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# On the radius of convergence of an entire function in a normed space

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**Abstract.** The *radius of boundedness* $R(x)$ of a numerical function $u$ defined in a normed space $E$, at a point $x$ in $E$, is the least upper bound of all numbers $r$ such that $u$ is bounded above in $x+rB$, where $B$ denotes the unit ball in $E$. If $f$ is entire, the radius of boundedness of $u = \log|f|$ is called the *radius of convergence* of $f$. We have

(A) $$|R(x)-R(y)| < \|x-y\|,$$

and if $u$ is plurisubharmonic,

(B) $$-\log R \text{ is plurisubharmonic.}$$

It is shown that there exist for $E = l^p$, $1 < p < +\infty$, functions $R\colon E \to ]0, 1]$ satisfying (A) and (B) such that $R$ is not the radius of boundedness of any plurisubharmonic function in $E$, a fortiori not the radius of convergence of any entire function. It is conjectured, on the other hand, that any function $R$ satisfying (A) and (B) in $E = l^1$ is the radius of convergence of some entire function, and partial results which support this conjecture are given.

Let $T$ be a topology on $E$. The *$T$-local radius of boundedness* $R_T(x)$ of $u$ is the supremum of all numbers $r$ such that $u$ is bounded above in $(x+rB)\cap W$ for some $T$-neighbourhood $W$ of $x$. It is shown that, for reasonable $T$, $-\log R_T$ is also plurisubharmonic when $u$ is plurisubharmonic. The above-mentioned results rely on an analysis of the relation between $R_T$ and $R$ for $T = \sigma(E, E')$, the weak topology.

**1. Introduction.** Let $E$ be a complex normed space and $f\colon E \to C$ an entire function (i.e. $f$ is continuous and its restriction to any finite-dimensional subspace of $E$ is entire). The *radius of convergence* $R(x)$ of $f$ at $x \in E$ is the least upper bound of all numbers $r$ such that the Taylor expansion of $f$ at $x$ converges uniformly in the ball of radius $r$ and center $x$. It is well known that $R$ has the following two properties:

(1.1) *$R$ is Lipschitz continuous with Lipschitz constant at most one*:

$$|R(y)-R(x)| \leqslant \|y-x\|;$$

(1.2) $$-\log R \textit{ is plurisubharmonic.}$$

It is natural to ask if a radius of convergence has other properties. Josefson [4] has shown that (1.1) and (1.2) are not sufficient for $R$ to be the radius of convergence of some entire function in $c_0$. In analogy with this we shall give in Propositions 3.9 and 3.11 a necessary condition (relating the behavior of $R$ at an arbitrary point to its decay at infinity) for $R$ to be the radius of boundedness of some plurisubharmonic function in $l^p$, $1 < p < +\infty$, or $c_0$. (For $c_0$ we get Josefson's condition.) Here the *radius of boundedness* of a function $u$: $E \to [-\infty, +\infty[$ at a point $x$ in $E$ is defined as the supremum of all $r$ such that $u$ is bounded above in $\{y;\ \|y-x\| \leqslant r\}$; it is well known that if $f$ is entire, then the radius of convergence of $f$ and the radius of boundedness of $\log |f|$ coincide (see Nachbin [7], p. 26).

Conversely one may try to construct an entire function with given radius of convergence. Simple examples like $f(x) = \Sigma x_k^k$, $x \in c_0$, show that the radius may well be finite (in the case mentioned it is equal to 1 everywhere). More generally, Dineen [2], Proposition 5, has shown that in any complex Banach space $E$ such that there is a sequence in the dual space which tends to zero in the weak star topology but not in norm, there is an entire function on $E$ whose radius of convergence is finite. By refining the construction of [2], Aron [1] proves that there exists an entire function on $E$ such that its radius of convergence becomes arbitrarily small in the unit ball. A theorem of Josefson [5] shows that the hypothesis in [2] and [1] concerning the dual is always satisfied in an infinite-dimensional Banach space.

In Section 4 of this paper we shall consider the problem of constructing an entire function with given radius of convergence. All evidence so far supports the conjecture that on $E = l^1$ any function $R$: $l^1 \to ]0, +\infty[$ satisfying (1.1) and (1.2) is the radius of convergence of some entire function. Theorem 4.4 gives a partial result in that direction: if $R$ depends only on finitely many coordinates in $l^1$ and satisfies (1.1) and (1.2), then it is a radius of convergence. In the other coordinate spaces $l^p$, $1 < p < +\infty$, and $c_0$, the problem can be solved approximately in the sense that if $R$ depends on finitely many variables and satisfies (1.1) and (1.2), then there is a radius of convergence $R_1$ such that

$$2^{-1+1/p} R \leqslant R_1 \leqslant R.$$

As to the methods used in this paper we only remark here that the crucial concept is that of the *local radius of boundedness*. We give its definition in Section 2 and its relation to the radius of boundedness in Theorem 3.8. The *inner and outer moduli* which measure the unit ball locally with respect to some (weak) topology are introduced in Section 3.

**2. The local radius of boundedness.** Let $E$ be a normed space, $\tau$ a topology on $E$ and $u$: $E \to [-\infty, +\infty[$ a numerical function on $E$. The *$\tau$-local* *radius of boundedness of $u$ at a point $x$* in $E$ is the least upper bound of all numbers $r$ such that $u$ is bounded above in $(x+rB)\cap W$ for some $\tau$-neighbourhood $W$ of $x$. Here $B$ is the closed unit ball of $E$, so that, with the usual conventions concerning operations in vector spaces, $x+rB$ is the ball of radius $r$ centered at $x$. We shall write $R_{\tau,u}(x)$ for the number just defined; when $\tau$ is the chaotic topology on $E$ we get the *radius of boundedness* and we shall then write $R_u(x)$. If $u$ is locally bounded above for the norm topology we have $0 < R_{\tau,u} \leqslant +\infty$; if $\tau$ is weaker than the norm topology, $R_{\tau,u}$ is clearly lower semicontinuous with respect to the latter.

We shall investigate the geometric relation between $R_u$, that is $R_{\tau,u}$ for $\tau$ chaotic, and $R_{\tau,u}$ for some other topology $\tau$. If $E$ is reflexive the weak topology $\tau = \sigma(E, E')$ will be of interest; if $E$ is the dual of $E_1$ the weak star topology $\tau = \sigma(E, E_1)$ may be used instead. The important property is that the unit ball in $E$ is $\tau$-compact in these cases. In any space we may use $\tau = \sigma(E, G^{\perp})$, where $G$ is a subspace of $E$ of finite codimension; hence $B$ is $\tau$-quasi-compact ($\tau$ is non-separated in all cases of interest).

A numerical function $u$ defined in a subset $\Omega$ of a complex vector space $E$ will be called *submedian* if

$$u(x) \leqslant \frac{1}{2\pi} \int_{0*}^{2\pi} u(x+e^{i\theta}y)\,d\theta \tag{2.1}$$

for all $x, y \in E$ such that $x+ty \in \Omega$ for all complex numbers $t$ with $|t| \leqslant 1$ Here $\int_*$ denotes the Lebesgue lower integral. When $E$ is normed, and $\Omega$ open in $E$, $u\colon \Omega \to [-\infty, +\infty[$ will be called *plurisubharmonic*, in symbols $u \in PSH(\Omega)$, if $u$ is submedian and upper semicontinuous with respect to the norm topology (the integral in (2.1) is then equal to the greatest lower bound of $\frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)\,d\theta$, where $\varphi$ is continuous and $\varphi(\theta) \geqslant u(x+e^{i\theta}y)$). A *holomorphic* function is by definition a continuous function $f\colon \Omega \to C$, $\Omega$ being open in $E$, such that its restriction to $\Omega \cap F$ is holomorphic in the usual sense whenever $F$ is a finite-dimensional subspace of $E$; the class of all such functions will be denoted by $\mathcal{O}(\Omega)$. If $f \in \mathcal{O}(\Omega)$, then $\log|f| \in PSH(\Omega)$. The notion of a pseudoconvex open set is extended in the same way: an open set $\Omega$ in a complex normed space $E$ is called *pseudoconvex* if $\Omega$ cuts every finite-dimensional subspace $F$ in a pseudoconvex set in $F$.

Lelong [6], p. 176, has proved that $-\log R_u$ is plurisubharmonic if $u$ is. We generalize his result as follows.

THEOREM 2.1. *Let $E$ be a complex normed space, $\tau$ a vector space topology on $E$ which is weaker than the norm topology, and $u$ a plurisubharmonic function on $E$. Then $-\log R_{\tau,u}$ is plurisubharmonic.*

The theorem is a consequence of the following more precise result.

PROPOSITION 2.2. *With $E$, $\tau$ and $u$ as in Theorem 2.1, define*

$$u_W(x, t) = \sup_y \left(u(x+y);\ y \in tW,\ \|y\| \leqslant |t|\right), \qquad x \in E,\ t \in C,$$

*where $W$ is a $\tau$-neighbourhood of the origin. Then*

$$u_\tau = \inf_W u_W = \lim_W u_W$$

*is plurisubharmonic in the interior $\Omega$ of the set where it is less than $+\infty$. Moreover, $\Omega$ is pseudoconvex.*

Proof. Consider the analytic map $\varphi_y\colon E \times C \to E$ defined by $\varphi_y(x, t) = x + ty$. The function $u \circ \varphi_y$ is plurisubharmonic in $E \times C$ so the upper regularization $u_W^*$ of

$$u_W = \sup_{y \in W \cap B} u \circ \varphi_y$$

is plurisubharmonic in the open set $\Omega_W$ where the family $u \circ \varphi_y$, $y \in W \cap B$, is locally bounded above. We claim that $u_\tau = \inf u_W = \inf u_W^*$ in $\Omega$ so that in particular $u_\tau$ is upper semicontinuous there. Let $(x_0, t_0)$ be an arbitrary point in $\Omega$, a set which is open by definition. Pick $\delta > 0$ such that $(x_0, |t_0| + \delta) \in \Omega$. The logarithmic convexity of $u_\tau$ in the variable $|t|$ implies that

$$u_\tau(x_0, t) \leqslant C_0 + C_1 \log |t| \quad \text{if} \quad |t_0| \leqslant |t| \leqslant |t_0| + \delta,$$

where the constants are chosen to yield equality when $|t| = |t_0|$ and when $|t| = |t_0| + \delta$. Hence $u_\tau(x_0, t) < a = u_\tau(x_0, t_0) + \varepsilon$ when $|t| \leqslant |t_0| + \delta_1$ for some $\delta_1 > 0$, $\varepsilon$ being given. By the definition of $u_\tau$ there exists a $\tau$-neighbourhood $W$ of the origin such that

$$u(x_0 + y) < a \quad \text{when } y \in (|t_0| + \delta_1)(W \cap B).$$

Hence

$$u(x + y) < a \quad \text{when } y \in t(W_1 \cap B) \tag{2.2}$$

provided $\|x - x_0\| \leqslant \delta_2$, $|t - t_0| \leqslant \delta_2$ and $\delta_2$ is so small that $2\delta_2 \leqslant \delta_1$ and $tW_1 + \delta_2 B \subset (|t_0| + \delta_1)W$ for all $t$ with $|t - t_0| \leqslant \delta_2$, $W_1$ being a suitable $\tau$-neighbourhood of the origin. It is possible to achieve this if $\tau$ is a vector space topology weaker than the norm topology. Now (2.2) implies that $u_{W_1}(x, t) \leqslant a$ when $\|x - x_0\| \leqslant \delta_2$ and $|t - t_0| \leqslant \delta_2$, hence $u_{W_1}^*(x_0, t_0) \leqslant a = u_\tau(x_0, t_0) + \varepsilon$. This proves that $u_\tau = \inf u_W^*$ and hence $u_\tau$ is an upper semicontinuous function. To see that $u_\tau$ is also submedian, consider elements $x, y$ of $E$ and complex numbers $t, s$ such that $(x + \zeta y, t + \zeta s) \in \Omega$ for all $\zeta$ with $|\zeta| \leqslant 1$. By a compactness argument there is a $\tau$-neighbourhood $W$ of the origin such that $(x + \zeta y, t + \zeta s) \in \Omega_W$ when $|\zeta| \leqslant 1$. Clearly, there is a denumerable family of $\tau$-neighbourhoods $W_j$ such that $W_{j+1} \subset W_j \subset W$ and

$$\inf_j u_{W_j}^*(x + e^{i\theta} y, t + e^{i\theta} s) = u_\tau(x + e^{i\theta} y, t + e^{i\theta} s)$$

for all $\theta \in [0, 2\pi]$, $x, y, t$ and $s$ being fixed. From this we get by monotone convergence

$$\inf_j \frac{1}{2\pi} \int_0^{2\pi} u^*_{W_j}(x + e^{i\theta} y, t + e^{i\theta} s)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \inf_j u^*_{W_j}(x + e^{i\theta} y, t + e^{i\theta} s)\, d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} u_\tau(x + e^{i\theta} y, t + e^{i\theta} s)\, d\theta.$$

On the other hand, the fact that $u^*_{W_j}$ is plurisubharmonic in $\Omega_W$ shows that

$$u_\tau(x, t) \leqslant \inf_j u^*_{W_j}(x, t) \leqslant \inf_j \frac{1}{2\pi} \int_0^{2\pi} u^*_{W_j}(x + e^{i\theta} y, t + e^{i\theta} s)\, d\theta$$

so combining these formulas we obtain that $u_\tau$ is submedian.

Finally, let

$$\Omega_{k,y} = \{(x, t) \in E \times C;\ u(x + ty) < k\},$$

a pseudoconvex open set in $E \times C$. Put

$$\Omega_{W,k} = \Big(\bigcap_{y \in W \cap B} \Omega_{k,y}\Big)^\circ$$

and

$$\Omega' = \bigcup_{W,k} \Omega_{W,k}.$$

It is well known that these operations preserve pseudoconvexity in a finite-dimensional space, and the proof extends to the case considered here. It is easily verified that $\Omega = \Omega'$ so Proposition 2.2 is proved.

Remark. It is the source of some difficulty that $u_\tau$ need not tend to $+\infty$ at the boundary of $\Omega$. In fact, putting

$$u(x) = \sup_{j \geqslant 1} j \log |x_j|, \qquad x \in c_0,$$

we have if $\tau$ is the chaotic topology $\{\emptyset, c_0\}$,

$$u_\tau(0, t) = \begin{cases} \log|t| & \text{if} \quad |t| \leqslant 1, \\ +\infty & \text{if} \quad |t| > 1, \end{cases}$$

and

$$\Omega = \{(x, t) \in c_0 \times C;\ |t| < 1\}$$

so that $(0, 1) \notin \Omega$ even though $u_\tau(0, 1) = 0$.

Proof of Theorem 2.1. With $\Omega$ as in the statement of Proposition 2.2 we know that $-\log d\big((x, t), \complement\Omega\big)$ is plurisubharmonic in $\Omega$, where $d\big((x, t), \complement\Omega\big)$ is the distance from $(x, t)$ to the complement of $\Omega$ measured along the direction $(0, 1)$. It is clear that if $(x, t) \in \Omega$, then $u_\tau(x, t) < +\infty$ and $R_{\tau,u}(x) \geqslant |t|$; on the other hand, if $R_{\tau,u}(x) > |t|$, then $u_\tau(x, t) < +\infty$ and the first part of the proof of Proposition 2.2 shows that $(x, s) \in \Omega$ for all $s$ with $|s| < |t|$ so that $d\big((x, 0), \complement\Omega\big) \geqslant |t|$. Thus $d\big((x, 0), \complement\Omega\big) = R_{\tau,u}(x)$ which proves the theorem.

**3. Necessary conditions.** Let as before $E$ be a normed space (for the moment real or complex) and let $B$ be its unit ball. If $\tau$ is a topology on $E$, $A_1$ and $A_2$ two subsets of $E$, and $x$ a point in $E$, we shall say that $A_1$ is *$\tau$-locally contained in $A_2$ at $x$*, in symbols $A_1 \subset_{\tau,x} A_2$, if

$$A_1 \cap W \subset A_2$$

for some $\tau$-neighbourhood $W$ of $x$; in other words, if the inclusion relation holds between the $\tau$-germs of $A_1$ and $A_2$ at $x$. To measure how the unit ball tapers off we introduce the *inner* and *outer moduli of $E$*, written $m(x)$ and $M(x)$, respectively. To define them, consider positive numbers $m$ and $M$ such that

$$x + mB \subset_{\tau,x} B, \tag{3.1}$$

and

$$B \subset_{\tau,x} x + MB. \tag{3.2}$$

Then $m(x)$ is the least upper bound of all $m$ such that (3.1) holds, and $M(x)$ is the greatest lower bound of all $M$ such that (3.2) holds.

If all balls $x + \varepsilon B$ are $\tau$-neighbourhoods of $x$ we clearly have $m(x) = +\infty$ and $M(x) = 0$ for $\|x\| < 1$, so this case is without interest. If, on the other hand, every $\tau$-neighbourhood of $x$ contains a line through $x$, such as with the various weak topologies one considers in an infinite-dimensional space, then

$$1 - \|x\| \leqslant m(x) \leqslant M(x) \leqslant 1 + \|x\| \quad \text{for } \|x\| < 1. \tag{3.3}$$

In general we cannot improve these inequalities:

EXAMPLE 3.1. If $E = L^1(]0, 1[)$, then

$$m(x) = 1 - \|x\|, \quad M(x) = 1 + \|x\| \quad \text{for } \|x\| < 1,$$

where the moduli are taken with respect to the weak topology $\sigma(E, E')$.

EXAMPLE 3.2. If $E = l^\infty(J)$, the space of all bounded functions on an arbitrary infinite set $J$, then

$$m(x) = 1 - \limsup |x_j|, \quad M(x) = 1 + \limsup |x_j| \quad \text{for } \|x\| < 1.$$

Here we may use the weak topology $\sigma(E, E')$ or the even weaker topology $\sigma(E, \oplus C)$ of pointwise convergence (or any between these). The limits are with respect to the Fréchet filter on $J$.

EXAMPLE 3.3. Let $J$ be an arbitrary infinite index set, and $E = l^p(J)$, $1 \leqslant p < +\infty$, or $E = c_0(J)$. Then

$$(3.4) \qquad m(x) = M(x) = (1-\|x\|^p)^{1/p} \quad \text{for } \|x\| < 1,\ 1 \leqslant p < +\infty,$$

and

$$(3.5) \qquad m(x) = M(x) = 1 \quad \text{for } \|x\| < 1,\ E = c_0(J).$$

(The case $p = 2$ covers all infinite-dimensional Hilbert spaces.) Here, again, we may use any topology weaker than $\sigma(E, E')$ but stronger than that of coordinatewise convergence. For $p > 1$ they agree on bounded sets so there is in fact no choice; for $p = 1$ we may take $\tau = \sigma(l^1, c_0)$ or $\tau = \sigma(l^1, l^\infty)$.

LEMMA 3.4. *If $\tau$ is a locally convex vector space topology on $E$, then the inner modulus $m$ with respect to $\tau$ is concave on $B$ and $m(tx)$ is a decreasing function of $t \geqslant 0$.*

Proof. Let $x_0, x_1$ be elements of the open unit ball $B^\circ$, and take $m_j$ such that $0 < m_j < m(x_j)$, $j = 0, 1$. Then by the definition of the inner modulus there is a $\tau$-zero-neighbourhood $W$, which we may assume convex, such that

$$x_j + m_j(B \cap W) \subset B, \quad j = 0, 1;$$

hence if $x = (1-\lambda)x_0 + \lambda x_1$ and $m = (1-\lambda)m_0 + \lambda m_1$, $0 \leqslant \lambda \leqslant 1$, we get

$$\begin{aligned} x + m(B\cap W) &= (1-\lambda)x_0 + \lambda x_1 + (1-\lambda)m_0(B\cap W) + \lambda m_1(B\cap W) \\ &= (1-\lambda)\big(x_0 + m_0(B\cap W)\big) + \lambda\big(x_1 + m_1(B\cap W)\big) \subset (1-\lambda)B + \lambda B = B. \end{aligned}$$

Therefore $m(x) \geqslant m = (1-\lambda)m_0 + \lambda m_1$ and letting $m_j$ tend to $m(x_j)$ we see that $m$ is concave on $B^\circ$. Since $m = 0$ on the unit sphere the concavity on $B$ follows. Finally, $m(-x) = m(x)$ so we see also that $t \mapsto m(tx)$ decreases for $t \geqslant 0$. This proves the lemma.

Concerning the outer modulus $M$ we note that $M(tx)(1-t)^{-1}$ is increasing in $t \in [0, 1[$ if $\|x\| = 1$ and $\tau$ is locally convex. We shall not need this, however, and omit the simple proof.

We shall now prove our first relation between the radius of boundedness and the $\tau$-local radius of boundedness.

PROPOSITION 3.5. *If $u$ is any numerical function on a normed space $E$, locally bounded above, then*

$$(3.6) \qquad \inf_{\|y\|<1} \frac{R_{\tau,u}\big(x + R_u(x)y\big)}{m(y)} \geqslant R_u(x)$$

*for all $x \in E$, $m$ denoting the inner modulus of $E$ with respect to some locally convex topology $\tau$.*

Proof. Let $x, y \in E$ be fixed with $\|y\| < 1$. Pick $r < m(y)R_u(x)$. Then in view of Lemma 3.4 there is a number $\lambda < 1$ such that $r < m(\lambda^{-1}y)R_u(x)$

and hence there exists a $\tau$-neighbourhood $W$ of the origin such that

$$\lambda^{-1}y+\left(rR_u(x)^{-1}B\cap W\right)\subset B,$$

or, after multiplying by $\lambda R_u(x)$ and adding $x$,

$$x+R_u(x)y+(r\lambda B\cap W_1)\subset x+\lambda R_u(x)B,$$

with $W_1=\lambda R_u(x)W$ a new neighbourhood of the origin. Since $u$ by definition is bounded above in $x+\lambda R_u(x)B$ we get $R_{\tau,u}\left(x+R_u(x)y\right)\geqslant r\lambda$, and letting $r$ tend to $m(y)R_u(x)$ and $\lambda$ to one we arrive at the desired conclusion.

We shall now consider the inequality opposite to (3.6). Here it becomes necessary to impose some condition on $\tau$.

PROPOSITION 3.6. *If $\tau$ is a locally convex topology on a normed space $E$ such that the unit ball is $\tau$-quasi-compact, then*

$$\inf_{\|y\|<1}\frac{R_{\tau,u}(x+\lambda y)}{M(y)}\leqslant\lambda \tag{3.7}$$

*for every $x\in E$ and every $\lambda>R_u(x)$. Here $M$ denotes the outer modulus of $E$ with respect to $\tau$, and $u$ is any function on $E$.*

Proof. Fix $x\in E$ and $\lambda>R_u(x)$. If (3.7) does not hold we can find numbers $r_1$ and $r_2$ such that

$$\inf_{\|y\|<1}\frac{R_{\tau,u}(x+\lambda y)}{M(y)}>r_2>\lambda>r_1>R_u(x).$$

We shall prove that $u$ is bounded above in $x+r_1B$ which contradicts the inequality $r_1>R_u(x)$. Since $x+r_1B$ is $\tau$-quasi-compact, it suffices to prove that $u$ is $\tau$-locally bounded above in $x+r_1B$. So let $a$ be an arbitrary point $x+r_1B$. Then $a=x+\lambda y$ where $\|y\|=\|(a-x)/\lambda\|\leqslant r_1/\lambda<1$ so that by our hypothesis

$$\frac{R_{\tau,u}(a)}{M(y)}>r_2.$$

By the definition of the $\tau$-local radius, $u$ is bounded above in

$$a+\left(r_2M(y)B\cap W_1\right)$$

for some $\tau$-zero-neighbourhood $W_1$, and by the definition of the outer modulus, since $r_2\lambda^{-1}>1$,

$$B\cap(y+W_2)\subset\left(y+r_2\lambda^{-1}M(y)B\right)\cap(y+W_2),$$

for some $\tau$-zero-neighbourhood $W_2$. We may of course assume that $\lambda W_2\subset W_1$. Multiplying the last inclusion relation by $\lambda$ and adding $x$ we obtain

$$(x+\lambda B)\cap(a+W_3)\subset\left(a+r_2M(y)B\right)\cap(a+W_3)=a+\left(r_2M(y)B\cap W_3\right),$$

where $W_3 = \lambda W_2$; hence

$$(x+r_1B)\cap(a+W_3) \subset a+\big(r_2M(y)B\cap W_3\big)$$

which shows that the restriction of $u$ to $x+r_1B$ is bounded above in a $\tau$-neighbourhood of $a$. This proves the proposition.

While the quasi-compactness requirement in Proposition 3.6 is of course necessary for (3.7) to hold for all functions $u$ we do not know if (3.7) holds when $u = \log|f|$, $f$ entire, and $E$ is (e.g.) $l^1$ or $c_0$ and $\tau$ is the weak topology. Sometimes the following refinement of Proposition 3.6 will serve instead:

PROPOSITION 3.7. *Assume in addition to the hypotheses of Proposition* 3.6 *that there is a projection $\pi$ in $E$ of norm one such that all $\tau$-open sets are of the form $\pi^{-1}\big(\pi(W)\big)$ for some $\tau$-open set $W$. Then*

$$\inf_{\substack{\|y\|<1\\ y\in\pi(E)}} \frac{R_{\tau,u}(x+\lambda y)}{M(y)} \leqslant \lambda \tag{3.8}$$

*for* $x\in\pi(E)$ *and* $\lambda > R_u(x)$.

Proof. We need only note that in the proof of Proposition 3.6 it now suffices to prove that the restriction of $u$ to $x+r_1B$ is bounded above in some $\tau$-neighbourhood of an arbitrary point $a\in\pi(x+r_1B) = x+r_1\pi(B)$.

We now assume that $E$ is a complex normed space and introduce the open set in $E\times C$

$$\Omega = \{(x,t)\in E\times C;\ |t|< R_{\tau,u}(x)\}. \tag{3.9}$$

It is easy to see that $\Omega$ is pseudoconvex if and only if $-\log R_{\tau,u}$ is plurisubharmonic, and this is the case, by Theorem 2.1, if $u$ is plurisubharmonic. It then follows that if $d\big((x,t),\complement\Omega\big)$ is the distance to the complement of $\Omega$ measured in some more or less arbitrary way, then $-\log d\big((x,t),\complement\Omega\big)$ is plurisubharmonic. In particular, we shall let $E_1$ denote the space $E\times C$ normed by taking as the open unit ball

$$\{(x,t)\in E\times C;\ \|x\|<1,\ |t|< m(x)\}; \tag{3.10}$$

similarly $E_2$ shall be the normed space obtained by taking as the open unit ball the convex hull of

$$\{(x,t)\in E\times C;\ \|x\|<1,\ |t|< M(x)\}. \tag{3.11}$$

The identity mapping $E_1\to E_2$ then has norm one, and its inverse $E_2\to E_1$ norm at most three (see (3.3)). We can now rephrase Propositions 3.5 and 3.6:

THEOREM 3.8. *Is $u\colon E\to[-\infty,+\infty[$ is a numerical function in a (real or complex) normed space $E$ which is locally bounded above for the norm topology, and $\tau$ a locally convex vector space topology on $E$ such that* *the closed unit ball is $\tau$-quasi-compact, then*

$$d_2\big((x,0), \complement\Omega\big) \leqslant R_u(x) \leqslant d_1\big((x,0), \complement\Omega\big), \tag{3.12}$$

*where $\Omega$ is the open set defined by* (3.9) *and $d_j\big((x,0), \complement\Omega\big)$ is the distance from $(x,0)\in E_j$ to $\complement\Omega$, measured by the norm in $E_j$, $j=1,2$.*

For Hilbert spaces, and more generally for the spaces $l^p(J)$, $1 \leqslant p < +\infty$, where the inner and outer moduli coincide by (3.4), (3.12) turns into an equation for $R_u$.

We shall conclude this section by giving an explicit necessary condition for a function in one of the classical coordinate spaces to be the radius of boundedness of some plurisubharmonic function (and a fortiori the radius of convergence of some entire function).

PROPOSITION 3.9. *Let $E = l^p(J)$, $1 < p < +\infty$, and let $u \in PSH(E)$ be such that*

$$\liminf_{\|x\|\to\infty} \|x\|^a R_u(x) > 0$$

*for some $a > 0$. Then*

$$R_u(x) \geqslant \varphi(\|x\|), \qquad x\in E,$$

*where $\varphi(\|x\|)$ is the distance from $(x,0)$ to the boundary of*

$$\omega = \{(x,t)\in E_1;\ |t| < A\,\|x\|^{-a}\}$$

*measured by the norm in $E_1$ defined by* (3.10), *the number $A$ being chosen to make $\varphi(0) = R_u(0)$. In particular*

$$|R_u(x) - R_u(y)| \leqslant \left(\frac{a}{1+a}\right)^{1-1/p} \|x-y\|. \tag{3.13}$$

COROLLARY 3.10. *Let $a > 0$ and $1 < p < +\infty$. The functions*

$$s(x) = (1+c\,|x_1|)^{-a}, \qquad S(x) = (1+c\,\|x\|)^{-a}, \qquad x\in l^p(J),$$

*are not radii of boundedness of any plurisubharmonic function in $l^p(J)$ when $c > a^{-1/p}(1+a)^{-1+1/p}$ in spite of the fact that $-\log s$ and $-\log S$ are plurisubharmonic for every $c > 0$, and $s$ and $S$ have Lipschitz constant $ac \leqslant 1$ when $c \leqslant 1/a$. For every $p$, $1 < p < +\infty$, we therefore get examples of functions satisfying* (1.1) *and* (1.2) *which are not radii of convergence by taking $c$ in the interval*

$$a^{-1/p}(1+a)^{-1+1/p} < c \leqslant 1/a.$$

Proof. We have

$$\liminf_{\|x\|\to 0} \frac{s(x)-s(0)}{\|x\|} = -ac < -(1+1/a)^{-1+1/p}$$

and similarly for $S$ so the corollary follows easily from (3.13).

Proof of Proposition 3.9. Put

$$v(t) = \sup_{\|x\|=|t|} -\log R_{\tau,u}(x), \qquad t \in C,$$

where $\tau$ is the weak topology. Then $v$ is subharmonic in $C$ and for some constant $C$,

$$v(t) \leqslant C + a\log|t|$$

when $|t|$ is large. On the other hand, assuming as we may that $R_u(0) = 1$, we obtain from Proposition 3.5 and (3.4),

$$R_{\tau,u}(y) \geqslant m(y) = (1-\|y\|^p)^{1/p} \tag{3.14}$$

for all $y$ of norm less than one. In view of the definition of $\omega$ and $A$ this means that for $|t| < 1$,

$$v(t) \leqslant -\frac{1}{p}\log(1-|t|^p) \geqslant -\log A + a\log|t|,$$

where the second inequality is an equation for one value of $|t|$, say for $|t| = |t_0| < 1$. Now a logarithmically convex function which is majorized by $C + a\log|t|$ for large values of $|t|$ and by $-\log A + a\log|t|$ for $|t| = |t_0|$ must in fact satisfy

$$v(t) \leqslant -\log A + a\log|t|$$

for all $|t| \geqslant |t_0|$. This means that

$$R_{\tau,u}(x) \geqslant A\,\|x\|^{-a}, \qquad \|x\| \geqslant |t_0|,$$

and hence by Proposition 3.6 and the definition of $\varphi$,

$$R_u(x) \geqslant \varphi(\|x\|), \qquad x \in E,$$

for the values of $R_{\tau,u}(x)$ when $\|x\| < |t_0| < 1$ are clearly irrelevant as long as (3.14) holds. The last statement in the proposition follows from an elementary calculation which we omit and which shows that the rigth-hand derivative of $\varphi$ at the origin is $-(1+1/a)^{-1+1/p}$.

To get similar results on $c_0$ we cannot use the weak topology. We can therefore retain only $s$, not $S$, of Corollary 3.10 as a counterexample.

PROPOSITION 3.11. *Let $E = c_0(J)$ ($J$ an infinite index set) and let $u \in PSH(E)$ be such that*

$$\liminf_{\|x\|\to+\infty} \|\pi(x)\|^a R_u(x) > 0$$

*for some $a > 0$, $\pi$ denoting the projection on a finite number of coordinates in $c_0(J)$. Then*

$$R_u(x) \geqslant \varphi(\|\pi(x)\|),$$

*where $\varphi$ is defined as in Proposition* 3.9 *with* $p = \infty$ (*i.e.* $\varphi(s) = \big(s+\varphi(s)\big)^{-a}$); *in particular*

$$\liminf_{\|\pi(x)\|\to 0} \frac{R_u(x)-R_u(0)}{\|\pi(x)\|} \geqslant -\frac{a}{a+1}.$$

COROLLARY 3.12. *Let* $a>0$. *The function*

$$s(x) = (1+c|x_1|)^{-a}, \qquad x\in c_0(J),$$

*is not the radius of boundedness of any plurisubharmonic function in* $c_0(J)$ *when* $c>1/(1+a)$ (*and* $c\leqslant 1/a$).

Proposition 3.11 is essentially due to Josefson [4] ($\pi(E)$ one-dimensional and $u=\log|f|$, $f\in\mathcal{O}(c_0)$). His methods are different from ours in that he uses Taylor expansions.

Proof of Corollary 3.12. Completely analogous to that of Corollary 3.10, assuming now Proposition 3.11.

Proof of Proposition 3.11. Define for $t\in C$,

$$v(t) = \sup_{\|\pi(x)\|=|t|} -\log R_{\tau,u}(x) = \sup_{\|\pi(x)\|=1} -\log R_{\tau,u}(tx),$$

where $\tau$ is the topology generated by the seminorm $x\mapsto\|\pi(x)\|$. Then $v$ is a function of $|t|$ only and satisfies

$$v(t)\leqslant C+a\log|t|, \qquad |t| \text{ large enough},$$

for some constant $C$. Using the second expression defining $v$ we see that $v$ is subharmonic, and Proposition 3.5 shows that

$$R_{\tau,u}(y)\geqslant m(y)R_u(0) = R_u(0) \quad \text{if} \quad \|y\|<1 \quad \text{and} \quad \pi(y)=y,$$

for the inner modulus $m$ of $c_0(J)$ with respect to $\tau$ satisfies $m(y)\leqslant 1$, $\|y\|<1$, with equality when $\pi(y)=y$. However, in view of the hypotheses and Liouville's theorem, $R_{\tau,u}$ is constant on the fibers $\pi^{-1}(y)$ so $R_{\tau,u}(y)\geqslant R_u(0)$ for all $y$ such that $\pi(y)$ has norm less than one; hence $v(1)\leqslant -\log R_u(0)$. By logarithmic convexity,

$$v(t)\leqslant -\log R_u(0)+a\log|t| \quad \text{if} \quad |t|\geqslant 1,$$

i.e.

$$R_{\tau,u}(x)\geqslant R_u(0)\,\|\pi(x)\|^{-a} \quad \text{if} \quad \|\pi(x)\|\geqslant 1.$$

We shall now apply Proposition 3.7 to this inequality, with $M$ as the outer modulus of $c_0(J)$ with respect to $\tau$. We note that $M(y)=1$ if $\|y\|<1$ and $\pi(y)=y$; if $\pi(y)=0$ we get $M(y)=1+\|y\|$, $\|y\|<1$, which is useless, hence the importance of using (3.8) rather than (3.7).

Thus

$$\inf_{\substack{\|y\|<1 \\ y\in\pi(E)}} R_{\tau,u}(x+\lambda y) \leqslant \lambda \quad \text{if} \quad \lambda > R_u(x), \quad x\in\pi(E),$$

so that, in view of the definition of $\varphi$,

$$R_u(x) \geqslant \varphi(\|x\|), \quad x\in\pi(E).$$

Since $R_u$ is constant on $\pi^{-1}(x)$ we finally get

$$R_u(x) \geqslant \varphi(\|\pi(x)\|), \quad x\in E = c_0(J).$$

**4. Construction of entire functions with given radius of convergence.** It is conjectured that to any given plurisubharmonic function in a reasonable infinite-dimensional Banach space $E$ there exists an entire function $h$ such that (3.12) holds with $u = \log|h|$ and $\Omega = \{(x,t)\in E\times C;\ v(x)+ +\log|t|<0\}$. Our results in this direction are, however, quite incomplete even in Hilbert spaces. The methods used impose conditions on the geometry of the unit ball in $E$, but the main obstacle to obtaining a more general result — and the only one in Hilbert spaces where the geometry causes no difficulty — is the Levi problem for functions of bounded type. (A function $f\in\mathcal{O}(\Omega)$ is said to be of *bounded type* if it is bounded when $\|x\|+d(x, \complement\Omega)^{-1}$ is bounded.)

THEOREM 4.1. *Let $E$ be an infinite-dimensional normed space, $G$ a subspace of $E$ of finite codimension such that there are projections of $E$ onto $G$ of norm arbitrarily close to one, and $V$ a plurisubharmonic function on $E$ which is constant on the cosets $x+G$. Then there exists an entire function $H$ on $E$ such that*

$$R_{\tau,\log|H|} = e^{-V}, \tag{4.1}$$

*where* $\tau = \sigma(E, G^{\perp})$.

Proof. Let $\Omega$ denote the open set in $E\times C$ defined by $|t|<\exp\big(-V(x)\big)$ $x\in E$, $t\in C$. Define a surjection $\theta\colon E\to C^n$ with kernel $G$ and let $v$ be the plurisubharmonic function in $C^n$ such that $V = v\circ\theta$; let

$$\omega = \{(z,t)\in C^n\times C;\ v(z)+\log|t|<0\}$$

so that $\Omega = \theta^{-1}(\omega)$. Since $v(z)+\log|t|$ is plurisubharmonic in $C^{n+1}$, $\omega$ is pseudoconvex, and by the solution of the Levi problem in $C^{n+1}$ there is $f\in\mathcal{O}(\omega)$ which cannot be continued beyond the boundary of $\omega$ (see Hörmander [3], p. 88). For every $z\in C^n$ we expand $f$ as a power series in $t$:

$$f(z,t) = \sum_{0}^{\infty} f_k(z)t^k;$$

it follows that the series converges uniformly for $|t|\leqslant r<\exp\big(-v(z)\big)$, hence

$$\limsup_{k\to\infty} |f_k(z)|^{1/k} \leqslant e^{v(z)}.$$

Using the fact that $v$ is upper semicontinuous we see that

$$\left[\limsup_{k\to\infty}\frac{1}{k}\log|f_k|\right]^* \leqslant v.$$

But here equality must hold, for otherwise the power series itself defines an extension of $f$, hence, introducing $F_k = f_k \circ \theta$, we get

$$\left[\limsup_{k\to\infty}\frac{1}{k}\log|F_k|\right]^* = V. \tag{4.2}$$

We now define

$$H(x) = \sum_0^\infty F_k(x)\,\xi_k(x)^k, \qquad x\in E, \tag{4.3}$$

where $\xi_k \in E'$ are linear forms of norm one, tending to zero in the weak star topology $\sigma(E', E)$. That such forms exist is a theorem of Josefson [5]; we remark, however, that their existence is trivial in a separable space. Since there is a projection of $E$ onto $G$ of norm arbitrarily close to one, we see that the restriction of $\xi_k$ to $G$ has norm tending to one as $k$ tends to infinity. Passing if necessary to a subsequence we may assume, and we shall assume in the sequel, that

$$|\xi_k(y_j)| \leqslant \tfrac{1}{2}, \quad j = 1, \ldots, k-1, \quad \text{and} \quad |\xi_k(y_k)| \geqslant 1 - 1/k \tag{4.4}$$

for some points $y_j \in G \cap B$.

We claim that $H$ is entire and that $R_{\tau,u} = e^{-V}$, where we have put for brevity $u = \log|H|$. Let $a$ be an arbitrary point of $E$, and let $r < \exp(-V(a))$. By Hartogs' theorem in $C^n$ (cf. Hörmander [3], p. 21) there is a number $\delta > 0$ such that $\log|F_k(x)|^{1/k} \leqslant -\log r$ for $x \in a + G + \delta B$, hence

$$|F_k(x)\,\xi_k(x)^k| \leqslant \frac{1}{r^k}\,|\xi_k(a) + \xi_k(x-a)|^k \leqslant \left(\frac{\varepsilon + \|x-a\|}{r}\right)^k,$$

when $k$ is large enough. If $r_1 + \varepsilon < r$ we see that the series in (4.3) converges uniformly for

$$x \in (a + G + \delta B) \cap (a + r_1 B).$$

Therefore the sum is analytic there, and, moreover, by the definition of the $\tau$-local radius of boundedness we see that $R_{\tau,u}(a) \geqslant r_1$; since $r_1$ can be chosen arbitrarily close to $\exp(-V(a))$ we have $R_{\tau,u} \geqslant \exp(-V)$.

It therefore remains to be proved that $R_{\tau,u} \leqslant \exp(-V)$; in other words that if $H$ is bounded in $a + (rB \cap W)$ for some $\tau$-neighbourhood $W$ of the origin, then $r \leqslant \exp(-V(a))$. So assume that $H$ is bounded in

$a+(rB\cap W)$, say

$$|H(b+zy)| = \Big|\sum F_k(b)\big(\xi_k(b)+\xi_k(y)z\big)^k\Big| \leqslant A$$

when $b\in a+\delta B$, $z\in C$, $|z|\leqslant r$, and $y\in B\cap G$. Let $\varepsilon>0$ be given. In view of (4.2) and the fact that $\xi_k(b)\to 0$ there is a number $m_b$ depending on $b$ such that

$$|F_k(b)|\leqslant e^{kV(b)+k\varepsilon} \quad \text{and} \quad |\xi_k(b)|<\varepsilon$$

for all $k>m_b$. For the terms in $H$ of index $k>m\geqslant m_b$ we then have in view of (4.4)

$$\Big|\sum_{k>m} F_k(b)\big(\xi_k(b)+\xi_k(y_m)z\big)^k\Big| \leqslant \sum_{k>m} e^{kV(b)+k\varepsilon}(\varepsilon+\tfrac{1}{2}r)^k \leqslant C$$

provided only

$$e^{V(b)+\varepsilon}(\varepsilon+\tfrac{1}{2}r)\leqslant e^{V(a)+2\varepsilon}(\varepsilon+\tfrac{1}{2}r)<1$$

which we obviously may assume, making $C$ independent of $b$. Hence

$$\Big|\sum_{k\leqslant m} F_k(b)\big(\xi_k(b)+\xi_k(y_m)z\big)^k\Big| \leqslant A+C, \qquad |z|\leqslant r.$$

But now the left-hand side is the modulus of a polynomial in $z$ of degree at most $m$, hence the leading coefficient may be estimated by Cauchy's inequalities:

$$|F_m(b)|\,|\xi_m(y_m)|^m \leqslant r^{-m}(A+C).$$

Considering (4.4) we may write this

$$|F_m(b)|^{1/m}\leqslant r^{-1}(1-1/m)^{-1}(A+C)^{1/m}\to 1/r$$

so that $\limsup|F_k(b)|^{1/k}\leqslant 1/r$. Since $b$ may vary in a neighbourhood of $a$ we get $[\limsup|F_k|^{1/k}]^*(a)\leqslant 1/r$, i.e. $e^{V(a)}\leqslant 1/r$ as claimed. The proof is complete.

THEOREM 4.2. *Let $E$ be an infinite-dimensional Hilbert space and $V$ a plurisubharmonic function in $E$ which factors through some finite-dimensional subspace. Then there exists $H\in\mathcal{O}(E)$ with $R_{\sigma,\log|H|}=e^{-V}$, where $\sigma=\sigma(E,E')$ is the weak topology.*

Proof. Let $G$ be a closed subspace of finite codimension in $E$ such that $V$ is constant on $x+G$. Then by Theorem 4.1 we may find $H\in\mathcal{O}(E)$ such that $R_{\sigma,\log|H|}\geqslant R_{\tau,\log|H|}=\exp(-V)$, where $\tau=\sigma(E,G^{\perp})$. However, now the proof applies also to any subspace $G_1$ of finite codimension in $G$: we let the forms $\xi_k$ be orthonormal, chosen once and for all, and then pick suitable vectors $y_k$ in $G_1$, e.g. $y_k=\pi(x_k)$, where $\pi$ is the orthogonal projection on $G_1$ and $x_k\in G$ are defined by $\xi_k(x)=\langle x,x_k\rangle$. Hence we obtain $R_{\tau_1,\log|H|}=\exp(-V)$, where $\tau_1=\sigma(E,G_1^{\perp})$ and, passing to the limit, $R_{\sigma,\log|H|}=\exp(-V)$.

THEOREM 4.3. *Let $G$ be a finite-codimensional subspace of an infinite-dimensional normed space $E$ such that there are projections of $E$ onto $G$ of norm arbitrarily close to one, and let $V \in PSH(E)$ be constant on every $x+G$. Assume that $S = \exp(-V)$ has Lipschitz constant $a \leqslant 1$. Then there exists $H \in \mathcal{O}(E)$ whose radius of convergence $R$ satisfies*

$$\tfrac{1}{3}S(x) \leqslant (2+a)^{-1}S(x) \leqslant R(x) \leqslant S(x), \quad x \in E. \tag{4.5}$$

Proof. By Theorem 4.1 there is an entire function $H$ with $\tau$-local radius of convergence $R_{\tau,\log|H|} = S$, where $\tau = \sigma(E, G^{\perp})$. Then $R = R_{\log|H|} \leqslant S$. By Proposition 3.6 and the trivial estimate $M(y) \leqslant 2$ we have

$$\frac{S(x)-a\lambda}{2} \leqslant \inf_{\|y\|<1} \frac{S(x+\lambda y)}{M(y)} \leqslant \lambda$$

for every $\lambda > R(x)$, hence $S(x) \leqslant 2\lambda + a\lambda$. Letting $\lambda$ tend to $R(x)$ we get $S(x) \leqslant (2+a)R(x)$ as claimed.

In spaces of known geometry we can of course be more precise:

THEOREM 4.4. *Let $G$ be the subspace of $c_0(J)$ or $l^p(J)$, $1 \leqslant p < \infty$, obtained by taking $n$ coordinates equal to zero. ($J$ denotes an infinite index set.) If $V$ is any plurisubharmonic function in $E = l^p(J)$, $c_0(J)$ which depends only on these coordinates, and if $S = \exp(-V)$ has Lipschitz constant $a \leqslant 1$, then there exists $H \in \mathcal{O}(E)$ with radius of convergence $R$ satisfying*

$$\tfrac{1}{2}S(x) \leqslant (1+a^q)^{-1/q}S(x) \leqslant R(x) \leqslant S(x), \quad x \in E, \tag{4.6}$$

*where $q = p(p-1)^{-1}$ and the case $E = c_0(J)$ corresponds to $q = 1$. In particular, for $E = l^1(J)$, we have $R = S$.*

Proof. We argue as in the previous proof but use Proposition 3.7 instead: it is enough to let $y$ vary in the subspace $F = \pi(E)$ spanned by the $n$ coordinates defining $G$, thus

$$\inf_{\substack{y \in F \\ \|y\|<1}} \frac{S(x+\lambda y)}{M(y)} \leqslant \lambda \quad \text{if} \quad \lambda = (1+\varepsilon)R(x) > R(x).$$

In other words, to every $\varepsilon > 0$ there exists $y \in F$ with $\|y\| < 1$ such that

$$S(x) - a(1+\varepsilon)R(x)\|y\| \leqslant S\big(x + (1+\varepsilon)R(x)y\big) \leqslant (1+2\varepsilon)R(x)M(y).$$

However, $M(y) = (1-\|y\|^p)^{1/p}$ for $y \in F$ so we get, putting $\|y\| = t$,

$$S(x) \leqslant (1+2\varepsilon)R(x)(1-t^p)^{1/p} + a(1+\varepsilon)R(x)t,$$

hence

$$S(x) \leqslant (1+2\varepsilon)R(x)\sup_{0\leqslant t<1}\big((1-t^p)^{1/p}+at\big) = (1+2\varepsilon)R(x)(1+a^q)^{1/q},$$

where the calculation is valid also in the extreme cases $p = 1$, $p = \infty$. The theorem is proved.

Remark. It is clear from the proof that for (4.5) or (4.6) to hold for a certain $x \in E$ it is sufficient that $a$ is a "local" Lipschitz constant for $S$ in the sense that

$$S(x+y) \geqslant S(x) - a\|y\| \quad \text{for} \quad \|y\| \leqslant S(x).$$

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