

On the coefficients of bounded real univalent functions

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Abstract. Let $S_1^R(a_1)$ denote the family of all functions of the form

$$f(z) = a_1 z + a_2 z^2 + \dots, \quad a_k = \operatorname{re} a_k \quad (k = 1, 2, \dots); \quad a_1 > 0,$$

univalent and regular in the circle $K = \{z: |z| < 1\}$, which are Löwner bounded (i.e., $|f(z)| < 1, z \in K$) and whose first coefficient a_1 has been fixed.

The aim of the paper is to give the relationships which define the set of the values of the functional

$$F(f) = F(a_2, a_3, a_4)$$

by the use of the parametrical-variational method.

I. INTRODUCTION

1. Let $S_1^R(a_1)$ denote the family of all functions of the form

$$(1.1) \quad f(z) = a_1 z + a_2 z^2 + \dots, \quad a_k = \operatorname{re} a_k \quad (k = 1, 2, \dots), \quad a_1 > 0,$$

univalent and regular in the circle $K = \{z: |z| < 1\}$, which are Löwner bounded (i.e., $|f(z)| < 1, z \in K$) and whose first coefficient a_1 has been fixed. Assign to every function $f \in S_1^R(a_1)$ the number

$$(1.2) \quad F(f) = F(a_2, a_3, a_4),$$

where $F(x, y, z)$ is an arbitrary analytic complex function of three real variables defined in the region $\{(x, y, z): |x| < \infty, |y| < \infty, |z| < \infty\}$.

The paper aims at giving the relationships which define the set D of the values of functional (1.2) when f is any function of the family $S_1^R(a_1)$.

The method we shall apply is a combination of the parametric method of representing some classes of univalent functions with the variational method. Research in this field has been initiated by P. P. Kufarev [6] and is being developed by I. A. Aleksandrov [1], M. I. Redkov [8].

2. It is easily verified that the family $S_1^R(a_1)$ is compact and connected (i.e., arc-wise connected with the topology of close to uniform convergence).

Thus it follows from the continuity of functional (1.2) that the set D is compact and connected. The compactness of the set D , which is a subset of the plane, means that this set is bounded and closed. Thus to characterise the set D , which in the sequel is called the *region of values of functional* (1.2), it suffices to determine its boundary Γ . The function f^* , for which $F(f^*) \in \Gamma$, is called the *boundary function* with respect to functional (1.2). The points of the boundary Γ have the following property: for an arbitrary external point \dot{H} of the region D there exists a boundary point H_0 such that

$$(1.3) \quad |H_0 - \dot{H}| \leq |H - \dot{H}|$$

for all points of the region D which belong to a sufficiently small neighbourhood of the point H_0 . The set of points of the type H_0 , Γ' , is dense in the set Γ (cf. [3]). Closing the first of them, we get the second. Hence it follows that to characterize the region D it suffices to determine all the boundary functions which correspond to the points of the set Γ' .

II. THE DIFFERENTIAL EQUATION OF THE BOUNDARY FUNCTIONS

M. Schiffer and O. Tammi [9] have obtained a variational formula valid in the family $S_1^R(a_1)$. This formula after simple modifications and the change of markers can be written down in the form:

$$(2.1) \quad f_*(z) = f(z) + \varepsilon b \varphi(z, a) + \varepsilon b \psi(z, a) + o(\varepsilon),$$

where

$$\begin{aligned} \varphi(z, a) &= f(z) \frac{1}{\mu^2(a) - 1} \sigma[f(z), f(a)] - z f'(z) \frac{\mu^2(a)}{\mu^2(a) - 1} \sigma(z, a), \\ \psi(z, a) &= f(z) \frac{1}{\mu^2(a) - 1} \sigma \left[f(z), \frac{1}{f(a)} \right] - z f'(z) \frac{\mu^2(a)}{\mu^2(a) - 1} \sigma \left(z, \frac{1}{\bar{a}} \right) \end{aligned}$$

with

$$\sigma(x, y) \equiv \frac{x^2 - 1}{(x - y)(x - 1/\bar{y})}, \quad \mu(a) = \frac{f(a)}{a f'(a)},$$

$$\varepsilon > 0, |a| < 1, \operatorname{re} b = 0.$$

The extremal property (1.3) of the boundary points of the region D together with the variational formula enables us to deduce the differential equation of the boundary function f^* of the class $S_1^R(a_1)$ with respect to functional (1.2).

Using the typical variational method (cf. e.g. [5]), we obtain the following theorem.

THEOREM 1. Every boundary function $w = f^*(z) = a_1 z + a_2^* z^2 + a_3^* z^3 + a_4^* z^4 + \dots$ with respect to functional (1.2) defined in the family $S_1^R(a_1)$ satisfies the equation

$$(2.2) \quad [M(w) + \lambda] \left(\frac{zw'}{w} \right)^2 = N(z) + \lambda,$$

where

$$(2.3) \quad M(w) = \sum_{s=2}^4 [a_s U_\zeta^{*(s)}(0, w) + \bar{a}_s \overline{U_\zeta^{*(s)}(0, 1/\bar{w})}],$$

$$(2.4) \quad N(z) = \sum_{s=2}^4 [a_s V_\zeta^{*(s)}(0, z) + \bar{a}_s \overline{V_\zeta^{*(s)}(0, 1/\bar{z})}],$$

with

$$U^*(\zeta, w) = v\sigma(v, w), \quad V^*(\zeta, z) = \zeta v' \sigma(\zeta, z), \quad \sigma(x, y) \equiv \frac{w^2 - 1}{(x - y)(x - 1/\bar{y})},$$

$$a_2 = \beta p, \quad a_3 = \beta q, \quad a_4 = \beta r,$$

$$p = \frac{\partial F}{\partial a_2} \Big|_{(a_2^*, a_3^*, a_4^*)}, \quad q = \frac{\partial F}{\partial a_3} \Big|_{(a_2^*, a_3^*, a_4^*)}, \quad r = \frac{\partial F}{\partial a_4} \Big|_{(a_2^*, a_3^*, a_4^*)},$$

$$v = f^*(\zeta), \quad v' = f^{*'}(\zeta),$$

$\beta, |\beta| = 1$, and $\lambda, \bar{\lambda} = \text{re } \lambda$, are unknown parameters.

III. THE PROPERTIES OF THE EQUATION OF THE BOUNDARY FUNCTIONS AND OF ITS SOLUTIONS

1. The functions $M(w)$ and $N(z)$ appearing in equation (2.2) are rational functions of their arguments and therefore the function $w = f^*(z)$ maps the circle K onto a region which is contained in the circle $R = \{w: |w| < 1\}$ whose boundary is interval-wise analytic [4].

Proceeding in the usual way we find that the function $f^*(z)$ maps the circle K onto a region which is obtained from the circle R by removing a finite number of analytic arcs one of which at least passes through a point lying on the circumference $\delta = \{w: |w| = 1\}$.

2. Denote by B_{t_0} the region onto which the boundary function $f^*(z)$ maps the circle K and by L_{t_0} the complement of the set of points of the circumference δ to the set G_{t_0} which is the boundary of the region B_{t_0} . The set L_{t_0} is the sum of analytic arcs $L_{t_0}^{(1)}, \dots, L_{t_0}^{(p)}$ ($1 \leq p < \infty$) with disjoint interiors. The point $\tilde{w}, \tilde{w} \in \delta$ will be called the end-point of each of the arcs $L_{t_0}^{(k)}$ ($1 \leq k \leq p$) which contain this point. If $w = w(t^*), a \leq t^* \leq b$ is the equation of any of the arcs, then the point $w(t^*), a < t^* < b$ does not belong to any of the remaining arcs. If the point $w(a)$ or $w(b)$ belongs to two or more arcs, it will be called the end-point of each of them.

If $w(a) \neq \bar{w}$ ($w(b) \neq \bar{w}$) and $w(a)$ ($w(b)$) belongs to one arc only, then $w(a)$ ($w(b)$) will be called the origin of this arc.

3. Observe that (2.3) and (2.4) imply the identities

$$(3.1) \quad M(w) = \overline{M\left(\frac{1}{\bar{w}}\right)}, \quad N(z) = \overline{N\left(\frac{1}{\bar{z}}\right)}.$$

Thus the functions $M(w)$ and $N(z)$ are real on the circumferences $|w| = 1$, $|z| = 1$, respectively. If the number u is a root of one of these functions, then the number $1/\bar{u}$ is also its root.

IV. THE PARAMETRIC FORM OF THE EQUATION OF BOUNDARY FUNCTIONS

Let $w = f(z)$ be a function of the class $S_1^R(a_1)$ which maps the circle K onto the region H obtained by removing a Jordan arc J with the equation $w = w(t^*)$, $0 \leq t^* \leq t_0$, $w(0) \in \delta$, which does not pass through the point $w = 0$, from the circle R . For fixed t , $0 \leq t \leq t_0$, consider the region H_t obtained by removing the arc $J_t \subset J$ with the equation $w = w(t^*)$, $0 \leq t^* \leq t$, from the circle R . Denote by $g(z, t)$ a function of the class $S_1^R(a_1)$ which maps the circle K onto the region H_t with $g'_z(0, t) = e^{-t}$ [7]. Further, let $h(w, t)$ denote the converse function of the function $w = g(z, t)$. With the above notation the following relationships hold:

$$(4.1) \quad g(z, 0) = z, \quad g(z, t_0) = f(z), \quad h(w, 0) = w, \quad h[f(z), t_0] = z, \\ h'_w[f(z), t] = [g'_z(z, t)^{-1}], \quad h'_w[f(z), t_0] = \frac{1}{f'(z)}, \quad h'_w(0, t) = e^t.$$

Let the set L_{t_0} consist of one arc. Thus in the above argument we may take the boundary function $f^*(z)$ for $f(z)$, we find that $g(z, t_0) = f^*(z)$, which by the fact that $f(z) \in S_1^R(a_1)$ yields the equality $t_0 = \log 1/a_1$.

We apply the variational formula (2.1) to the function $g(z, t)$, and we get a function $g_*(z, t)$, in which we substitute $h(w, t)$ for z , which gives the formula for the function $g_*[h(w, t), t]$. Putting $w = f(z)$, we get the variation of the function $\hat{f}_*(z) = g_*[h(f(z), t), t]$.

The variational formula thus obtained together with inequality (1.3) enables us to deduce the differential equation of the function $h(w, t)$ converse to $g(z, t)$ with $g(z, t_0) = f^*(z)$.

THEOREM 2. *The function $h = h(w, t)$ with an arbitrary t , $0 \leq t \leq \log \frac{1}{a_1}$, satisfies the equation*

$$(4.2) \quad M(w) + \lambda = \left(\frac{wh'}{h}\right)^2 [N(h, t) + \lambda],$$

where

$$(4.3) \quad N(h, t) = \sum_{s=2}^4 \left[\alpha_s T_\zeta^{(s)}(0, h, t) + \overline{\alpha_s} T\left(0, \frac{1}{h}, t\right) \right],$$

with

$$T(\zeta, h, t) = \frac{h[f^*(\zeta), t]}{h'_w[f^*(\zeta), t]} \sigma[h(f^*(\zeta), t), t],$$

$M(w)$, α_s , $\sigma(x, y)$ are the same as in theorem 1 and λ is a real parameter.

Finding the derivatives of the functions $U(\zeta, w)$, $T(\zeta, h, t)$, with respect to ζ at the $\zeta = 0$ (up to the fourth order inclusively) and making use of the relationships

$$(4.4) \quad f^*(0) = 0, \quad f^{*'}(0) = a_1, \quad h[f^*(0), t] = 0, \quad h'_w(0, t) = e^t,$$

we obtain by (2.3) and (4.3)

$$(4.5) \quad M(w) = \bar{d}w^3 + \bar{c}w^2 + \bar{b}w + a + b \frac{1}{w} + c \frac{1}{w^2} + d \frac{1}{w^3},$$

$$(4.6) \quad N(h, t) = \bar{n}h^3 + \bar{m}h^2 + \bar{l}h + k + 1 \frac{1}{h} + m \frac{1}{h^2} + n \frac{1}{h^3},$$

where

$$(4.7) \quad \begin{aligned} d &= 48\beta r a_1^4, & c &= 12\beta a_1^2(a_1 q + 6r v''), \\ b &= 4\beta [a_1^2 p + 3a_1 q v'' + r(4a_1 v''' + 3(v'')^2)], \\ a &= 2 \operatorname{re} \beta (p v'' + q v''' + r v^{IV}), \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} n &= \bar{d} e^{3t}, & m &= \bar{c} e^{2t} + 24a_1^4 e^t h''_w(0, t), \\ 1 &= \bar{b} e^t + 12a_1^4 e^{-t} [h''_w(0, t)]^2 - 8a_1^4 h''_w(0, t), \\ k &= 2 \operatorname{re} \beta [p S'_\zeta(0, t) + q S''_\zeta(0, t) + r S^{IV}_\zeta(0, t)] + \mu(t); & \operatorname{re} \mu(t) &= \mu(t) \end{aligned}$$

with

$$v^{(k)} = f^{*(k)}(0), \quad k = 2, 3, 4, \quad S(\zeta, t) = \frac{h[f^*(\zeta), t]}{h'_w[f^*(\zeta), t]}.$$

Remark. One obtains analogous conclusions if the set L_{t_0} is the sum of an arbitrary finite number of arcs.

V. BASIC THEOREM

Consider an arbitrary arc of the set J_t which has an origin. Denote this point by w_t . The point $\varkappa(t) = h(w_t, t)$ on the circumference $|z| = 1$ corresponds to w_t . $\varkappa(t)$ is a two-fold root of the function $N(h, t)$ and thus by (4.6) analytic arcs which form the set L_{t_0} can have at most three origins.

Consequently the set L_{t_0} consists of at most three analytic arcs which have their end-points on the circumference δ .

Simultaneously a non-empty set W of points which are the origins of the arcs belonging to L_{t_0} is assigned to L_{t_0} . The following cases are possible.

I. The set W has one element, then the set L_{t_0} consists of one arc C_1 (Fig. 1).

II. W consists of two points; then L_{t_0} consists of:

a) two disjoint arcs C_1, C_2 (Fig. 2a) or

b) three converging arcs $\hat{C}_1, \hat{C}_2, \hat{C}_3$ (Fig. 2b) (i.e., arcs whose common end-point is some point $\hat{w}, \hat{w} \in R$).

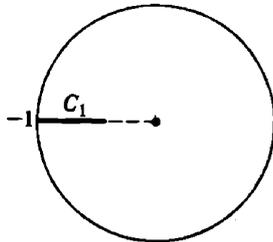


Fig. 1

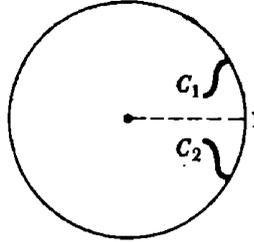


Fig. 2a

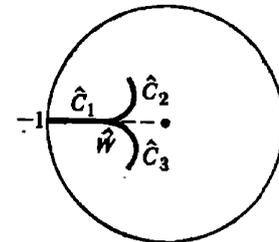


Fig. 2b

III. The set W contains three elements, then the following cases are possible:

a) L_{t_0} is the sum of three disjoint arcs C_1, C_2, C_3 (Fig. 3a);

b) L_{t_0} is the sum of disjoint elements: an arc C_1 and a curve C , which in its turn is the sum of three converging arcs (Fig. 3b);

c) L_{t_0} consists of four converging arcs $\tilde{C}_k, k = 1, \dots, 4$ (Fig. 3c).

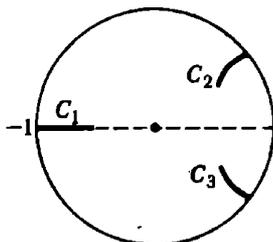


Fig. 3a

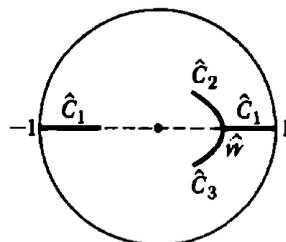


Fig. 3b

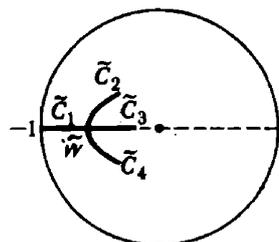


Fig. 3c

Now we shall consider all the above cases.

I. In the first case the function $N(h, t)$ may be represented in the form

$$(5.1) \quad N(h, t) = \frac{\bar{n}(h - \kappa(t))^2(h - d_1)(h - d_2) \left(h - \frac{1}{d_1}\right) \left(h - \frac{1}{d_2}\right)}{h^3}$$

Equating the coefficients at h^k ($k = -3, -2, -1, 1, 2$) in (5.1) and in (4.6), we get a system of equations which results in the following equation with the unknown function $\kappa = \kappa(t)$

$$(5.2) \quad 3\bar{n}\kappa^6 + 2\bar{m}\kappa^5 + \bar{l}\kappa^4 - 1\kappa^2 - 2m\kappa - 3n = 0.$$

In the case under consideration the function $\kappa(t)$ is continuous in the interval $0 \leq t \leq \log 1/a_1$ (cf. [7]).

IIa. The function $N(h, t)$ may now be written in the form

$$(5.3) \quad N(h, t) = \frac{\bar{n}(h - \kappa_1(t))^2 (h - \kappa_2(t))^2 (h - d_3) \left(h - \frac{1}{d_3}\right)}{h^3}.$$

Thus the function $\kappa(t)$, $0 \leq t \leq \log 1/a_1$, has one discontinuity point of the first kind $t = s$, which corresponds to the origin of one of the arcs C_1, C_2 and the end-point of the other of them. Thus the function $\kappa(t)$ has the form

$$(5.4) \quad \kappa(t) = \begin{cases} \kappa_1(t), & 0 \leq t \leq s, \\ \kappa_2(t), & s < t \leq \log \frac{1}{a_1}. \end{cases}$$

Equating the coefficients in (5.3) and (4.6) and eliminating the parameter d_3 , we get the system of equations

$$(5.5) \quad \begin{aligned} 2\bar{n}\kappa_1\kappa_2(\kappa_1 + \kappa_2) + \bar{m}\kappa_1\kappa_2 - \bar{m}\overline{\kappa_1\kappa_2} - 2n(\overline{\kappa_1\kappa_2})^2(\kappa_1 + \kappa_2) &= 0, \\ \bar{n}(\kappa_1 + \kappa_2)^2\kappa_1\kappa_2 + \bar{n}(\kappa_1\kappa_2)^2 + 1 - \bar{l}\kappa_1\kappa_2 - n(\kappa_1 + \kappa_2)^2(\kappa_1\kappa_2)^2 - n\kappa_1\kappa_2 &= 0 \end{aligned}$$

with $\kappa_1 = \kappa_1(t)$, $\kappa_2 = \kappa_2(t)$ from which the function $\kappa(t)$ can be found.

IIb. Let three arcs converge at the point $\hat{w} = w(s_1)$, $0 < s_1 < t_0$.

Thus, if $t \in [0, s_1]$, then the function $N(h, t)$ is of the form (5.1), while if $t \in (s_1, \log 1/a_1]$, it is of the form (5.3).

Thus the function $\kappa(t)$ shall be determined by equation (5.2) for $t \in [0, s_1]$, while for $t \in (s_1, \log 1/a_1]$ it will be of form (5.4), $\kappa_1(t)$, $\kappa_2(t)$ being the solutions of the system of equations (5.5).

IIIa. In this case we have

$$(5.6) \quad N(h, t) = \frac{\bar{n}(h - \kappa_1(t))^2 (h - \kappa_2(t))^2 (h - \kappa_3(t))^2}{h^3}$$

and

$$\kappa(t) = \begin{cases} \kappa_1(t), & 0 \leq t \leq s_1, \\ \kappa_2(t), & s_1 < t \leq s_2, \\ \kappa_3(t), & s_2 < t \leq \log \frac{1}{a_1}, \end{cases}$$

s_1, s_2 being the corresponding discontinuity points of the function $\kappa(t)$.

The functions $\kappa_j = \kappa_j(t)$ ($j = 1, 2, 3$) satisfy the system of equations

$$(5.7) \quad \begin{aligned} \kappa_1^2 \kappa_2^2 \kappa_3^2 &= \frac{n}{\bar{n}}, \\ 2(\kappa_1 + \kappa_2 + \kappa_3) &= -\frac{\bar{m}}{\bar{n}}, \\ 2\kappa_1 \kappa_2 \kappa_3 (\kappa_1 \kappa_2 + \kappa_1 \kappa_3 + \kappa_2 \kappa_3) &= -\frac{m}{\bar{n}}. \end{aligned}$$

The above system is obtained by equating the coefficients in (5.6) and (4.6) appearing at h^{-4} , h^{-2} , h^2 .

IIIb. If $\hat{w} = w(s_3)$, $0 < s_3 < \log 1/a_1$, denotes the point at which the arcs \hat{C}_k , $k = 1, 2, 3$, converge, then for $t \in [0, s_3]$ the function $N(h, t)$ is of form (5.1). On the other hand, if $t \in (s_3, s_4]$ ($t = s_4$ corresponds to the origin of one of the arcs \hat{C}_2, \hat{C}_3 and to the end-point of the arc C_1), then $N(h, t)$ is of form (5.3) and for $s_4 < t \leq \log 1/a_1$ again it is of form (5.1). The function $\kappa(t)$ will be defined by equation (5.2) or it will be of form (5.4) accordingly as $0 \leq t \leq s_3$, $s_4 < t \leq \log 1/a_1$, or $s_3 < t \leq s_4$.

IIIc. If $\tilde{w} = w(s_5)$, $0 < s_5 < \log 1/a_1$, is a point at which the arcs \tilde{C}_k , $k = 1, \dots, 4$, converge, then the function $\kappa(t)$ is defined by equation (5.2) for $t \in [0, s_5]$ and it is determined by the functions $\kappa_j(t)$ ($j = 1, 2, 3$), which satisfy system (5.7) for $t \in (s_5, \log 1/a_1]$.

Observe that in case IIa the function

$$\kappa(t) = \begin{cases} \kappa_2(t), & 0 \leq t \leq s, \\ \kappa_1(t), & s < t \leq \log \frac{1}{a_1}, \end{cases}$$

can be taken into account, to which, however, the same boundary function corresponds as to function (5.4) (the same boundary function may correspond to various functions $\kappa(t)$).

It is known that to characterize the region D of values of functional (1.2) defined in the family $S_1^R(a_1)$ it suffices to determine its boundary. This aim can be achieved with the aid of the present results and of those which follow from the Löwner theory.

Let $S_1^{*R}(a_1)$ denote a subclass of functions $f(z)$ which belong to the family $S_1^R(a_1)$ and map the circle K onto the region obtained by removing from the circle R a finite number of analytic arcs at least one of which has its end-point on the circumference δ . (The class $S_1^{*R}(a_1)$ is not empty, because $f^* \in S_1^{*R}(a_1)$.) The family $S_1^{*R}(a_1)$ is a dense subclass of the class $S_1^R(a_1)$ [7]; thus to characterize the region D it suffices to determine its boundary in the case where $f \in S_1^{*R}(a_1)$.

If $f^* \in S_1^{*R}(a_1)$, then it follows from the Löwner theory that the coefficients of the function $f^*(z)$ have parametrical representations: namely, they are expressed by some integrals of the function $\kappa = \kappa(t)$. It is easily verified that in the case of the family $S_1^{*R}(a_1)$ we have the following relationships (one has to use Löwner's equation about function with real coefficients, which were given by Bazilewic in [2]):

$$(5.8) \quad a_2^* = 2a_1^2 J_1,$$

$$(5.9) \quad a_3^* = 2a_1^3(2J_1^2 - J_2),$$

$$(5.10) \quad a_4^* = a_1^4(-8J_1^3 + 12J_1J_2 - 2J_3 - P),$$

where

$$J_1 = \int_0^{t_0} d(t)e^t dt, \quad J_2 = \int_0^{t_0} [2d^2(t) - 1]e^{2t} dt, \quad J_3 = \int_0^{t_0} [4d^3(t) - 3d(t)]e^{3t} dt,$$

and

$$P = \int_0^{t_0} [2d^2(t) - 1]X(t)e^{2t} dt$$

with

$$X(t) = 4e^t \int d(t)e^t dt, \quad X(0) = 0; \quad d(t) = \operatorname{re} \kappa(t), \quad t_0 = \log \frac{1}{a_1}.$$

The above considerations imply the following basic result of the paper.

THEOREM 3. *The boundary of the region of values of functional (1.2) defined in the family $S_1^R(a_1)$ is determined by the equation*

$$\Omega = F(a_2^*, a_3^*, a_4^*),$$

where a_2^*, a_3^*, a_4^* are defined by relationships (5.8)–(5.10). The function $\kappa(t)$, $|\kappa(t)| = 1$, is a solution of equation (5.2) or it is determined by the solution of the system of equations (5.5) or (5.7) or it is determined by solving simultaneously the equation and one of the systems.

Remark. When using the parametric-variational method here presented, we easily obtain the results concerning the functional

$$F(f) = a_4, \quad f \in S_1^R(a_1)$$

which were given by M. Schiffer and O. Tammi in [9].

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