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An upper estimate for the reciprocal sum of a sum-free sequence

by

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1. Introduction. Unless otherwise stated, a sequence will mean a strictly increasing sequence of positive integers. A sequence $\mathscr{A}$ with terms $a_1 < a_2 < \ldots$ is called *sum-free* if

$$a_n \neq \sum_{k=1}^{n-1} \varepsilon_k a_k \quad \text{with} \quad \varepsilon_k = 0, 1. \tag{1.1}$$

Letting $\varrho(\mathscr{A}) = \sum_{k=1}^{\infty} \frac{1}{a_k}$, Erdös [1] has shown that if $\mathscr{A}$ is sum-free, then

$$\varrho(\mathscr{A}) < 103. \tag{1.2}$$

It is apparent from the details of Erdös' proof that a slight modification can decrease the bound in (1.2) considerably. However, since the best bound for $\varrho(\mathscr{A})$ was not determined, apparently no attempt was made in [1] to improve the estimate of 103. Although we too do not find the best bound, we do improve (1.2) to an extent beyond that which can be achieved by direct application of Erdös' method. In particular, it is shown that if $\mathscr{A}$ is sum-free, then $\varrho(\mathscr{A}) < 4$. In achieving this bound, a central idea of [1] is retained but is combined with results found in [2].

Perhaps the simplest sum-free sequence that comes to mind is the sequence 1, 2, 4, ... consisting of the powers of 2. Denoting this sequence by $\mathscr{T}$ it follows that $\varrho(\mathscr{T}) = 2$. A naive guess would be to suppose that the reciprocal sum of any other sum-free sequence is dominated by the reciprocal sum of $\mathscr{T}$. The following example illustrates that this is not the case. Let $\mathscr{V}$ be a sequence whose first terms $v_1, v_2, \ldots, v_{14}$ are

$$1, 2, 4, 8, 19, 37, 55, 73, 91, 109, 127, 145, 163, 181.$$

Let $v_{15} = 1+\sum_{k=1}^{14} v_k = 1016$, and $v_k = 2^{k-15}v_{15}$ for $k > 15$. It is not difficult to verify that $\mathscr{V}$ is sum-free and

$$\varrho(\mathscr{V}) = \sum_{k=1}^{14} \frac{1}{v_k} + \frac{2}{v_{15}} = 2.035\ldots$$

Letting $\lambda = \text{l.u.b.}\,\varrho(\mathscr{A})$ where $\mathscr{A}$ ranges over all sum-free sequences, it follows from (1.2) and the above example that $2.03 < \lambda \leqslant 103$. It is suspected that λ is quite close to the lower estimate. However, as already indicated, we only show that $\lambda < 4$.

2. Inequalities. Let $A(x) = \sum_{a_k \leqslant x} 1$ be the counting function of sequence $\mathscr{A}$. When $\mathscr{A}$ is sum-free, an inequality established in [1] provides

$$A(x) \leqslant \frac{x}{k+1} + \sum_{i=1}^{k} a_i + k \quad (k = 1, 2, \ldots;\ x \geqslant 0). \tag{2.1}$$

It is then shown that (1.2) follows from (2.1). Inequality (2.1) can be substantially improved, and this is the substance of the following theorem.

THEOREM 1. *If $\mathscr{A}$ is sum-free, then*

$$A(x) \leqslant \frac{x}{k+1} + \frac{1}{k+1}\sum_{i=1}^{k} a_i + \frac{k}{2} \quad (k = 1, 2, \ldots;\ x \geqslant 0). \tag{2.2}$$

Proof. Let k be a positive integer, x a non-negative real, and let $A(x) = n$. If $n \leqslant k$,

$$A(x) \leqslant k = \frac{1}{k+1}\sum_{i=1}^{k} i + \frac{k}{2} \leqslant \frac{1}{k+1}\sum_{i=1}^{k} a_i + \frac{k}{2}$$

and (2.2) easily follows.

When $n > k$, consider the integers

$$b_{pq} = \sum_{i=1}^{p} a_i + a_q \quad (p = 0, 1, \ldots, k;\ q = p+1, \ldots, n). \tag{2.3}$$

The number of integers defined by (2.3) is $N = (k+1)n - k(k+1)/2$ and

$$0 < b_{pq} \leqslant b_{kn} \tag{2.4}$$

with p and q in the defined range.

Further, the integers b_{pq} are distinct, for suppose $b_{pq} = b_{rs}$ with $p \leqslant r$. Then

$$a_q = \sum_{i=p+1}^{r} a_i + a_s$$

which violates the assumption of sum-freeness unless $p = r$ and $q = s$. The distinctness of the b_{pq} together with (2.4) imply

$$\sum_{i=1}^{k} a_i + a_n = b_{kn} \geqslant N = (k+1)n - k(k+1)/2;$$

hence

$$A(x) = n \leqslant \frac{a_n}{k+1} + \frac{1}{k+1}\sum_{i=1}^{k} a_i + \frac{k}{2}. \tag{2.5}$$

Noting that $A(x) = n$ yields $a_n \leqslant x$, the result easily follows from (2.5).

Inequality (2.2) is somewhat awkward, and instead we focus on the following weaker version.

THEOREM 2. *If $\mathscr{A}$ is sum-free, then*

$$A(x) \leqslant \frac{x}{k+1} + a_k \quad (k = 1, 2, \ldots;\ x \geqslant 0). \tag{2.6}$$

Proof. Since $\mathscr{A}$ is strictly increasing, $a_i \leqslant a_k - (k-i)$ for $i \leqslant k$. Then an application of Theorem 1 yields

$$\begin{aligned} A(x) &\leqslant \frac{x}{k+1} + \frac{1}{k+1}\sum_{i=1}^{k} (a_k - k + i) + \frac{k}{2} \\ &= \frac{x}{k+1} + \frac{k}{k+1} a_k + \frac{k}{k+1} \leqslant \frac{x}{k+1} + a_k \end{aligned}$$

where $k \leqslant a_k$ is used to obtain the last inequality.

Sequences satisfying (2.6) will be of primary interest in what follows, and such sequences will be called *$\varkappa$-sequences*. In particular, every sum-free sequence is a $\varkappa$-sequence. Letting $\mu = \text{l.u.b.}\,\varrho(\mathscr{A})$ where the l.u.b. is over all $\varkappa$-sequences, obviously $\lambda \leqslant \mu$, and the objective of what follows is to show that $\mu < 4$.

One can easily produce examples of $\varkappa$-sequences which are not sum-free. However, there is a specific example which can be singled out which by its nature suggests it would have a large reciprocal sum. Namely, define the sequence $\mathscr{Q}$ with terms $q_1, q_2, \ldots$ as follows. Let $q_1 = 1$, then assuming $q_1, \ldots, q_{n-1}$ have been defined, let q_n be the least integer such that (2.6) is not violated for $k = 1, \ldots, n-1$, i.e.

$$q_n = \max_{1 \leqslant k \leqslant n-1} (k+1)(n - q_k).$$

The initial terms of $\mathscr{Q}$ are 1, 2, 4, 6, 9, 12, 15, 18, 21, 24, 28, 32, ... It happens that $\varrho(\mathscr{Q}) \approx 3.01$ and what one might naively guess is that $\varrho(\mathscr{Q})$ dominates the reciprocal sum of any other $\varkappa$-sequence. Here, we believe this to be true, i.e., we conjecture that $\mu = \varrho(\mathcal{Q})$, but we are unable to prove it. However, subsequent results provide some evidence in support of this conjecture. Nevertheless, this example does show that $\mu > 3$. A more detailed discussion of the sequence $\mathcal{Q}$ is postponed to Section 4.

3. A general estimate for μ. In this section a central idea already found in [1] is combined with inequality (2.6) to obtain an estimate for μ in terms of parameters which will be specified when appropriate.

To begin, consider a sequence $\mathscr{A}$ with terms $a_1 < a_2 < \dots$ which is a $\varkappa$-sequence, i.e., which satisfies

$$A(x) \leqslant \frac{x}{k+1} + a_k \quad (k = 1, 2, \dots;\ x \geqslant 0). \tag{3.1}$$

In order to use (3.1) effectively, summing $\varrho(\mathscr{A})$ by parts yields

$$\varrho(\mathscr{A}) = \sum_{i=1}^{\infty} \frac{1}{a_i} = \sum_{n=1}^{\infty} \frac{A(n)-A(n-1)}{n} = \sum_{n=1}^{\infty} \frac{A(n)}{n(n+1)}. \tag{3.2}$$

Next, the positive integers are partitioned into intervals

$$J(r) = \{n \mid 2^{r-1} < n \leqslant 2^r\}, \quad r = 0, 1, 2, \dots, \tag{3.3}$$

and the quantities

$$\varrho(r) = \sum_{n \in J(r)} \frac{A(n)}{n(n+1)} \tag{3.4}$$

are introduced, so that

$$\varrho(\mathscr{A}) = \sum_{r=0}^{\infty} \varrho(r). \tag{3.5}$$

For $r > 0$,

$$\varrho(r) \leqslant A(2^r) \sum_{n \in J(r)} \frac{1}{n(n+1)} = A(2^r)\left(\frac{1}{2^{r-1}+1} - \frac{1}{2^r+1}\right)$$

hence,

$$\varrho(r) < \frac{A(2^r)}{2^r} \quad (r \geqslant 0). \tag{3.6}$$

Certain parameters are now introduced which will be specified as the need arises. Namely, let N and M be positive integers and for simplicity of later computation, we specify that M be even. Also let $\gamma > 0$ be real. Then, consider the sets of integers

$$\Delta = \left\{r \mid r > N \text{ and } A(2^r) \leqslant \frac{\gamma 2^r}{r(r+1)}\right\}, \tag{3.7}$$

$$\Gamma = \left\{r \mid r > N \text{ and } A(2^r) > \frac{\gamma 2^r}{r(r+1)}\right\}. \tag{3.8}$$

Equation (3.5) can be rewritten as

$$\varrho(\mathscr{A}) = \sum_{r=0}^{N} \varrho(r) + \sum_{r \in \Gamma} \varrho(r) + \sum_{r \in \Delta} \varrho(r). \tag{3.9}$$

When $r \in \Delta$, $\varrho(r)$ can be expected to be "small" and as a matter of fact, the last term in (3.9) can be estimated using (3.6) as follows.

$$\sum_{r\in\Delta} \varrho(r) < \sum_{r\in\Delta} \frac{A(2^r)}{2^r} \leqslant \sum_{r\in\Delta} \frac{\gamma}{r(r+1)}$$
$$= \sum_{r>N} \frac{\gamma}{r(r+1)} - \sum_{r\in\Gamma} \frac{\gamma}{r(r+1)} = \frac{\gamma}{N+1} - \sum_{r\in\Gamma} \frac{\gamma}{r(r+1)}.$$

Substituting into (3.9) yields

$$\varrho(\mathscr{A}) < \sum_{r=0}^{N} \varrho(r) + \frac{\gamma}{N+1} + \sum_{r\in\Gamma} \left(\varrho(r) - \frac{\gamma}{r(r+1)}\right). \tag{3.10}$$

Next, let $r_1 < r_2 < r_3 \dots$ be the integers in Γ. Then

$$\begin{cases} r_1 \geqslant N+1, \\ r_i - r_j \geqslant i - j \quad (i \geqslant j), \\ r_i \geqslant i + N, \end{cases} \tag{3.11}$$

where the last inequality follows from the previous two by setting $j = 1$.

Now, focusing on an i and j with $i \geqslant j \geqslant 1$, let

$$q = \left[\frac{\gamma 2^{r_j}}{r_j(r_j+1)}\right].$$

Then from (3.8), $q < A(2^{r_j})$, hence $a_q < 2^{r_j}$ (if $q = 0$, interpret $a_0 = 0$) and from (3.1)

$$A(2^{r_i}) \leqslant \frac{2^{r_i}}{q+1} + a_q < \frac{2^{r_i}}{\gamma\left(\dfrac{2^{r_j}}{r_j(r_j+1)}\right)} + 2^{r_j}.$$

From the above and (3.6)

$$\varrho(r_i) < \frac{r_j(r_j+1)}{\gamma 2^{r_j}} + \frac{1}{2^{r_i - r_j}}.$$

Using (3.11) and the fact that $r(r+1)/2^r$ is monotonically decreasing for $r \geqslant 2$,

$$\varrho(r_i) < \frac{(N+j)(N+j+1)}{\gamma 2^{N+j}} + \frac{1}{2^{i-j}}. \tag{3.12}$$

Our interest is in the last term of (3.10). We have

$$\sum_{r\in\Gamma}\left(\varrho(r)-\frac{\gamma}{r(r+1)}\right)\leqslant\sum_{i=1}^{M}\left(\varrho(r_i)-\frac{\gamma}{r_i(r_i+1)}\right)+\sum_{i=M+1}^{\infty}\varrho(r_i). \tag{3.13}$$

Letting $j=\left[\frac{i}{2}\right]$ in (3.12) and setting $M=2H$,

$$\sum_{i=M+1}^{\infty}\varrho(r_i)\leqslant\sum_{i=M+1}^{\infty}\left(\frac{(N+j)(N+j+1)}{\gamma 2^{N+j}}+\frac{1}{2^{i-j}}\right)$$
$$=\frac{(N+H)(N+H+1)}{\gamma 2^{N+H}}+\frac{2}{\gamma}\sum_{p=N+H+1}^{\infty}\frac{p(p+1)}{2^p}+2\sum_{p=H+1}^{\infty}\frac{1}{2^p}$$
$$=\frac{(N+H)(N+H+1)}{\gamma 2^{N+H}}+\frac{2}{\gamma}\left(\frac{(N+H+1)^2+3(N+H+1)+4}{2^{N+H}}\right)+\frac{1}{2^{H-1}},$$

where use has been made of the identity

$$\sum_{p=n}^{\infty}\frac{p(p+1)}{2^p}=\frac{n^2+3n+4}{2^{n-1}}.$$

Letting $R=N+H$, the last inequality can be written as

$$\sum_{i=M+1}^{\infty}\varrho(r_i)\leqslant\frac{3R^2+11R+16}{\gamma 2^R}+\frac{1}{2^{H-1}}.$$

Combining (3.10), (3.13) and the above,

$$\varrho(\mathscr{A})\leqslant\sum_{r=0}^{N}\varrho(r)+\frac{\gamma}{N+1}+\frac{3R^2+11R+16}{\gamma 2^R}+\frac{1}{2^{H-1}}+ \tag{3.14}$$
$$+\sum_{i=1}^{M}\left(\varrho(r_i)-\frac{\gamma}{r_i(r_i+1)}\right).$$

At this point, a choice is made for the parameter γ. Let γ (which can be chosen freely) be

$$\gamma=\sqrt{\frac{(N+1)(3R^2+11R+16)}{2^R}}.$$

This is the value minimizing the sum of the second and third term in the right-hand side of (3.14). Upon substituting this value of γ into (3.14), an upper estimate for $\varrho(\mathscr{A})$ is obtained. This estimate is provided by the following theorem.

THEOREM 3. *Let $\mathscr{A}$ be a $\varkappa$-sequence and let N, M, and H be positive integers with $M=2H$. Then*

$$\varrho(\mathscr{A}) \tag{3.15}$$
$$\leqslant\sum_{r=0}^{N}\varrho(r)+2\sqrt{\frac{3R^2+11R+16}{(N+1)2^R}}+\frac{1}{2^{H-1}}+\sum_{i=1}^{M}\left(\varrho(r_i)-\frac{\gamma}{r_i(r_i+1)}\right)$$

where

$$R=N+H\quad\textit{and}\quad\gamma=\sqrt{\frac{(N+1)(3R^2+11R+16)}{2^R}}.$$

COROLLARY. *Using the notation of the theorem,*

$$\varrho(\mathscr{A})\leqslant\sum_{r=0}^{N}\varrho(r)+2\sqrt{\frac{3R^2+11R+16}{(N+1)2^R}}+\frac{1}{2^{H-1}}+\sum_{i=1}^{M}\varrho(r_i). \tag{3.16}$$

An application of the corollary provides an immediate improvement over (1.2). For this purpose, it is first observed that $A(n)\leqslant n$, hence

$$\varrho(r)=\sum_{n\in J(r)}\frac{A(n)}{n(n+1)}\leqslant\sum_{n=2^{r-1}+1}^{2^r}\frac{1}{n+1}<\log 2.$$

Then, setting $N=4$ and $M=2$, the corollary yields

$$\varrho(\mathscr{A})\leqslant\sum_{r=0}^{4}\varrho(r)+1.92+1+2\log 2<\sum_{n=1}^{16}\frac{1}{n+1}+4.32<6.76$$

for any $\varkappa$-sequence. Thus,

$$\mu<6.76. \tag{3.17}$$

A more extensive effort will show $\mu<4$. In anticipation of this result, this section will conclude with the following lemma.

LEMMA 1. *For any $\varkappa$-sequence $\mathscr{A}$,*

$$\varrho(r)<\frac{\log 2}{k+1}+\frac{a_k}{2^r}\quad(r=0,1,2,\ldots;\ k=1,2,\ldots). \tag{3.18}$$

Proof.

$$\varrho(r)=\sum_{n\in J(r)}\frac{A(n)}{n(n+1)}\leqslant\sum_{n=2^{r-1}+1}^{2^r}\frac{\frac{n}{k+1}+a_k}{n(n+1)}$$
$$=\frac{1}{k+1}\sum_{n=2^{r-1}+1}^{2^r}\frac{1}{n+1}+a_k\left(\frac{1}{2^{r-1}+1}-\frac{1}{2^r+1}\right)<\frac{\log 2}{k+1}+\frac{a_k}{2^r}.$$

4. A specific $\varkappa$-sequence. The $\varkappa$-sequence $\mathcal{Q}$ which was previously introduced has some interest in view of the conjecture that $\mu = \varrho(\mathcal{Q})$. By computer, all terms not exceeding 2^{18} were determined and a few of these results are summarized as follows.

$$(4.1)\qquad \begin{cases} A(2^{18}) = 3360, \\ \sum_{i=1}^{3360} \frac{1}{q_i} = 3.008466\ldots, \\ \sum_{r=0}^{18} \varrho(r) = 2.995648\ldots, \\ q_{410} = 8964. \end{cases}$$

The term q_{410} is specifically mentioned for reasons soon to become apparent.

Application of (3.18) to the sequence $\mathcal{Q}$ yields

$$(4.2)\qquad \sum_{i=1}^{M} \varrho(r_i) \leqslant \frac{M(\log 2)}{k+1} + q_k \sum_{i=1}^{M} \frac{1}{2^{r_i}} < \frac{M(\log 2)}{k+1} + \frac{q_k}{2^N}$$

where the last inequality follows by observing that $r_1, r_2, \ldots, r_M$ are distinct integers each greater than N.

Estimating $\varrho(\mathcal{Q})$ using (3.16) with $N = 18$ and $M = 20$ (where the latter selection is quite arbitrary) yields

$$\varrho(\mathcal{Q}) < 2.99565 + .00145 + \frac{1}{2^9} + \sum_{i=1}^{M} \varrho(r_i) < 3 + \sum_{i=1}^{M} \varrho(r_i).$$

Employing (4.2), the above becomes

$$\varrho(\mathcal{Q}) < 3 + \frac{20 \log 2}{k+1} + \frac{q_k}{2^{18}}.$$

Letting $k = 410$ (which was selected to balance the last two terms) and recalling that $q_{410} = 8964$, the estimate $\varrho(\mathcal{Q}) < 3.07$ results. Thus

$$3 < \varrho(\mathcal{Q}) < 3.07.$$

5. The initial terms of a $\varkappa$-sequence. There is a considerable gap between the estimate for μ provided by (3.17) as opposed to the conjectured value $\mu = \varrho(\mathcal{Q})$. To close this gap, this section is devoted to showing that if a $\varkappa$-sequence has a "large" reciprocal sum, then a few of its initial terms (the first three terms) must coincide with initial terms of $\mathcal{Q}$. Section 6 will then proceed to use this meager information to obtain a more effective use of Theorem 3 to estimate μ.

To begin, a lemma which is rather obvious is introduced for completeness. Roughly, this lemma states that a sequence $\mathcal{A}$ is a $\varkappa$-sequence if (2.6) is true when x is restricted to the terms of the sequence.

LEMMA 2. *A sequence $\mathcal{A}$ is a $\varkappa$-sequence if and only if*

$$(5.1)\qquad i \leqslant \frac{a_i}{k+1} + a_k \quad (i = 1, 2, 3, \ldots;\ k = 1, 2, 3, \ldots).$$

Proof. By taking $x = a_i$ in (2.6), the condition of the lemma is clearly seen to be necessary. To show sufficiency, consider an $x \geqslant 0$. Let a_i be such that $a_i \leqslant x < a_{i+1}$ (and define $a_0 = 0$ for $x < a_1$). Then for any k,

$$A(x) = i \leqslant \frac{a_i}{k+1} + a_k \leqslant \frac{x}{k+1} + a_k.$$

Recalling the terms of $\mathcal{Q}$ are denoted by $q_1, q_2, \ldots$ (and have values $1, 2, 4, 6, \ldots$, respectively); also define $q_0 = 0$. The next lemma, though complicated in detail, is crucial to the remaining results of this section.

LEMMA 3. *Let h, w, r, m be integers with $r > 0$ and $0 \leqslant h < w < m+r$ such that*

(i) $\frac{r-1}{r} \leqslant \frac{h+1}{w+1}$, *and*

(ii) $m \leqslant rq_{h+1} - (r-1)(q_h+1)$.

Let $\mathcal{B}$ be a $\varkappa$-sequence with terms $b_1 < b_2 < \ldots$ such that $b_i = q_i$, $i = 1, \ldots, h$ and $b_w = t \geqslant q_w$. Let $\mathcal{A}$ be a sequence obtained from $\mathcal{B}$ as follows. The values of the terms $b_{h+1}, \ldots, b_w$ are altered to become $q_{h+1}, \ldots, q_w$ respectively; and the $d = t - q_{h+1}$ terms b_{rp+m} with $p = 1, \ldots, d$ are deleted. Then $\mathcal{A}$ is a $\varkappa$-sequence and

$$(5.2)\qquad \varrho(\mathcal{A}) - \varrho(\mathcal{B}) = \sum_{i=h+1}^{w} \left(\frac{1}{q_i} - \frac{1}{b_i}\right) - \sum_{p=1}^{d} \frac{1}{b_{rp+m}}.$$

Proof. Equality (5.2) simply follows from the manner in which $\mathcal{A}$ is obtained from $\mathcal{B}$. The difficulty here is in showing that $\mathcal{A}$ is a $\varkappa$-sequence.

Let $A(x)$ and $B(x)$ be the counting functions of $\mathcal{A}$ and $\mathcal{B}$ respectively. From the construction of $\mathcal{A}$, $A(a_i) \leqslant B(a_i)$ except possibly for $i = h+1, \ldots, w$, and $a_k \geqslant b_k$ except possibly for $k = h+1, \ldots, w$. Thus, except for $i = h+1, \ldots, w$ and $k = h+1, \ldots, w$ we can immediately verify (5.1) since

$$i = A(a_i) \leqslant B(a_i) \leqslant \frac{a_i}{k+1} + b_k \leqslant \frac{a_i}{k+1} + a_k.$$

Inequality (5.1) is also immediate when $i = h+1, \ldots, w$. In these cases, if $k > w$, then (5.1) certainly holds. If $k \leqslant w$, then $a_i = q_i$ and $a_k = q_k$; hence (5.1) holds in view of the fact that $\mathcal{Q}$ is a $\varkappa$-sequence. In fact this argument is valid for all $i \leqslant w$. Thus the proof is reduced to verifying (5.1) for $k = h+1, \ldots, w$ and $i > w$.

Fixing k with $h+1 \leqslant k \leqslant w$, (5.1) must be verified for $i > w$. For such an i, it follows from the construction of $\mathcal{A}$ that $a_i = b_s$ (with $s \leqslant i$). Three cases are now considered.

Case I. Let i be such that a_i is beyond the last deleted term (i.e., $s > rd+m$). Then

$$i = A(a_i) = B(a_i) - d \leqslant \frac{a_i}{w+1} + b_w - (t - q_{h+1}) = \frac{a_i}{w+1} + q_{h+1}$$

$$\leqslant \frac{a_i}{k+1} + q_k = \frac{a_i}{k+1} + a_k$$

which verifies (5.1) in this case.

Case II. Let i be such that $b_m \leqslant a_i \leqslant b_{rd+m}$ (i.e., $m \leqslant s \leqslant rd+m$). In this range, the number of terms of $\mathcal{B}$ deleted up to b_s is $\left[\dfrac{s-m}{r}\right]$. Using (ii) and (i),

$$i = A(a_i) = B(a_i) - \left[\frac{s-m}{r}\right] = s - \left[\frac{s-m}{r}\right] \tag{5.3}$$

$$\leqslant s - \left(\frac{s-m}{r}\right) + \left(\frac{r-1}{r}\right) = s\left(\frac{r-1}{r}\right) + \frac{m+(r-1)}{r}$$

$$\leqslant s\left(\frac{r-1}{r}\right) + q_{h+1} - \left(\frac{r-1}{r}\right) q_h \leqslant \frac{(h+1)(s-q_h)}{w+1} + q_{h+1}.$$

But $\mathcal{B}$ is a $\varkappa$-sequence, hence $s \leqslant b_s/(h+1) + b_h$, hence $b_s \geqslant (h+1)(s-b_h) = (h+1)(s-q_h)$. Thus, from (5.3)

$$i \leqslant \frac{b_s}{w+1} + q_{h+1} = \frac{a_i}{w+1} + a_{h+1} \leqslant \frac{a_i}{k+1} + a_k$$

which verifies (5.1) in this case.

Before proceeding to the last case, an observation is needed. The term b_m occurs prior to the first deleted term (which is b_{r+m}). If $m > w$, then the latter implies $a_m = b_m$ (i.e., when $i = m$ then $s = m$). For this special situation, (5.3) yields

$$m \leqslant \frac{(h+1)(m-q_h)}{w+1} + q_{h+1} \quad (\text{for } m > w). \tag{5.4}$$

Case III. Let i be such that $a_i < b_m$. It is still assumed that $i > w$. This means that a_i occurs beyond the last altered term but prior to the first deleted term. Thus $i = s$ in this case and it follows that $w < i < m$. (It may generally happen that $m \leqslant w$, in which case no such i exists and this case would hold vacuously.) Using the fact that $\mathcal{B}$ is a $\varkappa$-sequence, $i \leqslant b_i/(h+1) + b_h$, hence,

$$a_i = b_i \geqslant (h+1)(i-b_h) = (h+i)(i-q_h).$$

Thus letting $f = m-i$ and use of (5.4) provides

$$i = m - f \leqslant \frac{(h+1)(m-q_h)}{w+1} + q_{h+1} - f \leqslant \frac{(h+1)(m-q_h)}{w+1} + q_{h+1} - \left(\frac{h+1}{w+1}\right) f$$

$$= \frac{(h+1)(i-q_h)}{w+1} + q_{h+1} \leqslant \frac{a_i}{w+1} + q_{h+1} \leqslant \frac{a_i}{k+1} + q_k = \frac{a_i}{k+1} + a_k$$

which again verifies (5.1) and completes this case.

This completes the proof of the lemma.

THEOREM 4. *Let $\mathcal{B}$ be a $\varkappa$-sequence. Then there exists a $\varkappa$-sequence $\mathcal{A}$ with $a_1 = 1$ such that $\varrho(\mathcal{A}) \geqslant \varrho(\mathcal{B})$.*

Proof. Letting $h = 0$, $w = 1$, $r = 1$ and $m = 1$ in Lemma 3, we obtain a $\varkappa$-sequence $\mathcal{A}$ such that $a_1 = 1$ and

$$\varrho(\mathcal{A}) - \varrho(\mathcal{B}) = \left(1 - \frac{1}{b_1}\right) - \sum_{p=1}^{t-1} \frac{1}{b_{p+1}}$$

where $t = b_1$. Since each $b_i \geqslant t$,

$$\varrho(\mathcal{A}) - \varrho(\mathcal{B}) \geqslant 1 - \sum_{p=1}^{t} \frac{1}{t} = 0.$$

At this point, it is mentioned that a result like Theorem 4 should be true for sum-free sequences, but we have not been able to prove this.

THEOREM 5. *Let $\mathcal{B}$ be a $\varkappa$-sequence. Then there is a $\varkappa$-sequence $\mathcal{A}$ such that $a_1 = 1$, $a_2 = 2$ and $\varrho(\mathcal{A}) \geqslant \varrho(\mathcal{B})$.*

Proof. From Theorem 4, it may be assumed that one already has $b_1 = 1$. Applying Lemma 3 with $h = 1$, $w = 2$, $r = 3$ and $m = 2$ yields a $\varkappa$-sequence $\mathcal{A}$ with $a_1 = 1$, $a_2 = 2$ and

$$\varrho(\mathcal{A}) - \varrho(\mathcal{B}) = \left(\frac{1}{2} - \frac{1}{b_2}\right) - \sum_{p=1}^{t-2} \frac{1}{b_{3p+2}} = \frac{1}{2} - \sum_{p=0}^{t-2} \frac{1}{b_{3p+2}}$$

where $t = b_2$. (The application of Lemma 3 requires one to verify $t \geqslant q_2 = 2$. But this holds for obvious reasons.) It must now be shown that the last expression is non-negative for $t \geqslant 2$.

Since $\mathscr{B}$ is strictly increasing, $b_{3p+2} \geqslant b_2 + 3p = t+3p$. But (2.6) provides $3p+2 \leqslant b_{3p+2}/2+1$, or $b_{3p+2} \geqslant 6p+2$. Letting $c_p = \max(t+3p, 6p+2)$, it follows that $b_{3p+2} \geqslant c_p$ and

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) \geqslant \frac{1}{2} - \sum_{p=0}^{t-2} \frac{1}{c_p} = D(t).$$

For a specific t, one can determine $D(t)$ explicitly and upon doing this for $2 \leqslant t \leqslant 14$, one finds $D(t) \geqslant 0$ in this range. For $t \geqslant 15$,

$$\begin{aligned} D(t) &= \frac{1}{2} - \sum_{p=0}^{t-2} \frac{1}{c_p} \geqslant \frac{1}{2} - \sum_{p=0}^{t-2} \frac{1}{t+3p} \\ &= \left(\frac{1}{2} - \frac{1}{t}\right) - \sum_{p=1}^{t} \frac{1}{t+3p} + \frac{1}{4t-3} + \frac{1}{4t} \\ &> \left(\frac{1}{2} - \frac{1}{t}\right) - \frac{1}{3} \sum_{p=1}^{3t} \frac{1}{t+p} + \frac{1}{2t} \\ &> \frac{1}{2} - \frac{1}{2t} - \frac{\log 4}{3} \geqslant \frac{1}{2} - \frac{1}{30} - \frac{\log 4}{3} > 0. \end{aligned}$$

THEOREM 6. *Let $\mathscr{B}$ be a $\varkappa$-sequence. Then there is a $\varkappa$-sequence $\mathscr{A}$ such that $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ and $\varrho(\mathscr{A}) \geqslant \varrho(\mathscr{B})$.*

Proof. From Theorem 5, we may assume $b_1 = 1$ and $b_2 = 2$. It then follows from (2.6) that $b_3 \geqslant 4$. Two cases are now considered.

Case I. $4 \leqslant b_3 \leqslant 27$. Letting $h = 2$, $w = 3$, $r = 4$, and $m = 7$, the conditions of Lemma 3 are easily verified. This guarantees the existence of a $\varkappa$-sequence $\mathscr{A}$ such that $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) = \left(\frac{1}{4} - \frac{1}{t}\right) - \sum_{p=1}^{t-2} \frac{1}{b_{4p+7}} \quad \text{where } t = b_3.$$

The strict monotonicity of $\mathscr{B}$ provides $b_{4p+7} \geqslant b_3 + (4p+4) = t + (4p+4)$; while (2.6) with $k = 2$ yields $b_{4p+7} \geqslant 12p+15$. Thus

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) \geqslant \left(\frac{1}{4} - \frac{1}{t}\right) - \sum_{p=1}^{t-4} \frac{1}{\max(t+4p+4, 12p+15)} = D(t).$$

Upon evaluating $D(t)$ in the range $4 \leqslant t \leqslant 27$ it can be determined that $D(t) \geqslant 0$.

Case II. $b_3 \geqslant 28$. Application is now made of Lemma 3 with $h = 2$, $w = 5$, $r = 2$, and $m = 5$. Letting $t = b_5$, the application of Lemma 3 requires one to verify that $t \geqslant 9$. This follows easily by observing that $t = b_5 > b_3 \geqslant 28$. Lemma 3 now yields a sequence $\mathscr{A}$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ (as well as $a_4 = 6$ and $a_5 = 9$) such that

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) = \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{9}\right) - \left(\frac{1}{b_3} + \frac{1}{b_4} + \frac{1}{b_5}\right) - \sum_{p=1}^{t-4} \frac{1}{b_{2p+5}}.$$

Using an argument similar to that in Case I, the strict monotonicity of $\mathscr{B}$ yields $b_{2p+5} \geqslant t+2p$, while (2.6) yields $b_{2p+5} \geqslant 6p+9$. Letting $u = \left[\frac{t-6}{4}\right]$, it then follows that

$$\begin{aligned} (5.5)\quad \varrho(\mathscr{A}) - \varrho(\mathscr{B}) &= \frac{19}{36} - \left(\frac{1}{b_3} + \frac{1}{b_4} + \frac{1}{t}\right) - \sum_{p=1}^{u} \frac{1}{b_{2p+5}} - \sum_{p=u+1} \frac{1}{b_{2p+5}} \\ &\geqslant \frac{19}{36} - \left(\frac{1}{b_3} + \frac{1}{b_4}\right) - \frac{1}{t} - \sum_{p=1}^{u} \frac{1}{t+2p} - \sum_{p=u+1}^{t-4} \frac{1}{6p+9}. \end{aligned}$$

But

$$\begin{aligned} (5.6)\quad \frac{1}{t} + \sum_{p=1}^{u} \frac{1}{t+2p} &= \sum_{p=0}^{u} \frac{1}{t+2p} \leqslant \frac{1}{2} \sum_{p=-1}^{2u} \frac{1}{t+p} \\ &< \frac{1}{2} \log\left(\frac{2u+t}{t-2}\right) \leqslant \frac{1}{2} \log\frac{3}{2}, \end{aligned}$$

where the last inequality results from observing that

$$2u + t \leqslant 2\left(\frac{t-6}{4}\right) + t = \frac{3}{2}(t-2).$$

Also

$$\begin{aligned} (5.7)\quad \sum_{p=u+1}^{t-4} \frac{1}{6p+9} &\leqslant \frac{1}{6} \sum_{p=u+1}^{t-4} \frac{1}{p+1} \\ &= \frac{1}{6} \sum_{p=u+1}^{t-7} \frac{1}{p+1} + \frac{1}{6}\left(\frac{1}{t-6} + \frac{1}{t-5} + \frac{1}{t-4}\right) \\ &< \frac{1}{6} \log\left(\frac{t-6}{u+1}\right) + \frac{1}{2(t-6)} \leqslant \frac{1}{6} \log 4 + \frac{1}{2(t-6)} \end{aligned}$$

where the last inequality follows from $u+1 > (t-6)/4$.

Combining (5.5), (5.6), and (5.7) and using the fact that $b_3 \geqslant 28$, $b_4 \geqslant 29$, and $t = b_5 \geqslant 30$,

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) > \frac{19}{36} - \left(\frac{1}{28} + \frac{1}{29}\right) - \frac{1}{2} \log\frac{3}{2} - \frac{1}{6} \log 4 - \frac{1}{48} > 0.$$

It would be advantageous to be able to obtain further theorems like those presented above. We do not see a general procedure for doing this, and in fact we are unable to show that a $\varkappa$-sequence $\mathscr{A}$ with a "large" reciprocal sum must have $a_4 = 6$ which would be the next natural result along these lines.

It is also mentioned that slight modification of the above proofs would show that any $\varkappa$-sequence $\mathscr{A}$ such that $\varrho(\mathscr{A}) > \mu - \varepsilon$ with ε sufficiently small must have $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. It can also be shown, but we do not include the details, that for small enough ε, either $a_4 = 6$ or $28 \leqslant a_4 \leqslant 64$. A partial result concerning a_4 is included and will be used in estimating μ.

THEOREM 7. *Let $\mathscr{B}$ be a $\varkappa$-sequence. Then there is a $\varkappa$-sequence $\mathscr{A}$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ and either $a_4 = 6$ or $a_4 \geqslant 28$ such that $\varrho(\mathscr{A}) \geqslant \varrho(\mathscr{B})$.*

Proof. In view of Theorem 6, it may be assumed that $b_1 = 1$, $b_2 = 2$ and $b_3 = 4$. If $b_4 \geqslant 28$, then there is nothing further to prove. It follows from (2.6) with $k = 2$ that $b_4 \geqslant 6$. Thus the proof deals only with b_4 in the range $6 \leqslant b_4 \leqslant 27$.

Lemma 3 is now invoked with $h = 3$, $w = 4$, $r = 5$ and $m = 10$. This provides a $\varkappa$-sequence $\mathscr{A}$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ and $a_4 = 6$ such that

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) = \left(\frac{1}{6} - \frac{1}{t}\right) - \sum_{p=1}^{t-6} \frac{1}{b_{5p+10}}$$

where $t = b_4$ (and thus $6 \leqslant t \leqslant 27$). From (2.6) with $k = 3$, it follows that $b_{5p+10} \geqslant 20p + 24$, hence

$$\varrho(\mathscr{A}) - \varrho(\mathscr{B}) \geqslant \left(\frac{1}{6} - \frac{1}{t}\right) - \sum_{p=1}^{t-6} \frac{1}{20p+24}.$$

By direct computation, one determines that the last expression is non-negative for $6 \leqslant t \leqslant 27$, which completes the proof.

6. A refined estimate for μ. Letting $\mathscr{A}$ be a $\varkappa$-sequence, the objective here is to obtain an upper bound for $\varrho(\mathscr{A})$ which improves upon (3.17). In view of Theorem 7, it is assumed throughout this section that $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ and either $a_4 = 6$ or $a_4 \geqslant 28$. It then follows that $A(1) = 1$, $A(2) = 2$, $A(3) = 2$ and $A(4) = 3$, hence

$$\sum_{r=0}^{2} \varrho(r) = \sum_{n=1}^{4} \frac{A(n)}{n(n+1)} = \frac{23}{20}. \tag{6.1}$$

An application of Theorem 3 with $N = 6$ and $M = 6$ yields

$$\varrho(\mathscr{A}) \leqslant \sum_{r=0}^{6} \varrho(r) + \frac{1}{8}\sqrt{\frac{179}{7}} + \frac{1}{4} + \sum_{i=1}^{6}\left(\varrho(r_i) - \frac{\gamma}{r_i(r_i+1)}\right) \tag{6.2}$$

where $\gamma = \frac{1}{16}\sqrt{1253} \approx 2.2$ and $r_i \geqslant 7$ for $i = 1, \ldots, 6$. Combining (6.1) and (6.2),

$$\varrho(\mathscr{A}) < 2.0322 + \sum_{n=5}^{64} \frac{A(n)}{n(n+1)} + \sum_{i=1}^{6}\left(\varrho(r_i) - \frac{\gamma}{r_i(r_i+1)}\right). \tag{6.3}$$

For estimating $\varrho(\mathscr{A})$, two cases are now considered.

Case I. $a_4 \geqslant 28$. In this case, $A(n) = 3$ for $5 \leqslant n \leqslant 27$. For $n > 27$, $A(n) \leqslant A(n-1)+1$, hence $A(n) \leqslant n-24$. But it also follows from (2.6) that $A(n) \leqslant n/4+4$. Thus

$$A(n) \leqslant \min(n-24, [\tfrac{1}{4}n+4]) = s_n \quad (\text{for } n > 27),$$

hence

$$\sum_{n=5}^{64} \frac{A(n)}{n(n+1)} \leqslant \sum_{n=5}^{27} \frac{3}{n(n+1)} + \sum_{n=28}^{64} \frac{s_n}{n(n+1)} < .75. \tag{6.4}$$

Next, Lemma 1 with $k = 3$ provides

$$\varrho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{4} + \left(\frac{4}{2^r} - \frac{\gamma}{r(r+1)}\right).$$

Recalling that $\gamma \approx 2.2$, it follows that $\dfrac{4}{2^r} - \dfrac{\gamma}{r(r+1)} < 0$ for $r \geqslant 7$, hence

$$\varrho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{4} \quad (\text{for } r \geqslant 7). \tag{6.5}$$

From (6.3), (6.4) and (6.5),

$$\varrho(\mathscr{A}) < 2.0322 + .75 + \tfrac{3}{2}\log 2 < 3.84 \tag{6.6}$$

in the case $a_4 \geqslant 28$.

Case II. $a_4 = 6$. From (2.6) with $k = 2, 3, 4$ it follows that

$$A(n) \leqslant \tfrac{1}{3}n+2, \quad A(n) \leqslant \tfrac{1}{4}n+4 \quad \text{and} \quad A(n) \leqslant \tfrac{1}{5}n+6.$$

Since $A(n)$ must also be an integer, it follows that

(6.7)
$$\sum_{n=5}^{64} \frac{A(n)}{n(n+1)} \leqslant \sum_{n=5}^{23} \frac{[\frac{1}{3}n+2]}{n(n+1)} + \sum_{n=24}^{39} \frac{[\frac{1}{4}n+4]}{n(n+1)} + \sum_{n=40}^{64} \frac{[\frac{1}{5}n+6]}{n(n+1)} < 1.0926.$$

As in Case I, (6.5) holds here too (since it uses only the fact that $a_3 = 4$ and does not depend on a_4). But it also follows from Lemma 1 with $k = 4$ that

$$\varrho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{5} + \left(\frac{6}{2^r} - \frac{\gamma}{r(r+1)}\right).$$

In this case $\frac{6}{2^r} - \frac{\gamma}{r(r+1)} < 0$ for $r \geqslant 8$ (where $\gamma \approx 2.2$), thus

$$\varrho(r) - \frac{\gamma}{r(r+1)} < \frac{\log 2}{5} \quad (\text{for } r \geqslant 8). \tag{6.8}$$

Noting that $r_1 \geqslant 7$ and $r_i \geqslant 8$ for $i \geqslant 2$, combining (6.5) and (6.8) yields

$$\sum_{i=1}^{6} \left(\varrho(r_i) - \frac{\gamma}{r_i(r_i+1)} \right) < \frac{\log 2}{4} + \frac{5 \log 2}{5} = \frac{5}{4} \log 2 < .875. \tag{6.9}$$

From (6.3), (6.7) and (6.9) it then follows that

$$\varrho(\mathscr{A}) < 2.0322 + 1.0926 + .875 = 3.9998 \tag{6.10}$$

for the case $a_4 = 6$.

The two estimates (6.6) and (6.10) allow us to conclude $\mu \leqslant 3.9998$, hence

$$\mu < 4. \tag{6.11}$$

7. Concluding remarks. Throughout, various questions were posed. Of these, the primary problem is that of determining the precise value of λ and those sum-free sequences $\mathscr{A}$ such that $\varrho(\mathscr{A}) = \lambda$ (if they exist). In contrast with Theorem 4, it would be of interest to prove that if a sum-free sequence $\mathscr{A}$ is such that $\varrho(\mathscr{A})$ is sufficiently close to λ, then $a_1 = 1$. In connection with $\varkappa$-sequences, the problem of determining whether or not $\varrho(\mathscr{Q}) \leqslant \mu$ remains unanswered. If this were resolved, one would have a much better estimate of μ than that provided by (6.11), hence a much better estimate of λ.

Perhaps the difficulty in determining μ results from the fact that throughout, sequences were constrained to be integer-valued. We conclude by posing a problem which avoids such a constraint. Consider the class of real-valued sequences $\mathscr{A}$ with counting function $A(x)$ and terms $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots$ which satisfy

$$A(x) \leqslant \frac{x}{k} + a_k \quad (k = 1, 2, 3, \ldots;\ x \geqslant 0). \tag{7.1}$$

What is the best bound for $\varrho(\mathscr{A})$ over this class of sequences?

References

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О поведении функций $\mathrm{Re}\{\zeta(s)\}$, $\mathrm{Im}\{\zeta(s)\}$ в критической полосе

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1. Пусть

$$\begin{cases} s = \sigma + it, \ \sigma = \frac{1}{2} + \delta, \\ 0 < \delta_1 \leqslant \delta \leqslant \delta_2 \leqslant \frac{1}{4} - \Delta, \ \Delta < \frac{1}{4}, \end{cases} \tag{1}$$

и $Q(\delta)$ обозначает промежуток

$$T < t < T + (2\pi)^{-\delta} T^{\Delta+\delta} \psi(T), \tag{2}$$

где $\psi(T)$ — сколь угодно медленно возрастающая к $+\infty$ функция. Пусть, дальше,

$$S(a, b) = \sum_{a \leqslant n < b \leqslant 2a} e^{it \ln n}, \quad b \leqslant \sqrt{t/2\pi}, \tag{3}$$

обозначает элементарную тригонометрическую сумму (ср. [2], стр. 34). Положим

$$V(t, \delta) = \mathrm{Im}\{\zeta(\tfrac{1}{2} + \delta + it)\}. \tag{4}$$

Покажем, что имеет место

Теорема 1. *Если*

$$|S(a, b)| < A(\Delta) \sqrt{a}\, t^{\Delta}, \tag{5}$$

то для каждого $\delta \in \langle \delta_1, \delta_2 \rangle$ *существует* $T_1(\Delta, \psi, \delta) > 0$, *такое, что при* $T \geqslant T_1$ *промежуток* $Q(\delta)$ *содержит значение* $\bar{t}$ *для которого*

$$V(\bar{t}, \delta) = 0. \tag{6}$$

Пусть

$$U(t, \delta) = \mathrm{Re}\{\zeta(\tfrac{1}{2} + \delta + it)\}. \tag{7}$$

Относительно этой функции имеет место

Теорема 2. *Если* $S(a, b)$ *удовлетворяет условию* (5), *то для каждого* $\delta \in \langle \delta_1, \delta_2 \rangle$ *существует* $T_2(\Delta, \psi, \delta) > 0$, *такое, что при* $T \geqslant T_2$ *промежуток* $Q(\delta)$ *содержит значение* $\bar{\bar{t}}$, *для которого*

$$U(\bar{\bar{t}}, \delta) = 1. \tag{8}$$