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ON A CLASS OF STOCHASTIC FUNCTIONAL INTEGRAL EQUATIONS

BY

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1. Introduction. Ordinary random or stochastic integral equations play a major role in the characterizations of many problems in life sciences and engineering (see, e.g., [1] and [4]-[8]). It is only recently that a concerted effort has been made to develop and unify the theory of such equations, see Bharucha-Reid [1].

The object of the present paper* is to introduce a study of a random or stochastic functional integral equation in the subject area. To the knowledge of the authors there have been no studies of such equations from the stochastic point of view. More specifically, we shall be concerned with a stochastic functional integral equations of the form

$$x(t;\omega) = h(t;\omega)+\int_0^t k(t,\tau;\omega)f\big(\tau, x_\tau(\omega)\big)\,d\tau, \tag{1.1}$$

$$x(t;\omega) = h(t;\omega)+\int_0^\infty k(t-\tau;\omega)f\big(\tau, x_\tau(\omega)\big)\,d\tau, \tag{1.2}$$

where

(i) $t\in R_+$, and $\omega\in\Omega$, the supporting set of a complete probability measure space (Ω, A, μ);

(ii) $x(t;\omega)$ is the unknown random function defined for $t\in R_+$ and $\omega\in\Omega$;

(iii) $h(t;\omega)$ is the stochastic free term defined for $t\in R_+$ and $\omega\in\Omega$;

(iv) $k(t,\tau;\omega)$ is the stochastic kernel defined for $0\leqslant\tau\leqslant t<\infty$ and $\omega\in\Omega$;

(v) $x_t(\omega)$ is the restriction of the function $x(\tau)$ to the interval $[0, t]$, $t>0$, with $x_0(\omega) = x(0;\omega)\in L_2(\Omega, A, \mu)$.

Note that the similar case where x_t denotes the restriction of the function $x_\tau(\omega)$ to the interval $[t-\tau, t]$, with fixed $\tau>0$, can also be

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studied analogously. Of course, in this case we need an initial condition of the form $x(t;\omega) = \varphi(t;\omega)$, $t \in [-\tau, 0]$, $\omega \in \Omega$.

In this study we shall state several existence theorems and significant special cases concerning equations (1.1) and (1.2). A deterministic version of the stochastic functional integral equation (1.1) has been studied by Corduneanu [2], using functional analytic techniques. Stochastic functional integral equations occur quite often in the study of feed-back systems with hysteresis, [2] and [3], among others.

In Section 2 we shall state some definitions and the fixed-point theorem of Schauder and Tychonoff which are essential in the study. The main results are given in Section 3.

2. Preliminaries. The function space which provides a natural framework is the space $C_c[R_+, L_2(\Omega, A, \mu)]$ defined as follows:

Definition 2.1. $C_c[R_+, L_2(\Omega, A, \mu)]$ will denote the space of all continuous maps $x(t;\omega)$ from R_+ into $L_2(\Omega, A, \mu)$ with the topology defined by the family of semi-norms

$$\|x(t;\omega)\|_n = \sup_{0 \leqslant t \leqslant n}\Big(\int_\Omega |x(t;\omega)|^2 d\mu(\omega)\Big)^{1/2}.$$

It is known that this space is locally convex and that the topology is metrizable. For convenience, we write

$$\Big\{\int_\Omega |x(t;\omega)|^2 d\mu(\omega)\Big\}^{1/2} = \|x(t;\omega)\|_{L_2(\Omega, A, \mu)}.$$

Throughout the study of equations (1.1) and (1.2) we shall assume that $h(t;\omega) \in C_c[R_+, L_2(\Omega, A, \mu)]$ and the stochastic kernel is product measurable in all the variables, μ-essentially bounded, continuous from the set $\Delta = \{(t, \tau)\colon 0 \leqslant \tau \leqslant t < \infty\}$ into $L_\infty(\Omega, A, \mu)$. We write

$$|||k(t, \tau;\omega)||| = \mu\text{-}\operatorname*{ess\,sup}_{\omega \in \Omega} |k(t, \tau;\omega)|.$$

Definition 2.2. We shall call $x(t;\omega)$ a *random solution to equation* (1.1) if, for every fixed $t \in R_+$, $x(t;\omega) \in L_2(\Omega, A, \mu)$ and satisfies equation (1.1) μ-a.e.

THEOREM 2.1 (Schauder and Tychonoff). *Let W be a closed, bounded convex set in a Banach space, and let T be a completely continuous operator on W such that $T(W) \subset W$. Then T has at least one fixed point in W. That is, there is at least one $x^* \in W$ such that $T(x^*) = x^*$.*

3. Main results.

THEOREM 3.1. *Let the stochastic functional integral equation* (1.1) *satisfy the following conditions*:

(i) *the mapping $x(t;\omega) \to f\big(t, x_t(\omega)\big)$ is a completely continuous map from $C_c[R_+, L_2(\Omega, A, \mu)]$ into $C_c[R_+, L_2(\Omega, A, \mu)]$*;

(ii) *there exist two continuous non-negative real functions* $g(t)$ *and* $l(t)$, *defined for* $t \in R_+$, *such that*

(a) $\|f(t, x_t(\omega))\|_{L_2(\Omega, A, \mu)} \leqslant l(t)$ *whenever* $\|x(t;\omega)\|_{L_2(\Omega, A, \mu)} \leqslant g(t)$

and

(b) $$\|h(t;\omega)\|_{L_2(\Omega, A, \mu)} + \int_0^t |||k(t, \tau; \omega)||| l(\tau) d\tau \leqslant g(t), \quad t \in R_+.$$

Then there exists at least one solution $x(t;\omega)$ *of* (1.1) *in the space* $C_c[R_+, L_2(\Omega, A, \mu)]$ *such that*

$$\|x(t;\omega)\|_{L_2(\Omega, A, \mu)} \leqslant g(t), \quad t \in R_+.$$

Proof. In the space $C_c[R_+, L_2(\Omega, A, \mu)]$ consider the set

$$\Lambda = \{x(t;\omega): \|x(t;\omega)\|_{L_2(\Omega, A, \mu)} \leqslant g(t),\ t \in R_+\}.$$

It is clear that Λ is a closed convex set. For each $x(t;\omega) \in \Lambda$, we define the operator U by

$$(Ux)(t;\omega) = h(t;\omega) + \int_0^t k(t, \tau; \omega) f\big(\tau, x_\tau(\omega)\big) d\tau.$$

It is obvious that U maps Λ into $C_c[R_+, L_2(\Omega, A, \mu)]$. We show that $U\Lambda \subset \Lambda$.

Let $x(t;\omega) \in \Lambda$. Then

$$\begin{aligned}\|(Ux)(t;\omega)\|_{L_2(\Omega, A, \mu)} &\leqslant \|h(t;\omega)\|_{L_2(\Omega, A, \mu)} + \int_0^t \big\|k(t, \tau; \omega) f\big(\tau, x_\tau(\omega)\big)\big\|_{L_2(\Omega, A, \mu)} d\tau \\ &\leqslant \|h(t;\omega)\|_{L_2(\Omega, A, \mu)} + \int_0^t |||k(t, \tau; \omega)||| l(\tau) d\tau \leqslant g(t),\end{aligned}$$

by assumption (ii) (b). Thus $U\Lambda \subset \Lambda$.

From the behavior of $k(t, \tau; \omega)$ and assumption (i) it can be easily shown that U, as a map from Λ into Λ, is continuous. Furthermore, since f is completely continuous, it follows that $U\Lambda$ is relatively compact in $C_c[R_+, L_2(\Omega, A, \mu)]$. This allows us to apply the Schauder-Tychonoff fixed-point theorem to the pair (Λ, U), which proves that there exists at least one random solution to the stochastic functional integral equation (1.1) in Λ.

As an application to Theorem 3.1 we shall consider the problem of obtaining sufficient conditions for the second moments of the random solutions to be bounded.

COROLLARY 3.1. *Assume that the random functional integral equation* (1.1) *satisfies the following*:

(i) *condition* (i) *of Theorem* 3.1;

(ii) $\|h(t;\omega)\|_{L_2(\Omega,A,\mu)}$ *is bounded on* $t\in R_+$;

(iii) $\|f(t, x_t(\omega))\|_{L_2(\Omega,A,\mu)} \leqslant \lambda(t)\|x_t(\omega)\|$, *where* $\lambda(t)$ *is a positive continuous function on* R_+ *and*

$$\|x_t(\omega)\| = \sup_{0\leqslant s\leqslant t} \|x(s;\omega)\|_{L_2(\Omega,A,\mu)};$$

(iv) *there exists a real number* $m<1$ *such that*

$$\int_0^t |||k(t,\tau;\omega)|||\lambda(s)ds \leqslant m, \qquad t\in R_+.$$

Then, there exists at least one random solution of equation (1.1) *in* $C_c[R_+, L_2(\Omega, A, \mu)]$ *such that* $\|x(t;\omega)\|_{L_2(\Omega,A,\mu)} \leqslant M$ *for some real number* $M>0$.

Proof. We need to show that condition (ii) of Theorem 3.1 is satisfied. Choose $g(t) = M$ and let $l(t) = M\lambda(t)$, where M is some sufficiently large positive real number. Then (ii) (a) is trivially verified. Also,

$$\|h(t;\omega)\|_{L_2(\Omega,A,\mu)} + \int_0^t |||k(t,\tau;\omega)|||l(\tau)d\tau \leqslant \sup_{0\leqslant t} \|h(t;\omega)\|_{L_2(\Omega,A,\mu)} + mM \leqslant M$$

$$\text{if } M \geqslant (1-m)^{-1}\{\sup_{0\leqslant t}\|h(t;\omega)\|_{L_2(\Omega,A,\mu)}\}.$$

Hence, the assertion now follows from Theorem 3.1.

We shall now state a corollary concerning the exponential decay of the second moments of the random solutions of equation (1.1).

COROLLARY 3.2. *Assume that equation* (1.1) *satisfies the following*:

(i) *condition* (i) *of Theorem* 3.1;

(ii) $\|h(t;\omega)\|_{L_2(\Omega,A,\mu)} \leqslant h_0 e^{-at}$, $t\in R_+$, $h_0, a>0$;

(iii) $\|f(t, x_t(\omega))\|_{L_2(\Omega,A,\mu)} \leqslant Le^{-\beta t}\|x_t\|$, $t\in R_+$, *with* $a<\beta$, $L>0$;

(iv) $|||k(t,\tau;\omega)||| \leqslant ke^{-a(t-\tau)}$, $0\leqslant \tau\leqslant t<\infty$.

Then there exists at least one random solution to equation (1.1) *such that*

$$\|x(t;\omega)\|_{L_2(\Omega,A,\mu)} \leqslant \gamma e^{-at}, \qquad t\in R_+,$$

provided $h_0 + KL\gamma(\beta-a)^{-1} \leqslant \gamma$.

For the proof, choose $g(t) = \gamma e^{-at}$ and $l(t) = \gamma Le^{-\beta t}$. The result follows from the application of Theorem 3.1.

We now consider the stochastic functional integral equation (1.2) whose stochastic kernel of the convolution type is important in applications.

THEOREM 3.2. *Assume that equation* (1.2) *satisfies the following conditions*:

(i) $\sup\limits_{0\leqslant t} \|h(t;\omega)\|_{L_2(\Omega,A,\mu)} < M$, $M \in R_+$;

(ii) $k(t;\omega)$ *is* μ*-essentially bounded and continuous from* R_+ *into* $L_\infty(\Omega, A, \mu)$, *and such that*

$$\int_0^\infty |||k(t;\omega)|||\,dt < \infty;$$

(iii) $f\colon x \to f(t, x_t)$ *is a completely continuous map from the space* $C_c[R_+, L_2(\Omega, A, \mu)]$ *into* $C_c[R_+, L_2(\Omega, A, \mu)]$ *such that*

$\|x(t;\omega)\|_{L_2(\Omega,A,\mu)} \leqslant M + K\varphi(M)$ *implies* $\|f(t, x_t(\omega))\|_{L_2(\Omega,A,\mu)} \leqslant \varphi(M)$, $t \in R_+$,

where $K = \int_0^\infty |||k(t;\omega)|||\,dt$, *and* $\varphi(M)$ *is a positive real-valued function defined for sufficiently large* M.

Then there exists at least one random solution to equation (1.2) *in* $C_c[R_+, L_2(\Omega, A, \mu)]$ *such that* $\|x(t;\omega)\|_{L_2(\Omega,A,\mu)}$ *is bounded.*

Proof. We shall sketch the proof, since the details are similar to those of Theorem 3.1. Write

$$\Lambda = \{x(t;\omega)\colon x(t;\omega)\in C_c[R_+, L_2(\Omega, A, \mu)] \text{ and } \|x(t;\omega)\|_{L_2(\Omega,A,\mu)} \leqslant M + \sqrt{K}\varphi(M)\}$$

and

$$(Ux)(t;\omega) = h(t;\omega) + \int_0^\infty k(t-\tau;\omega) f\big(\tau, x_\tau(\omega)\big)\,d\tau.$$

In view of assumptions (ii) and (iii), it follows that U is continuous. Also, if $x(t;\omega)\in\Lambda$, then

$$\begin{aligned}\|(Ux)(t;\omega)\|_{L_2(\Omega,A,\mu)} &\leqslant \|h(t;\omega)\|_{L_2(\Omega,A,\mu)} + \Big\|\int_0^\infty k(t-\tau;\omega) f\big(\tau, x_\tau(\omega)\big)\,d\tau\Big\|_{L_2(\Omega,A,\mu)} \\ &\leqslant M + K\varphi(M),\end{aligned}$$

which implies $U\Lambda \subset \Lambda$. The rest of the proof is similar to that of Theorem 3.1.

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