$, then $|f(z)| \leqslant_r |F(z)|$ if and only if there exists a function $\omega(z)$ regular in K_r, bounded $|\omega(z)| \leqslant 1$ for $z \in K_r$ and such that $f(z) = \omega(z) \cdot F(z)$.

We say that f is ***domain subordinate*** to F in K_r or that F is ***domain majorant*** of f in K_r if there exists a function $\omega(z)$ regular in K_r, $|\omega(z)| \leqslant |z|$ for $z \in K_r$ and such that

$$f(z) = F\big(\omega(z)\big).$$

We write then

$$f(z) \prec_r F(z).$$

If F is univalent in K_r, then $f(z) \prec_r F(z)$ if and only if $f(K_r) \subset F(K_r)$ and $f(0) = F(0)$.

Many authors (Biernacki [7], [8], Robinson [24], Goluzin [15], Alenicin [1], Lewandowski [19]–[21], Bielecki and Lewandowski [2]–[6], Jabłoński [16], [17], Bogucki [11], Bogucki and Waniurski [12], [13], ...) investigated some problems which can be expressed by the following theorems:

THEOREM A. *If $f \in H_n$, $F \in T$, then there exists a number $r(H_n, T)$, depending only on the classes H_n and T not depending on the functions f and F, such that*

$$f(z) \prec_1 F(z) \Rightarrow |f(z)| \leqslant_{r(H_n, T)} |F(z)|.$$

THEOREM B. *If $f \in H_n$, $F \in T$, then there exists a number $R(H_n, T)$, depending only on the classes H_n and T not depending on the functions f and F, such that*

$$|f(z)| \leqslant_1 |F(z)| \Rightarrow f(z) \prec_{R(H_n, T)} F(z).$$

These two theorems may be generalized in the following way:

THEOREM C. *If $f \in H_n$, $F \in T$, then there exists a real positive function $M(r;\, H_n, T)$ such that*

$$f(z) \prec_1 F(z) \Rightarrow |f(z)| \leqslant |F(z)| \cdot M(|z|;\, H_n, T) \quad \textit{for } z \in K_1.$$

THEOREM D. *If $f \in H_n$, $F \in T$, then there exists a real positive function $R(r;\, H_n, T)$ such that*

$$|f(z) \leqslant_1 |F(z)| \Rightarrow f(K_{R(r; H_n, T)}) \subset F(K_r) \quad \textit{for } 0 < r \leqslant 1.$$

The problems which are conected with Theorem C were investigated by Z. Bogucki and J. Waniurski [14]. The present authors investigated some problems in connection with Theorem D. In papers [22], [23] is presented the method of finding a function $R(r;\, H_n, T)$ and is solved this problem for some special cases ($T = S$, $T = S^*$).

THEOREM 1, [22]. *The function $R(r;\ H_n, T)$ in Theorem D may be defined in the following way*

$$R(r;\ H_n, T) = \sup\{R\colon\ 0 < R < 1,\ O_R^n \cap D(r, R, T) = \emptyset\}$$

or

$$R(r;\ H_n, T) = \inf\{R\colon\ 0 < R < 1,\ O_R^n \cap D(r, R, T) \neq \emptyset\},$$

where

$$D(r, R, T) = \{w\colon\ w = F(z)/F(\zeta),\ |z| = r,\ |\zeta| = R,\ F \in T\} = \{w\colon \underset{|z|=r}{\exists}\ \underset{|\zeta|=R}{\exists}\ \underset{F\in T}{\exists}\ w = F(z)/F(\zeta)\}$$

and

$$O_R^n = \{w\colon\ w = \omega(z),\ |z| = R,\ \omega \in \Omega_n\} = \{w\colon \underset{|z|=R}{\exists}\ \underset{\omega\in\Omega_n}{\exists}\ w = \omega(z)\},$$

Ω_n is the class of functions $\omega(z) = a_{n-1}z^{n-1} + a_n z^n + \ldots$ which are regular and bounded by 1 *in the unit disc K_1, in addition if $n = 1$ we suppose that $a_0 \geqslant 0$.*

Using this theorem we obtained ([22], [23]) the following results:

THEOREM 2.

$$R(r;\ H_1, S^*) = \begin{cases} r & \text{for } 0 < r \leqslant r_0, \\[4pt] \dfrac{\sqrt{r}}{1+\sqrt{r}+r} & \text{for } r_0 < r \leqslant 1, \end{cases}$$

where $r_0 = .29\ldots$ is the unique positive root of the equation $r^3 + r^2 + 3r - 1 = 0$.

COROLLARY 1. *If $f \in H_1$ and $F \in S^*$, then*

$$|f(z)| \leqslant_1 |F(z)| \Rightarrow f(K_{1/3}) \subset F(K_1).$$

THEOREM 3. *For $n \geqslant 2$ the function $R(r;\ H_n, S) = R(r;\ H_n, S^*)$ and is equal to the inverse function to the function*

$$r(R) = \frac{(1-R)^2 - 2R^n - (1-R)\sqrt{(1-R)^2 - 4R^n}}{2R^n}$$

defined in the interval $0 < R \leqslant R_n$, where R_n is the smallest positive root of the equation $(1-R)^2 - 4R^n = 0$.

COROLLARY 2. *If $f \in H_n$ $(n \geqslant 2)$ and $F \in S$ (or $F \in S^*$), then*

$$|f(z)| \leqslant_1 |F(z)| \Rightarrow f(K_{R_n}) \subset F(K_1),$$

where R_n defined in Theorem 3.

Remarks. 1. For $n \geqslant 2$ the number $R(r;\ H_n, T)$ may be greater than r.

We can easily check that

$$R(1/4;\ H_2, S) = 2/7 > 1/4.$$

2. F. Bogowski and Z. Stankiewicz [9], [10] found the function $R(r;\ H_n, T)$ for $T = S^c$ — convex univalent functions and for $T = S^*_{1/2}$ — starlike of order $1/2$. They also generalized this problem on the class H_0 of functions $f(z)$ such that $f \in H_1$ and $f(z)/z \neq 0$ for $z \in K_1$.

3. F. Bogowski and Z. Stankiewicz [9] proved that

$$R(1;\ H_1, S^*_{1/2}) = 1/2.$$

4. J. G. Krzyż [18] proved that

$$R(1;\ H_1, S) = 1/3.$$

5. The function $R(r;\ H_1, S)$ is not known yet.

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On a class of capacities on complex manifolds endowed with an hermitian structure and their relation to elliptic and hyperbolic quasi-conformal mappings

by J. Ławrynowicz (Łódź)

A class of capacities is introduced on complex manifolds endowed with an hermitian structure and an inhomogeneity function. The finiteness, algebraic properties, behaviour under holomorphic and biholomorphic mapping as well as connections with quasi-conformal capacities of condensers and quasi-conformal mappings on real submanifolds endowed with the induced riemannian or pseudo-riemannian structure are investigated. In particular an extensive generalization of the Schwarz lemma on holomorphic mappings and a theorem on invariance of capacities under biholomorphic mappings are proved. An interpretation in the physics of elementary particles is given.

This research has been inspired by papers of S. Kobayashi (J. Math. Soc. Japan, 1967), R. Kühnau (J. Reine Angew. Math., 1968 and 1970), S. S. Chern, H. I. Levine and L. Nirenberg (in *Global analysis*, Papers in honor of K. Kodaira, Tokyo 1969), and P. A. M. Dirac (in Proc. Internat. Conf. on relativistic theories of gravitation, Warsaw and Jabłonna 1962).

Approximation of analytic operators by polynomials in Fréchet spaces with bounded approximation property

by Cz. Matyszczyk (Warszawa)

Let X be a complex Fréchet space, i.e. complete locally convex, metric space.

Definition 1. We say that X has the *polynomial approximation property* (PAP) if for every open polynomially convex subset Q of X, for every complex Fréchet space Y, for every analytic operator $U: Q \to Y$ and for every compact subset K of Q there is a sequence of polynomials (P_n) convergent uniformly to U on K. If the sequence (P_n) does not depend on K, then we say that X has the *strong polynomial approximation property* (SPAP).

Every Fréchet space with the Grothendieck approximation property has the PAP. In my paper *Approximation*..., Bull. Acad. Polon. Sci. 1972, I proved that every complex Banach space X with the BAP (the identity on X is the pointwise limit of a sequence of finite rank operators) has the SPAP. I now announce the solution to the problem of deciding when a Fréchet space X with the BAP has the SPAP.

THEOREM 1. *If a Fréchet space X has the BAP, then the following conditions are equivalent*:

(i) *X has the SPAP*;

(ii) *the topology on X may be defined by a sequence of norms*;

(iii) *there is no subspace of X topologically isomorphic to the space s of all complex sequences endowed with the product topology.*

It is interesting to note that: *an open polynomially convex subset Q of s has the property*:

every analytic complex valued function defined on Q is the limit of a sequence of polynomials convergent uniformly on each compact subset of Q,

if and only if the subset Q has a finite number of connected components.

Remark. Theorem 1 is more strong in the "real" case. We may replace, in Theorem 1 and thus in Definition 1, "complex" by "real", "open polynomially convex" by "open" and "analytic" by "continuous".

The above results will be published.

Eine Normalform für dreifach-zusammenhängende in einem Kreisring eingebettete Gebiete

von C. MICHEL und U. PIRL (Berlin)

In der konformen Geometrie auf einer Riemannschen Fläche F ist es wichtig, stetige Normalformen $\mathcal{N}$ bezüglich gewisser normiert konformer Abbildungen zu kennen, bei denen die Gebiete $G' \in \mathcal{N}$ auf F relativ schlicht liegen. Wir betrachten hier den Fall, dass $F = F_r$ eine konzentrische Kreisringfläche

$$F_r: \{z \mid r < |z| < 1\}$$

ist. Es sei $\mathscr{G}_n$ die Menge aller $(n+2)$-fach zusammenhängenden schlichten Gebiete $G \subset F_r$, die $\mathscr{K}_1$: $\{z \mid |z| = 1\}$ und $\mathscr{K}_2$: $\{z \mid |z| = r\}$ als Randkomponenten haben, und bei denen keine Randkomponente punktförmig ist. Zwei Gebiete $G_1, G_2 \in \mathscr{G}_n$ heissen dann normiert konform äquivalent, wenn es eine schlichte konforme Abbildung $f(z)$ von G_1 auf G_2 gibt, bei der $f(\mathscr{K}_v) = \mathscr{K}_v$, $v = 1, 2$, gilt.

Unter einer stetigen Normalform für $\mathscr{G}_n$ bezüglich dieser normiert konformen Äquivalenz verstehen wir eine Teilmenge $\mathscr{N} \subset \mathscr{G}_n$ mit folgenden Eigenschaften:

1. Zu jedem $G \in \mathscr{G}_n$ gibt es genau ein normiert konform äquivalentes $G' \in \mathscr{N}$,

2. Ist $\{G'_v\}$, $G'_v \in \mathscr{N}$, eine Folge, die im Sinne Carathéodoryscher Kernkonvergenz gegen ein Gebiet $G' \in \mathscr{N}$ konvergiert, dann gilt $G' \in \mathscr{N}$.

SATZ. *Für $n = 1$ ist die folgende Menge $\mathscr{N}$ von dreifach zusammenhängenden Gebieten $E \subset \mathscr{N}$ eine stetige Normalform:*

E hat die Randkomponenten $\mathscr{K}_1, \mathscr{K}_2$ und n, wobei n das Bild einer in der ζ-Ebene gelegenen achsenparallelen Ellipse n^* bei der Abbildung $w = e^\zeta$ ist und der Mittelpunkt von n^* auf der negativ reellen Achse liegt (n^* darf auch zu einer achsenparallelen Strecke oder zu einem Kreis ausgeartet sein) —n ist also in der logarithmischen Metrik $ds = \frac{dw}{w}$ eine eventuell ausgeartete Ellipse.

Dieser Satz kann mit Hilfe einer Verfeinerung der Koebeschen Kontinuitätsmethode bewiesen werden, wobei zum Beweis der konformen Inäquivalenz zweier verschiedener Gebiete aus $\mathscr{N}$ das Argumentprinzip verwendet wird.

Ein ausführliche Darstellung mit Beweisen erscheint in den „Mathematischen Nachrichten".

On some method of conditional extremum determination

by L. MIKOŁAJCZYK (Łódź)

Denote by R a set of real measurable functions φ satisfying the condition $m \leqslant \varphi(t) \leqslant M$ for $t \in [a, b]$. Let $\mathscr{M}$ be a family of functions f defined as follows: $f(z) = \int_a^b g(z, t)\varphi(t)\,dt$, where $z \in D$, D is a certain region in C, the function $g(z, t)$ is holomorphic in D with respect to z and continuous in $D \times (a, b)$ with respect to $\{z, t\}$.

Let $F_0, F_1, \ldots, F_m$ be a sequence of real functionals of the form $F_j(f) = F_j\big(f(\xi), \bar{f}(\xi), \ldots, f^{(k)}(\xi), \overline{f^{(k)}}(\xi)\big)$, $j = 0, 1, \ldots, m$, defined in the family $\mathscr{M}$, where ξ is a fixed point in D. Consider now Problem 1 which consists in determining the extremal values of the functional F_0 under the condition $(F_1(f), \ldots, f_m(f)) \in \Delta$, where Δ is a fixed set in $\boldsymbol{R}^n$.

Using Dubowicki–Miljutin theorem we have proved the following

THEOREM 1. *If $f^*(z) = \int_a^b g(z, t)\varphi^*(t)\,dt$ is a solution to Problem 1 and a set Δ is a convex polyhedron, then the function φ^* satisfies the condition $c(t)\cdot\varphi^*(t) = \min_{m\leqslant\varphi\leqslant M} c(t)\cdot\varphi$, where $c(t) = \sum_{r=0}^{k} c_r\cdot \operatorname{re} g^{(r)}(\xi, t)$, c_r — real constants.*

Let M_1 be a family of functions defined by an integral $f(z) = \int_a^b g(z, t)\,d\alpha(t)$, where α — a non-decreasing function over $[a, b]$ satisfying the condition $\int_a^b d\alpha(t) = 1$. By the use of Theorem 1 we can prove

THEOREM 2. *A function $f(z)$ being a solution to Problem 1 in the family $\mathcal{M}_1$ and a set Δ — a convex polyhedron then $f^*(z) = \sum_{r=1}^{N} \lambda_r g(z, t_r)$, where $\lambda_r \geqslant 0$, $\sum_{r=1}^{N} \lambda_r = 1$, $N \leqslant k+1$.*

Evaluations de la seconde et troisième dérivée dans la famille des fonctions en moyenne p-multivalentes

par K. MIŚTA (Katowice)

Considérons une fonction holomorphe de la forme $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ dans le cercle unité. Soit $n(\omega) = n(\omega, f)$ le nombre des solutions de l'équation $f(z) = \omega$ dans le cercle unité.

DÉFINITION. La fonction f est appelée en *moyenne p-multivalente* dans le sens de Biernacki, si

$$\tfrac{1}{2}\int_0^{2N} n(R\cdot e^{i\varphi}, f)\,d\varphi \leqslant p \quad \text{pour tout } R > 0.$$

Dans le cas $p = 1$ les évaluations de premier et du second coefficient ainsi que les estimations concernants le valeur absolu de cette fonction et de sa dérivée sont connues. Soit $\varphi(z) = z + a_2z^2 + \ldots$ une fonction en moyenne univalente; alors

$$|a_2| \leqslant 2 \quad \text{et} \quad |a_3| \leqslant 3$$

et

$$(1) \qquad |\varphi(z)| \leqslant \frac{r}{(1-r)^2}, \quad |\varphi'(z)| \leqslant \frac{1+r}{(1-r)^3}$$

pour $|z| \leqslant r$.

Considérons maintenant la fonction $g(\xi) = \dfrac{z+\xi}{1+\bar{z}\xi}$, où $z \neq 0$ est un point du cercle unité et une fonction arbitraire φ en moyenne univalente. A l'aide de la fonction $h(\xi) = \dfrac{(g(\xi)) - \varphi(z)}{\varphi'(z)(1-|z|^2)}$ et (1) nous obtenons les estimations exactes de la seconde et troisième dérivée de la fonction en moyenne univalente, soit

$$|\varphi''(z)| \leqslant \frac{2(2+r)}{(1-r)^4}, \tag{2a}$$

$$|\varphi'''(z)| \leqslant \frac{6(3+r)}{(1-r)^5}, \qquad \text{pour } |z| < r. \tag{2b}$$

En utilisant les évaluations (2a) et (2b) et la relation fondamentale entre les fonctions en moyenne univalentes et en moyenne p-multivalentes nous obtiendronds l'évaluation exacte de la seconde et troisième dérivée de la fonction en moyenne p-multivalente, soit

$$|f''(z)| \leqslant \frac{r^{p-2}\cdot p}{(1-r)^{2p+2}} [(p+1)r^2 + 2(p+1)r + p - 1],$$

$$|f'''(z)| \leqslant \frac{r^{p-3}\cdot p}{(1-r)^{2p+3}} [(p_2+3p+2)r^3 + 3(p^2+3p+2)r^2 + \\ + 3(p^2+p-2) + p^2 - 3p + 2],$$

pour $|z| \leqslant r$.

Некоторые применения симметризационных методов в теории функций

И. П. Митюк (Краснодар)

Вводится понятие спиральной симметризации S_θ^1. Для спиральной симметризации имеет место следующая теорема (см. [1]):

Если G_θ^1-результат симметризации S_θ^1 области G относительно $z_0 \epsilon G$ то справедливо неравенство:

$$-(G_\theta^{(1)}, z_0) \geqslant \tau(G, z_0) \tag{1}$$

где $\tau(G, z_0)$-внутренний радиус области G относительно точки z_0.

Равенство в (1) имеет место тогда и только тогда, когда

$$g_{G_\theta^{(1)}}(z, z_0) = g_G(z, z_0), \qquad z \epsilon G \cap G_\theta^{(1)},$$

где $g_G(z, z_0)$-функция Грина (не обязательно классическая) для области G относительно $z = z_\theta$. Доказательство теоремы единственности дано в [2]. При $\theta = 0$ получаем теорему единственности для симметризации М. Маркуса.

Даются обобщения спиральной симметризации типа симметризации Сеге, формулируется теорема единственности для обобщения симметризации $S_\theta^{(n)}$. При $\theta = 0$ получается теорема единственности для симметризации Сеге.

Рассматриваются некоторые приложения $S_\theta^{(n)}$-симметризации к изучению экстремальных свойств функций, регулярных в круге, кольце и многосвязной области [3], [4].

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Estimation of coefficient a_{3k+1} of univalent and k-symmetric functions in the unit circle

by K. Petiie (Gliwice)

Let $S^{(k)}$ be a class of functions

$$f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1}$$

regular, univalent and k-symmetric in the circle $|z| < 1$.

Theorem. *If* $f(z) \in S^{(k)}$, *then*

$$|a_{3k+1}| \leqslant \frac{2}{3k} \quad \textit{for } k \geqslant 8, \tag{1}$$

where the equality holds for the function

$$f(z) = z(1+\eta z^{3k})^{-2/3k}, \qquad |\eta| = 1,$$

$$|a_{3k+1}| \leqslant \frac{4[3(k+1)^2(3k+2)+k^2(3k^2+3k+1)]^{3/2}}{9\sqrt{3}(3k+2)(3^{k^2}+3k+1)k^3} \tag{2}$$

for $2 \leqslant k \leqslant 7$.

In the proof I availed myself of the idea of L. E. Ahlfors which consists in choosing a matched polynomial for the function $f(z) \in S^{(k)}$ and in applying, for so obtained p-valent function, the area theorem of G. M. Goluzin.

The work will be published.

Algebraic criterion of quasi-conformality for Riemannian differential spaces

by A. M. Pierzchalski (Łódź)

Let (M, C) be a Riemannian differential space (R. Sikorski, Colloq. Math. 1972; P. Walczak, Bull. Acad. Polon. Sci. 1974). For any positive integer k let $A_k(M, C)$ be the totality of functions $u \in C$ satisfying two conditions: (i) u is bounded on M, (ii) $D_k(u) = \left(\int_M |\operatorname{grad} u|^k d\mu^k(M, C)\right)^{1/k} < \infty$, where $\mu^k(M, C)$ is the k-dimensional Lebesgue measure on (M, C) (cf. the detailed version, Bull. Acad. Polon. Sci. 1974, two papers). With the usual algebraic operations $A_k(M, C)$ becomes an algebra over $\boldsymbol{R}$. The completion $\bar{A}_k(M, C)$ of $A_k(M, C)$ with respect to the norm $\|u\|_k = \sup_{p \in M} |u(p)| + D_k(u)$ for $u \in A_k(M, C)$ is a Banach algebra with unit.

The author obtains the following criterion of quasi-conformality: A diffeomorphism f of (M, C) onto (M', C') is a (Q, k)-quasi-conformal (for the definition cf. the detailed version) with some $1 \leqslant Q < \infty$ iff the induced mapping $\bar{A}_k(M, C) \ni u \to u \circ f^{-1} \in \bar{A}_k(M, C)$ is an algebraic isomorphism.

Generalizations of Bazilevič functions

by M. O. Reade (Ann Arbor)

Let $f(z) = z + \ldots$ be analytic in the unit disc Δ, with $[f(z) \cdot f'(z)/z] \neq 0$ there, and let $U(w)$ be "suitable" function defined on $f(\Delta)$. If the inequality

$$\frac{\partial}{\partial\theta} \arg\left[zf'(z)\, U\big(f(z)\big)\right] \geqslant 0, \qquad z = re^{i\theta},$$

holds in Δ, then $f(z)$ is said to be *OST-convex in Δ*. If the inequality

$$\int_{\theta_1}^{\theta_2} d\left[\arg\big(zf'(z)\big)\, U\big(f(z)\big)\right] > -\pi$$

holds for all $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi$, and all $0 \leqslant r < 1$, then $f(z)$ is said to be *OST close-to-convex* (with respect to $U(w)$) in Δ. Ogawa (Journal of the Mathematical Society of Japan 13 (1961), p. 431–441) Sakaguchi (ibidem 14 (1962), p. 312–321) and Takatsuka (Duke Mathematical Journal 33 (1966), p. 583–593) have shown OST-convex and OST-close-to-convex functions are univalent. In this note we obtain (1) representation theorems for OST functions, using the technique of an earlier note (Publicationes Mathematicae (Debrecen) 11 (1964), p. 39–43), (2) geometric descriptions of OST functions that appear for special choices of the function U, and (3) some distorion theorems for special cases.

These results are related to earlier ones due to Bazilevič (Mat. Sb. 37 (1955), p. 471–476), the present author, and Sheil-Small (Quarterly Journal of Mathematics (Oxford) 2 (1972), p. 135–142), among others.

A class of extremum problems for analytic functions

by E. Reich (Minneapolis)

Let $\varkappa(z)$, $|z| < 1$, be a given complex-valued bounded measurable function. For functions f analytic for $|z| < 1$, with $\iint\limits_{|z|<1} |f(z)|\,dx\,dy \leqslant 1$, let

$$L_\varkappa[f] = \iint\limits_{|z|<1} \varkappa(z) f(z)\,dx\,dy.$$

We consider the extremum problem

$$(*) \qquad \sup_f |L_\varkappa[f]|.$$

Analytic function theory methods for investigating $(*)$ will be discussed with reference to special examples of $\varkappa$'s. It is known that the value of $(*)$ is related to a boundary value problem for quasi-conformal mapping, and this provides an alternative method for investigating $(*)$. The problem of characterizing those $\varkappa$ for which an extremal f exists is open.

Über eine Variants des Modulpotenzprodukts

von H. Renelt (Halle)

Das endlich vielfach zusammenhängende Gebiet B werde in gewisse nichtüberlappende Streifen F_i vorgegebenen topologischen Verlaufs eingeteilt. 1957 wurde von U. Pirl (mittels Koebescher Kontinuitätsmethode) und J. A. Jenkins (mittels einer Variationsmethode von Schiffer und Spencer) gezeigt, dass zu jeder Linearkombination $\sum a_i M_i \neq 0$ der konformen Moduln M_i der F_i mit beliebigen, aber festen $a_i \geqslant 0$ ein quadratisches Differential gehört, dessen kritische Trajektorien diejenige Einteilung von B in Streifen F_i liefern, für die $\sum a_i M_i$ maximal wird unter allen solchen Einteilungen von B.

Betrachtet man $\sum a_i/M_i \to \min$ anstelle von $\sum a_i K_i \to \max$, so ergibt sich unter Verwendung des Dirichletintegrals eine neue Beweismethode, die darüberhinaus auch bei unendlich vielfach zusammenhängenden Gebieten anwendbar ist.

Some boundary properties of series in Laguerre polynomials

by P. RUSSEV (Sofia)

The aim of the communication is to show that some classical results about power series as Hadamard's gap theorem, Fatou's theorem for convergence on the circumference of convergence and also Jentzsch's theorem, are valid for series in Laguerre's polynomials.

The Levi problem for domains spread over spaces with a basis

by M. SCHOTTENLOHER (München)

A proof of the following result will be presented:

A pseudoconvex domain spread over a Banach space with a basis is a domain of existence of a holomorphic function.

The proof is different from proofs in [1], [2], [3], although it uses one essential tool of [2] to apply a characterization of domains of existence in [4]. The same proof gives the corresponding result for domains over Silva spaces (i.e. $\mathcal{DFS}$-spaces) with a basis, over certain metrizable spaces with a basis or over complemented subspaces of such spaces, extending results of [1].

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Analytic continuation of positive definite functions

by M. L. SKWARCZYŃSKI (Warszawa)

A real analytic function is called *positive definite* at $z_0 \in C^n$ if the coefficients $a_{m,\bar{n}}$, $|m|$, $|n| \leqslant K$ of the Taylor series $f(z, \bar{z}) = \sum a_{m,\bar{n}}(z-z_0)^m \times \overline{\times(z-z_0)^n}$ form positive definite hermitian matrix for every $K = 0, 1, \ldots$ Some elementary properties and examples of such functions are presented. It is also shown that the property of function to be positive definite at a point in C^n is preserved under analytic continuation.

Faber polynomials in the theory of univalent bounded functions

by J. ŚLADKOWSKA-ZAHORSKA (Gliwice)

Let S_1 be the class of conformal mappings of the unit disc onto itself with $f(0) = 0$, $f'(0) > 0$. Let

$$\log\frac{z-\xi}{f(z)-f(\xi)} = \sum_{m,n=0}^{\infty} a_{mn} z^m \xi^n,$$

$$-\log\left(1-\overline{f(\xi)}f(z)\right) = \sum_{m,n=1}^{\infty} b_{mn} z^m \xi^n.$$

Let (x_n), $n = 1, \ldots, N$, denote an arbitrary sequence of complex numbers. We have the following

THEOREM. *Let $f \in S_1$ be a function for which the functional*

$$\Phi(f) = \mathrm{Re}\left\{\sum_{m,n=1}^{\infty} (a_{mn}x_m x_n + b_{mn} x_m \bar{x}_n)\right\}$$

assumes its maximum in S_1. Let $F_m(w)$ be the Faber polynomial of degree m for the function $1/f(z)$. Then

$$\sum_{m=1}^{N}\left(x_m F_m\left(\frac{1}{f(z)}\right) + \bar{x}_m \overline{F_m\left(\overline{f(z)}\right)}\right) = \sum_{m=1}^{N}(x_m z^{-m} + \bar{x}_m z^m) + \text{const}.$$

The proof is based on the formula which gives a family of comparison functions for any given $f \in S_1$.

On completeness of holomorphic principal bundles

by S. TAKEUCHI (Gifu-shi)

Main results are the followings:

DEFINITION. Let M be a q-complete complex manifold. We say M is *strictly q-complete* if M is not $(q-1)$-complete. We define the completeness of M (denoted by $\nu(M)$) by setting $\nu(M) = q$, when M is strictly q-complete.

DEFINITION. Let G be a connected complex Lie group, with a maximal compact subgroup K. G is called *q-compact* if $\dim_C k \cup \sqrt{-1}k = q$, where k denotes the Lie algebra of K. If the equality holds, we say that G is *strictly q-compact*. This number q is called the *compactness* of G (denoted by $\beta(G)$).

$\tilde{k}: = k+\sqrt{-1}k$ is the complex Lie subalgebra of $\mathscr{G}$, where $\mathscr{G}$ denotes the Lie algebra of G. Then the complex Lie subgroup $\tilde{K}$ of G corresponding the Lie algebra $\tilde{k}$ is closed in G and it is known that G is biholomorphically isomorphic to $\tilde{K} \times C^a$, where C^a denotes the complex euclidean space of dimension a. This number a is independent of the choice of K and hence may be called the *complex characteristic index* of G (denoted by $a(G)$). Denoting the complex dimension of G by $\delta(G)$, and the real dimension of K by $\gamma(G)$, we obtain the following formulae for G and its closed connected normal complex Lie subgroup N;

$$a(G)+\gamma(G) = \beta(G)+\delta(G), \tag{1}$$

$$a(G) \leqslant a(N)+a(G/N), \tag{2}$$

$$\beta(G) \leqslant \beta(N)+\beta(G/N), \tag{3}$$

$$\gamma(G) = \gamma(N)+\gamma(G/N), \tag{4}$$

$$\delta(G) = \delta(N)+\delta(G/N). \tag{5}$$

We shall prove for a holomorphic principal boundle $P(B, G)$ over a complex manifold B with a structure group G, that $\nu(P) \leqslant \beta(G)+\nu(B)$ which is a generalization of the theorem of Matsushima–Morimoto. We prove eventually compactness equals completeness for complex Lie groups, i.e., $\beta(G) = \nu(G)$, which generalizes a characterization of Stein group.

Resolution of Warschawski–Walsh–Sewell's problem and application to various questions of analysis

by P. M. TAMRAZOV (Kiev)

Let: $\mathfrak{G}$ be the class of all bounded simple-connected domains of complex plane C; $G \in \mathfrak{G}$; $A(G)$ be the class of all continuous functions on G, holomorphic in G; $f \in A(G)$; $\omega_{f,E,\xi}(\delta)$ and $f_E^{(n)}(\xi)$ be moduli of continuity (global and local) and derivatives of f along a set $E \subset \bar{G}$; $C(E)$ be the class of all functions continuous on E; $\mathfrak{M}$ be the class of all functions $\mu(\delta)$ of modulus of continuity type; c be constants not depending on δ; $K = \{\xi: |\xi| < 1\}$.

Let us consider the following implications:

S1. $\omega_{f,\partial G} \leqslant \mu \Rightarrow \omega_{f,\bar{G}} \leqslant c\mu$,

S2. $\omega_{f,\partial G,z_0} \leqslant \mu \Rightarrow \omega_{f,\bar{G},z_0} \leqslant c\mu \quad (z_0 \in \partial G)$,

S3. $\exists f'_{\partial G}(z_0) \Rightarrow \exists f'_{\bar{G}}(z_0) \quad (z \in \partial G)$,

S4. $\{f'_{\partial G} \in C(\partial G),\ \omega_{f,\partial G}(\delta) \leqslant c\delta\} \Rightarrow f'_{\bar{G}} \in C(G)$.

Hardy and Littlewood proved S1 in the case when $G = K$, $\mu(\delta) = \delta^\alpha$, $\alpha > 0$. In the case when G is a Jordan domain and $\mu(\delta) = \delta^\alpha$, $\alpha > 0$, Warschawski proved S2 and Walsh and Sewell – S3, S4 and S1 (analogously for $\mu(\delta) = \delta|\log\delta|$, $\delta|\log\delta|^2$).

In the well-known Sewell's monography (1942) the following problems have been posed:

P1. To omit the restriction that G is Jordan.

P2. To prove analogous results for $\mu(\delta)$ other than δ^α, $\delta|\log\delta|$.

P3. What is the most general modulus of continuity $\mu(\delta)$ for which S1 is valid in the case when G is Jordan (or even Jordan domain with analytic boundary)?

These problems form the general problem, and it was essentially posed by Warschawski, Walsh and Sewell.

Warschawski–Walsh–Sewell's problem is solved by the author (1971–1972): S1–S4 are proved for any $G \in \mathfrak{G}$ and $\mu \in \mathfrak{M}$. It also is established that in S1 and S2 as c one can take an absolute constant, and that the concavity of $\nu(t) \equiv \log\mu(\exp t)$ is necessary and sufficient for the validity of S2 (and is sufficient for validity of S1) with $c = 1$.

Suppose: $\boldsymbol{R}^+ = \{\delta: \delta > 0\}$; Λ be the class of all functions $\omega(\delta) > 0$ on $\boldsymbol{R}^+$ for which $\int_0^1 \omega(t)t^{-1}dt < \infty$; $C^n\Lambda(F)$ be the class of all functions on a set $F \subset \boldsymbol{C}$ for which there exist derivatives $\varphi_F^{(0)} \equiv \varphi, \ldots, \varphi_F^{(n)}$ along P such that $\omega_{\varphi_F^{(n)},F} \in \Lambda$; J be operator on Λ defined by

$$(J\omega)(\delta) = \int_0^\delta \omega(t)t^{-1}dt + \delta\int_\delta^\pi \omega(t)t^{-2}dt.$$

The solution of Warschawski–Walsh–Sewell's problem has many applications, for instance:

1. Let G be a Jordan domain with rectifiable boundary ∂G whose small arcs and their chords are infinitesimals of the same order. Then for any $\varphi \in C^n\Lambda(\partial G)$ the integral $\Phi(\xi) = (2\pi i)^{-1}\int_{\partial G}\varphi(w)(w-\zeta)^{-1}dw$ of

Cauchy type has in G derivatives $\Phi^{(0)} = \Phi, \ldots, \Phi^{(n)}$, $n \geqslant 0$, which can be continously prolonged onto $\bar{G}$ and prolonged function $\Phi^{(n)}$ satisfies the inequality $\omega_{\Phi^{(n)},\bar{G}} \leqslant cJ\omega_{\varphi^{(n)},\partial G}$.

2. Let G be a Jordan domain with rectifiable boundary possessing continuously turning tangent. Suppose that the angle $\tau(s)$ between the tangent and the positive real axis as a function of arc lenght s on ∂G belongs to $C^n\Lambda(\boldsymbol{R}^+)$, $n \geqslant 1$. Let $g(\xi)$ be a conformal homeomorphism of K onto G. Then

$$\omega_{g^{(n+1)},\bar{K}} \leqslant cJ\omega_{\tau^{(r)},\boldsymbol{R}^+}.$$

3. Suppose $h(\xi)$ be holomorphic in K, $\mathrm{Re}\,h(\xi)$ be continuous on K and $u(\theta) \equiv \mathrm{Re}\,h(e^{i\theta})$ in respect to θ belong to $C^n\Lambda(\boldsymbol{R}^+)$, $n \geqslant 0$. Then $h^{(n)}(\xi)$ can be prolonged onto $\bar{K}$ and the prolonged function satisfies the inequality

$$\omega_{h^{(n)},\bar{K}} \leqslant cJ\omega_{u^{(n)},\boldsymbol{R}^+}.$$

4. In inverse theorems of polynomial approximation we obtain statements about behaviour of functions along G (not only along ∂G).

5. In the direct theorems of approximation it is sufficient to demand functions be good only along ∂G (not along $\bar{G}$).

Qualitative properties of generalized analytic functions of several complex variables

by W. TUTSCHKE (Halle)

One of the most important aims of the theory of generalized analytic functions is the following: we want to get a connection between generalized analytic functions and analytic functions in the ordinary sense. That means in particular, that we want to construct such functions with help of analytic functions. Moreover, one can find many common properties of analytic and genefalized analytic functions. In the following we define a generalized analytic function $(w_1, \ldots, w_m)$ as a function, whose partial complex derivatives relative to the conjugate complex variables z_i^* are prescribed functions of all variables and the functions w_j self.

1) *Factorization.* If the right-hand sides of the differential equations satisfy a certain Lipschitz condition, then each w_j has the representation $w_j = \Phi_j \exp\omega_j$, where Φ_j are analytic. This representation shows, that the set of all zeros is an analytic set. In the case of one complex variable

we have a close connection between the factorization and the fact, that the zeros are isolated (cf. the papers of L. Bers and I. N. Vekua). Although in the case of several complex variables the zeros are not isolated, one can prove the possibility of the factorization (ДАН 214 (1974), p. 1276–1279).

2) With help of a solution of the well-known *Cousin-problems* for analytic functions of several complex variables one get results about generalized analytic functions. For example we get a global factorization from local representations. Another example: one gets in this manner a global solution of the inhomogeneous Cauchy–Riemann differential equations (in several complex variables). It is also possible to solve such Cousin-problems for generalized analytic functions (see also the paper of B. Goldschmidt).

3) A factorization in the sense of 1) is possible, iff w is a solution of a certain complex *differential-inequality*. In this manner we get a generalization of the notion of approximately analytic functions in the case of several complex variables (*Abspaltung holomorpher Faktoren aus Lösungen komplexer Differentialungleichungen*, Math. Nachr., in press; in the case of one complexe variable L. Bers was first to consider such functions).

4) With help of theorems about the factorization we can prove, that in certain cases the *stationary points* of (non-constant) solutions of real differential-inequalities of second order are isolated.

5) The *Hartogs theorem* (about continuity of seperately analytic functions) also holds for generalized analytic functions of several variables (example: $\partial w/\partial z_j^* = w \cdot h_j(z_1, \ldots, z_j, w)$, h_j is a holomorphic function of w).

We call a function *approximately analytic in weak sense*, if all $\partial w/\partial z_j = 0$ in all points with $w = 0$. Then the Hartogs theorem holds for approximately analytic functions in weak sense (under some additional assumptions about the derivatives (*Eine Erweiterung des Hartogsschen Stetigkeitssatzes*, to appear).

Квазиконформные отображения с якобианом переменного знака

Л. И. Волковыский (Ташкент)

Пусть Q карта, представляющая замкнутую риманову поверхность S, разбитую на области, допускающие правильную раскраску двумя красками. Пусть D_i^+, D_j^- соответствующие им области, γ_{ij}- смежные стороны, $\gamma = \bigcup \gamma_{ij}$. Мы рассматриваем отображения f поверхности S, сужения которых на D_i^+, D_j^- представляют квазиконформные отображения с якобианом $J < 0$ на D_i^+ и $J < 0$ на D_j^- с общими значениями на γ_{ij}, возможно с $|\mu| = 1$ на γ (μ-бельтрамовский коэффициент f на $S \setminus \gamma$). Доказывается, что при некоторой регулярности γ такие f существуют и существует естественный гомеоморфизм $\varphi: S \to S'$, $S' = S^{\mu^*}$, $\mu^* = \mu$ или $1/\bar{\mu}$, соотвественно $|\mu| < 1$, $|\mu| > 1$; такой что $f^* = f \circ \varphi^{-1}$ представляет конформно — антиконформное отображение карты $Q' = \varphi(Q)$. Решается задача о минимальнои отклонении $K[\varphi]$ для Q.

Для частных случаев рассматриваются интегральные теоремы и краевые задачи, связанные с возникающими римановыми поверхностями со складками.

Polynomial structure on principal fibre bundles

by P. Walczak (Łódź)

A complex manifold M can be equipped in the natural way with the $(1,1)$ tensor field f, which satisfies the algebraic equation

$$f^2 + I = 0, \tag{1}$$

where I is the identity mapping on TM. If M is a real C^∞-manifold and f satisfies (1), then f is said to be an *almost complex structure* on M. It is known that f is induced by a structure of a complex manifold on M if and only if the tensor field $[f,f]$ on M defined by the formula

$$[f,f](X, Y) = [fX, fY] - f[X, fY] - f[fX, Y] + f^2[X, Y]$$

vanishes. The second example of a "polynomial structure" is an almost contact structure, which is an $(1,1)$ tensor field f satisfying the equation

$$f^3 + f = 0$$

and such that

$$f^2 + I = X \otimes \omega,$$

where X is a vector field and ω is an 1-form on M. It is known that if f is an almost contact structure on M, then there is a canonical almost complex structure f' on the manifold $M \times R$ such that $[f', f'] = 0$ iff $[f, f] = 0$.

Generalizing these concepts Goldberg and Yano (Kodai Math. Sem. Rep., 1970) introduced the notion of polynomial structure of degree d, which is an $(1,1)$ tensor field of constant rank on M, satisfying the equation

$$Q(f) = f^d + a_d f^{d-1} + \ldots + a_2 f + a_1 I = 0$$

and such that $f^{d-1}(x), \ldots, f(x), I$ are lineary independent for any point x of M. In this situation we shall call f a *Q-structure*. We can prove that if the manifold M is parallelizable (e.g., M is a Lie group), Q is a polynomial of degree $d \leqslant \dim M$ and $\dim M$ is even or Q has a real root, then there is a Q-structure on M.

Here we will consider a principal fibre bundle P over a paracompact C^∞-manifold M with a structure group G and the projection $\pi\colon P \to M$ equipped with a connection Γ. If f is an $(1, 1)$ tensor field on M, then the formulae

$$f^h(Z^h) = \big(f(Z)\big)^h, f^h(A^*) = 0,$$

where Z is a vector field on M, A – a left invariant vector field on G, Z^h – the horizontal lift of Z and A^* – the fundamental vector field on P corresponding to A, define a tensor field f^h on P, which we shall call a horizontal lift of f. Similarly, if f is a left-invariant $(1, 1)$ tensor field on G, then the formulae

$$f^*(A^*) = f(A)^*, \quad f^*(Z^h) = 0,$$

define a tensor field f^* on P, which we shall call a fundamental tensor field corresponding to f.

It is easy to see that if f is a Q-structure on M, f' is a Q'-structure on G, and Q' is a divisor of Q, then $F = f^h + f'^*$ is a Q-structure on P. Hence, we obtain the following results:

THEOREM 1. *If M admits a Q-structure f of degree d and* $\dim G$ *is even or Q has a real root, then P admits a Q-structure F such that* $\operatorname{rank} F \geqslant \operatorname{rank} f$.

COROLLARY. *If M is an almost complex manifold and* $\dim G$ *is even, then P admits an almost complex structure. In particular, P is orientable.*

It is easy to see that if the connection Γ is flat, then

$$[f^h, f^h](Z^h, Z'^h) = \big([f, f](Z, Z')\big)^h$$

for any vector fields Z and Z' on M. The formula

$$[f^*, f^*](A^*, B^*) = \big([f, f](A, B)\big)^*$$

holds always. Thus we obtain the following

THEOREM 2. *If the connection Γ is flat, $[f, f] = 0$ and $[f', f'] = 0$, then $[F, F] = 0$.*

Application of Pontriagin's maximum principle to examining some extremal problems

by S. WALCZAK (Łódź)

We consider the following problem: given the sequence of functionals $F_0(h), F_1(h), \ldots, F_n(h)$ (F_0 – a real functional) defined in some family M of complex functions. We want to determine $\min\limits_{h \in M} F_0(h)$ under the conditions $F_j(h) = \lambda_j$, $j = 1, 2, \ldots, n$, where λ_j — constants

$$F_j(h) = F_j\big(h(\zeta_1), \overline{h(\zeta_1)}, \ldots, h^{n_1}(\zeta_1), \overline{h^{(n_1)}(\zeta_1)}; \ldots$$
$$\ldots; h(\zeta_k), \overline{h(\zeta_k)}, \ldots, h^{(n_k)}(\zeta_k), \overline{h^{(n_k)}(\zeta_k)}\big), \quad j = 0, 1, \ldots, n.$$

Such problem in the variational calculus is called an isoperimetric problem. Employing the Pontriagin maximum principle I have constructed the method of solving those problems in families of convex, starlike and Carathéodory functions as well as in families of univalent and bounded functions. For example, in the family of starlike functions the following theorem holds:

THEOREM. *If h^* is a solution to an isoperimetric problem, then*

$$h^*(z) = z\prod_{j=1}^{N}(1-e^{-it_j}z)^{-\lambda_j},$$

where $N \leqslant n_1+n_2+\ldots+n_k+k$, $\lambda_j \geqslant 0$, $\sum_{j=1}^{N} \lambda_j = 2$.

This method can also be applied in other families of functions (of one or more variables) having the structural representation.

Capacity and quasi-conformal mappings on Riemannian manifolds

by M. Z. WOJCIECHOWSKA (Łódź)

The author gives a systematic study of quasi-conformal mappings in Riemannian manifolds by means of conformal capacity. This is a counterpart of the approach proposed by J. Ławrynowicz (Rep. Math. Phys., 1974) for essentially pseudo-Riemannian manifolds and is motivated by inherent difficulties in extending the extremal length approach due to K. Suominen (Ann. Acad. Sci. Fenn., 1968) to the pseudo-Riemannian case.

In this research the key theorem gives estimates of the norm for the differential of quasi-conformal mapping by means of its jacobian and yields, in particular, the equivalence of quasi-conformal mappings in the sense of Suominen and in the author's sense. The paper will apear in the Rev. Roumaine Math. Pures Appl.

Некоторые вопросы комплексной „Теории потенциала"

В. П. ЗАХАРЮТА (Ростов)

Предполагается обсуждение следующих вопросов:

1. Экстремальные плюрисубгармонические функции и их свойства.

2. C^n-регулярность компактов и областей в C^n.
3. Различные характеризации C^n-полярных множеств.
4. Применения к изоморфной классификации пространств аналитических функций в областях и на компактах в C^n.
5. Базисы из ортогональных полиномов и интерполяционные ряды Ньютона–Лагранжа для компактов в C^n.
6. Наилучшие полиномиальные приближения на компактах.
7. Некоторые задачи.

Решение проблемы Лейа о трансфинитном диаметре произвольного компакта в C^n

В. П. Захарюта (Ростов)

Дается положительное решение проблемы Ф. Лейа (1957 г.) о существовании при $n \geqslant 2$ обычного предела у последовательности, определяющей трансфинитный диаметр для произвольного компакта в C^n:

$$d(K) = d^{(n)}(K) = \overline{\lim_{s\to\infty}} d_s(K).$$

Ранее был изучен случай декартова произведения плоских компактов (М. Шиффер, И. Сицяк, 1961 г.).

Результат достигается с помощью рассмотрения главной постоянной Чебышева $\tau(K)$, определяемой как среднее геометрическое семейства постоянных Чебышева по всевозможным „направлениям" $\tau(K, \theta)$.

В качестве приложения для произвольного компакта $K \subset C^n$ рассмотрен аналог классической теоремы Пойа об оценке коэффициентов степенного разложения функции, аналитической в окрестности бесконечности, через трансфинитный диаметр множества её особенностей. Для специальных классов компактов аналоги теоремы Пойа рассматривали М. Шиффер и Й. Сицяк, а также В. П. Шейнов.

Ставится ряд задач.

On the problem of orthogonality for Teichmüller quasi-conformal mappings in doubly connected domains

by J. Zając (Łódź)

An analogue of the N-class of complex dilatations due in the case of the unit disc to L. V. Ahlfors (Ann. Math., 1961) is discussed for annuli, what enables its applications for arbitrary plane doubly connected domains. In particular, a characterization of this class is given by means of the strong orthogonalization in the sense of E. Reich and K. Strebel (Ann. Acad. Sci. Fenn., 1970). This result is essential e.g. when deriving the parametrical differential equation for Teichmüller quasi-conformal mappings in an annulus.

Sur la courbure des lignes de niveau dans les classes des fonctions convexes et Σ-convexes d'ordre a

par J. ZDERKIEWICZ (Lublin)

Soit S_a^c, $0 \leqslant a < 1$, la classes des fonctions $f(z) = z + a_2 z^2 + \ldots$ holomorphes et univalentes dans le cercle $K = \{z\colon |z| < 1\}$ et satisfaisant la condition

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > a \quad \text{pour } z \in K. \tag{1}$$

Par $\sum\limits_a^c$, $0 \leqslant a < 1$, désignons la famille des fonctions $f(z) = z + a_0 + a_1/z + \ldots$, holomorphes et univalentes dans le domaine $|z| > 1$ et satisfaisant la condition (1) dans ce domaine. Soit encore $K_r(\vartheta)$ la courbure de l'image de la circonférence $z = re^{i\vartheta}$, $0 \leqslant \vartheta \leqslant 2\pi$ dans la représentation $w = f(z)$ au point $w = f(re^{i\vartheta})$.

THÉORÈME 1. *Soit* $f(z) \in S_a^c$, $0 \leqslant a < 1$, *et*

$$a(r) = \frac{1+r}{2r} - \frac{1}{(1-r)\log\frac{1+r}{1-r}}, \qquad S(r) = \frac{1+(1-2a)r^2}{2r} - \frac{1}{\log\frac{1+r}{1-r}}.$$

Alors

$$\frac{(1-r^2)^{1-a}}{r} \leqslant K_r(\vartheta) \leqslant \begin{cases} \dfrac{2}{(1-r^2)\log\frac{1+r}{1-r}}\left(\dfrac{1+r}{1-r}\right)^{S(r)}, & si\ \ 0 \leqslant a < a(r), \\[2ex] \dfrac{1-(2a-1)r}{r(1-r)^{2a-1}}, & si\ \ a(r) \leqslant a < 1. \end{cases} \tag{2}$$

Théorème 2. *Si* $f(z) \in \sum_{a}^{c}$, $0 \leqslant a < 1$, *on a*

$$(3)\qquad \frac{1+2a\dfrac{1}{r^2}+(2a-1)\dfrac{1}{r^4}}{r\left(1+\dfrac{1}{r^2}\right)^{3-a}} \leqslant K_r(\vartheta) \leqslant \frac{1-2a\dfrac{1}{r^2}+(2a-1)\dfrac{1}{r^4}}{r\left(1-\dfrac{1}{r^2}\right)^{3-a}}.$$

Les limitations (2) *et* (3) *sont exactes.*

Об отображении областей комплексных многообразий

Ю. Б. Зелинский (Киев)

Пусть M^n и N^n комплексные n-многообразии, $D \subset M$—открытая область.

Теорема. *Пусть $f\colon D \to N^n$—голоморфное отображение и пусть выполнены условия*

1) $f(\partial D) \cap f(D) = \emptyset$;

2) *найдется замкнутое множество F, принадлежащее границе и открытое множество $U \subset \partial D$ (где U—вещественное $(2n-1)$-многообразие) такое, что $U \setminus F$-несвязно, причем в любой окрестности U лежат внешние точки D;*

3) $f_0^*\colon H^{2n-2}(F_1) \to H^{2n-2}(F \cap U)$ - *эпиморфизм, где* $F_1 = f(F \cap U)$ *и гомоморфизм* f_0^* *групп когомологий индуцирован отображением* f;

4) $f(F) \cap f(\partial D \setminus F) = \emptyset$.

Тогда $f|_D$-монотонное отображение.

Замечание. Если $M^n = N^n = C^n$, то при выполнении условий теоремы $f|_D$—гомеоморфизм.

Получен ряд других результатов обобщающих принцип граничного соответствия.

Generating functions for some classes of univalent functions

by E. J. Złotkiewicz (Lublin)

Let $p(z) = e^{i\beta}+p_1z+p_2z^2+\ldots$ be regular in the unit disc Δ with $|\beta| < \pi/2$, let $\psi(u, v)$ be a complex continuous function defined in a domain of $C \times C$. With some simple restrictions on ψ the relation

$$\mathcal{R}\,\psi(p(z), zp'(z)) > 0 \Rightarrow \mathcal{R}\,p(z) > 0$$

is proved.

This result is then used to generate subclasses of starlike, spirallike and close-to-convex functions.