Comparing handle decompositions of homotopy equivalent manifolds

by

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Abstract. Denote by $C(M, \partial M)$ the chain complex associated with a handle decomposition of a smooth or PL manifold M. Our main theorem deals with homotopy equivalent manifolds M and X, $\dim M = \dim X > 4$. Given a handle decomposition of X, there exist a handle decomposition of M and a chain homomorphism $f\colon C(M, \partial M) \to C(X, \partial X)$ which is onto, and $\ker f$ is an h-cobordism (acyclic subcomplex). The torsion of $\ker f$ is computed. A corollary: the Morse number μM is an invariant of simple homotopy type for $\dim M \geqslant 6$ and the kth Morse number $\mu_k M$ is a homotopy invariant for $\dim M > 6$, $k = 0, 1, \ldots, \dim M$.

1. Introduction. Given a nice handle decomposition of a manifold M, we can associate with it, in a certain natural way, a chain complex

$$C(M, \partial M) = \{C_q(M, \partial M), \partial_q\}$$

consisting of free, based modules over the integer group ring $Z\pi_1 M$. In this paper we study $C(M, \partial M)$ for compact smooth or PL manifolds of the same homotopy type. For this purpose we define a class of chain complexes, so called algebraic decompositions, and prove that the class of chain complexes associated with handle decompositions of a fixed closed manifold is equal to a simple homotopy equivalence class of algebraic decompositions. This approach has made Milnor's duality theorem [7] applicable in the proof of our main result.

Our main results are geometrical in character and deal with the situation where there are given a homotopy equivalence of pairs $f\colon (X^n, \partial X^n) \to (M^n, \partial M^n)$ and a handle decomposition of M and we are looking for a handle decomposition of X with the associated chain complex similar to that of M as closely as possible. Thus, following the ideas of Kervaire [3], we prove the stable theorem: there exists a handle decomposition of X such that f induces base-preserving isomorphisms $C_q(X, \partial X)$ onto $C_q(M, \partial M)$ for $q \leqslant n-4$. Concerning handles of indices exceeding $n-4$, we prove the non-stable theorem: there exist a handle decomposition of X and a chain homomorphism $C(X, \partial X) \to C(M, \partial M)$ which is a base-preserving isomorphism modulo an acyclic subcomplex of $C(X, \partial X)$. The torsion of that subcomplex is expressed by the torsion of f and that of $f|\partial M$; in the closed case it reduces to the torsion of f.

For a smooth manifold M the Morse number μM (qth Morse number $\mu_q M$) is defined as the minimum over all Morse functions f on M of the number of critical points of f (the number of critical points of index q). For a simply-connected manifold M of dimension $n \geq 6$ Morse numbers can be expressed in terms of homology of M [11] but, as homology spheres show, this is no longer true for manifolds with nontrivial fundamental groups. However, our results imply the following corollary: if $n \geq 6$, then μM^n depends only on the simple homotopy type of M^n, and if $n > 6$, then $\mu_q M^n$ depends only on the homotopy type of M^n.

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2. Preliminaries. For the reader's convenience we list some symbols used in the sequel.

$D^n = \{x \in R^n: \|x\| \leq 1\}$, $S^{n-1} = \partial D^n$,

$p_X: \tilde{X} \to X$ universal covering of a space X,

$Z\pi$ integral group ring of a group π,

$(x_1, \ldots, x_k)$ submodule (subgroup) generated by the set $\{x_1, \ldots, x_k\}$. If $x_1, \ldots, x_k$ are not elements of a definite module (group), the symbol denotes the free module (group) freely generated by $\{x_1, \ldots, x_k\}$.

$(a_1, \ldots, a_k;\ R_1, \ldots, R_s)$ presentation of a group with generators $a_1, \ldots, a_k$ and relators $R_1, \ldots, R_s \in (a_1, \ldots, a_k)$.

$H_*(\ )$ homology with coefficients Z, i.e., the integers.

A decomposition of a manifold means a handle decomposition of M which starts on a collar of ∂M if the boundary is nonempty and which is nice (i.e., the indices of handles increase and handles of the same index are disjoint), polar (i.e., there is one handle of the maximal index, one handle of index 0 if $\partial M = \emptyset$, and no handles of index 0 if $\partial M \neq \emptyset$) and oriented (i.e., for each handle of index i there are chosen orientations of $D^i \times D^{n-i}$ and D^i). The union of all handles of indices $\leq i$ is denoted by M_i. In the terminology of handle theory we follow [3].

If a homology equivalence class of manifolds is fixed, then π denotes the fundamental group. More precisely, for every manifold M in that class and every base point $x \in M$ we assume a fixed isomorphism $\pi_1(M, x) \to \pi$.

The chain complex $C(M, \partial M) = \{C_q(M, \partial M), \partial_q\}$ is associated with a decomposition of a compact manifold M if

$$C_q(M, \partial M) = H_q(p_M^{-1} M_q, p_M^{-1} M_{q-1}),$$

and $\partial_q: C_q(M, \partial M) \to C_{q-1}(M, \partial M)$ is the connecting homomorphism of the triple $p_M^{-1} M_q$, $p_M^{-1} M_{q-1}$, $p_M^{-1} M_{q-2}$. Homology groups of $C(M, \partial M)$ are isomorphic to singular homology groups $H_q(\tilde{M}, \partial\tilde{M})$ [1].

For any handle h in M the set of handles in $\tilde{M}$ covering h is invariant under the action of π on $\tilde{M}$. Choosing one handle over each q-handle in M, we get a $Z\pi$-base of $C_q(M, \partial M)$. Such a base will be called a *preferred base*. In the sequel if we say that an isomorphism between two complexes preserves preferred bases, we mean that each preferred base from the first complex goes on to a preferred base of the second.

Let us recall some important operations on bases which are realizable by changes of decomposition. Suppose there is a preferred base in $C_q(M, \partial M)$ and let T be the matrix transforming another base of $C_q(M, \partial M)$ into the preferred one. We say that the change of bases with the matrix T can be realized (by a change of decompositions) if there is a decomposition of M with the associated complex $C'(M, \partial M)$ and an isomorphism

$$\varphi = \{\varphi_i\}: C'(M, \partial M) \to C(M, \partial M)$$

induced by id_M such that the matrix of φ_i (in some preferred bases) is equal to T for $i = q$ and to the identity matrix for $i \neq q$.

LEMMA 1. *Let $\{a_1, \ldots, a_t\}$ be a preferred base of $C_q(M, \partial M)$. Then the sets*

$$\{a_i, a_2, \ldots, a_{i-1}, a_1, a_{i+1}, \ldots, a_t\}, \quad 1 \leq i \leq t, \tag{1}$$

$$\{\pm a_1 g, a_2, \ldots, a_t\}, \quad g \in \pi, \tag{2}$$

$$\{a_1 + a_2, a_2, \ldots, a_t\}, \tag{3}$$

are bases of $C_q(M, \partial M)$. Changes (1) *and* (2) *are realizable and if $2 \leq q \leq n-2$, then also change* (3) *is realizable.*

Lemma 1 is that version of the Adding Lemma (cf. [10], 6.7 and p. 89; for a simply-connected version see [1], Proposition 9.4) which we need. In other words, it states that a change of bases with matrix T is realizable if T represents the zero elements in $\mathrm{Wh}\,\pi$ the Whitehead group of π [7].

COROLLARY. *The following changes of the matrix of ∂_q are realizable*:

(R_1) *permutation of rows*,

(C_1) *permutation of columns*,

(R_2) *multiplication of a row by $\pm g$, $g \in \pi$*,

(C_2) *multiplication of a column by $\pm g$, $g \in \pi$*,

(R_3) *addition of a row to another one, $2 \leq q < n-2$*,

(C_3) *addition of a column to another one, $2 < q \leq n-2$.*

In fact, the change (i) in $C_q(M, \partial M)$ induces (R_i) on ∂_q and (C_i) on ∂_{q+1} for $i = 1, 2, 3$. We shall show this for $i = 3$, the only case which needs a computation. The matrix of ∂_q can be described as follows [7]: if $\partial_q a_k^q = \sum_j a_j^{q-1} \alpha_{kj}$, then the coefficient at $g \in \pi$ in α_{kj} is equal to the intersection number of the left sphere of a_k^q with the right sphere of $a_j^{q-1} g$, computed in $\partial(a_j^{q-1} g)$ (a_k^q, a_j^{q-1} denote the elements of preferred bases of C_q or C_{q-1} as well as handles in $\tilde{M}$). The attaching map of a handle determining $a_1^q + a_2^q$ is formed by "piping" the left spheres of a_1^q and a_2^q with an arc. Assumptions on q in (R_3) and (C_3) show that an arc can be chosen such that its lifting to $\tilde{M}$ connects prescribed handles and that it can be isotoped to make it disjoint with the right sphere of any $(q-1)$-handle. Hence the coefficient at g in $\partial_q(a_1^q + a_2^q)$ is equal to the sum of coefficients at g in $\partial_q a_1^q$ and $\partial_a a_2^q$.

3. The stable theorem. Consider a decomposition of a compact manifold M^n. let $\check{\partial} M_q$ denote $\partial M_q - (\partial M \cup \bigcup_i \operatorname{Im} \varphi_i)$, where φ_i range over the attaching maps of $(q+1)$-handles. Let $f\colon S^q \to \check{\partial} M_q$ be an embedding and $\tilde{f}\colon S^q \to p_M^{-1} M_q$ its lifting, where $2 \leqslant q < n-3$. The following two lemmas are due to Kervaire [3] (Lemma 2 is equal to Kervaire's Lemma 5 and the considerations following its proof in [3] yield Lemma 3).

LEMMA 2. *For any* $a \in C_{q+1}(M, \partial M)$ *the embedding* $f\colon S^q \to \check{\partial} M_q$ *is isotopic in* ∂M_{q+1} *to an embedding* g *such that* $[\tilde{g}] = [\tilde{f}] + \partial_{q+1} a$ *in* $H_q(p_M^{-1} M_q, p_M^{-1} M_{q-1})$.

LEMMA 3. *If a generator of* $C_q(M, \partial M)$ *is given by a* q*-handle* h^q *and belongs to* $\operatorname{Im} \partial_{q+1}$, *then* h^q *can be replaced by a* $(q+2)$*-handle, i.e., the decomposition can be changed in a way such that* h^q *disappears and instead there appears a new* $(q+2)$*-handle.*

THEOREM 1. *Let* X^n, M^n *be compact connected manifolds and let* $f\colon (X, \partial X) \to (M, \partial M)$ *be a homotopy equivalence of pairs. For any decomposition of* M *there exist a decomposition of* X *and a homomorphism* $\varphi = \{\varphi_q\}\colon C(X, \partial X) \to C(M, \partial M)$ *induced by* f *such that* φ_q *are isomorphisms and preserve preferred bases for* $q \leqslant n-4$. *Fundamental groups of* X *and* Y *are identified via homomorphism induced by* f.

Proof. The proof is by induction on q. The case $q = 0$ follows by the assumption that the decomposition is polar. If $n \leqslant 4$, then either the theorem has no meaning or the only case to consider is $q = 0$. Thus for the rest of the proof we can assume $n > 4$.

Case $q = 1$. Let $\partial X = Y_1 \cup \ldots \cup Y_k$ and $\partial M = N_1 \cup \ldots \cup N_k$ be unions of components. Assume first that

(4) all 1-handles in M besides $h_2, \ldots, h_k$ are trivially attached to a collar of N_1 and h_j joins N_1 with N_j for $j = 2, \ldots, k$.

If g is a homotopy inverse to f, then it induces an isomorphism $\pi_1(M_0 + h_2 + \ldots + h_k, *) \to \pi_1 Y_1 * \ldots * \pi_1 Y_k$ (on the right $*$ denotes the free product). Let $(a_1, \ldots, a_\alpha;\ R_1, \ldots, R_t)$ be a presentation of $\pi_1(M_0 + h_2 + \ldots + h_k, *)$ and let

$$(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta;\ R_1, \ldots, R_t, Q_1, \ldots, Q_u) \tag{5}$$

be a presentation given by the decomposition under consideration (see [2]). Presentation (5) can be regarded as a presentation of $\pi_1 X$ via the isomorphism $g_\#$ induced by g. By [2] there exists a decomposition of X which determines that presentation of $\pi_1 X$. Let b_i correspond to ξ_i in $\pi_1(X, *)$ and to η_i in $\pi_1(M, *)$. Then $f_\# \xi_i = f_\# g_\# [b_i] = (fg)_\# \eta_i = \eta_i$. Since ξ_i and η_i are represented by left discs of some 1-handles, one can change f by a homotopy $\operatorname{rel} M$ such that the induced homomorphism $f\colon C_1(X, \partial X) \to C_1(M, \partial M)$ is an isomorphism preserving preferred bases.

In general, one can first pass to a decomposition satisfying (4), then find — as above — the needed decomposition of X, and finally change both decompositions in order to obtain the initial decomposition of M.

Case $2 \leqslant q \leqslant n-4$. Let $\{a_1, \ldots, a_r, a_{r+1}, \ldots, a_t\}$ be a preferred base of $C_q(X, \partial X)$ and let $\{b_1, \ldots, b_s, b_{s+1}, \ldots, b_u\}$ be a preferred base of $C_q(M, \partial M)$. Suppose $\partial_q a_i \neq 0$ if and only if $i \leqslant r$ and $\partial_q b_i \neq 0$ if and only if $i \leqslant s$. The map f induces a homomorphism $f_c\colon C(X, \partial X) \to C(M, \partial M)$ which, by the inductive assumption, is an isomorphism of $C_i(X, \partial X)$ for $i < q$. This implies that the set $\{f_c \partial_q a_1, \ldots, f_c \partial_q a_r\}$ generate $\operatorname{Im} \partial_q^M$. Thus there exist $w_{jk} \in Z\pi$ (π is the common fundamental group of X and M) such that

$$\partial_q b_j = \sum_k (f_c \partial_q a_k) w_{jk} = \partial_q f_c \big(\sum_k a_k w_{jk}\big), \quad j = 1, \ldots, s. \tag{6}$$

One can also find $x_j \in \ker \partial_q \subset C_q(X, \partial X)$ satisfying

$$f_*[x_j] = \big[\sum f_c(a_k) w_{jk} - b_j\big], \quad j = 1, \ldots, s, \tag{7}$$

because the homomorphism induced by f_c in homology is an isomorphism.

Add to X_q s trivial handles $h_1, \ldots, h_s$ of index q and s handles $g_1, \ldots, g_s$ of index $q+1$ such that h_i and g_i are complementary for $i = 1, \ldots, s$. A little more precise description of that addition shows that the resulting decomposition is nice. Add to X_{q-1} all the old handles of index q and trivially attached handles $h_1, \ldots, h_s$ of index q and new handles $g_1, \ldots, g_s$ with g_i complementary to h_i besides the old handles of indices $q+1$. By Lemma 1 one can change the base $h_1, \ldots, h_s, a_1, \ldots, a_t$ to the base $h'_1, \ldots, h'_s, a_1, \ldots, a_t$, where $h'_j = h_j + \sum_{k \geqslant r} a_k w_{jk} - x_j$. Consequently, $\partial_{q+1} g_j = h'_j - \sum_k a_k w_{jk} + x_j$. By (7) we get

$$f_c h'_j - b_j = f_c h'_j - \sum f_c(a_k) w_{jk} + f_c x_j = f_c\big(h'_j - \sum a_k w_{jk} + x_j\big) \equiv 0 \operatorname{mod} \operatorname{Im} \partial.$$

If $\xi \in \operatorname{Im} \partial_{q+1}$ and h is a generator of $C_q(X, \partial X)$, then f can be changed by homotopy $\operatorname{rel} X_{q-1}$ in such a way that the induced homomorphism carries h to $f_c h + \xi$. Thus there exists a map homotopic to f such that the induced homomorphism $\bar{f}_c$ is equal to f_c on $C_i(X, \partial X)$ for $i < q$ and $\bar{f}_c h'_j = b_j$.

Note that $\bar{f}_c^{-1} \partial_q b_1, \ldots, \bar{f}_c^{-1} \partial_q b_s$ generate $\operatorname{Im} \partial_q$. Hence the set $\{\partial_q h'_1, \ldots, \partial_q h'_s\}$ generates $\operatorname{Im} \partial_q$. By the addition of linear combinations of handles $h'_1, \ldots, h'_s$ to a_i (the addition allowed by Lemma 1) we change the attaching map of a_i in such a way that $\partial_q a_i = 0$.

Thus we have reduced the proof to the case where $r = s$ and f_c sends a_i to b_i for $i \leqslant r$. Let $\{a_{r+1}, \ldots, a_t, p_1, \ldots, p_k\}$ be a set of generators of $\ker \partial_q^X$, where $p_i \in (a_1, \ldots, a_r)$. Then

$$(b_{r+1}, \ldots, b_u, f_c p_1, \ldots, f_c p_k) = \ker \partial_q^M. \tag{8}$$

In fact, each $x \in \ker \partial_q$ can be written as $y + z$, where $z \in (b_1, \ldots, b_r)$ and $y \in (b_{r+1}, \ldots, b_u)$. Hence

$$z = \sum_{i \leqslant r} b_i \lambda_i = \sum_{i \leqslant r} f_c a_i \lambda_i = f_c\big(\sum_{i \leqslant r} a_i \lambda_i\big),$$
$$0 = \partial x = \partial z = f_c \partial\big(\sum_{i \leqslant r} a_i \lambda_i\big).$$

Moreover, $\partial(\sum_{i\leqslant r} a_i\lambda_i) = 0$ (for f_c is an isomorphism on $C_{q-1}(X, \partial X)$) and $\sum_{i\leqslant r} a_i\lambda_i = \sum p_j\mu_j$ (because the set $\{p_1, \ldots, p_k\}$ generates $\ker\partial_q \cap (a_1, \ldots, a_r)$). We get $z = \sum f_c p_j \mu_j$, which proves (8).

Since f_* is an isomorphism, it follows from (8) that there exist y_{jk}, $y'_{jl} \in Z\pi$ such that in homology

$$[b_j] = f_*(\sum_{k>r} [a_k] y_{jk} + \sum_l [p_l] y'_{jl}) .$$

Change the decomposition of X by the addition of complementary pairs $\bar{h}_i$, g_i of handles of indices q and $q+1$, $i = r+1, \ldots, u$. Replace, by Lemma 1, the handle $\bar{h}_j$ by the handle

$$h_j = \bar{h}_j + \sum_{k>r} a_k y_{jk} + \sum_l p_l y'_{jl}, \quad j = r+1, \ldots, u .$$

Then $\partial g_j = h_j - \sum_{k>r} a_k y_{jk} - \sum_l p_l y'_{jl}$. These operations lead to a decomposition with the following properties:

(9) $$f_c h_j \equiv b_j \bmod \operatorname{Im}\partial, \quad j>r,$$

(10) $$([a_{r+1}], \ldots, [a_t]) \subset ([h_{r+1}], \ldots, [h_u], [p_1], \ldots, [p_k]) .$$

The formula

$$f_c h_j - b_j \equiv f_c h_j - \sum_{k>r} f_c a_k y_{jk} - \sum_l f_c p_l y'_{jl} = f_c(\partial g_j) \equiv 0 \bmod \operatorname{Im}\partial$$

justifies (9). By (8) there exist $x_{jk}, x'_{jl} \in Z\pi$ such that

(11) $$f_*[a_j] = \sum_{k>r} [b_k] x_{jk} + \sum_l f_*[p_l] x'_{jl} = \sum_{k>r} f_*[h_k] x_{jk} + \sum_l f_*[p_l] x'_{jl} = f_*(\sum_{k>r} [h_k] x_{jk} + \sum_l [p_l] x'_{jl})$$

and since f_* is an isomorphism, we get (10).

From (9) it follows that f is homotopic in M_{q+1} to a map which sends h_i to b_i, $i = r+1, \ldots, u$. By Lemma 1 one can replace the base $\{a_1, \ldots, a_t, h_{r+1}, \ldots, h_u\}$ by the base

$$\{a_1, \ldots, a_r, a'_{r+1}, \ldots, a'_t, h_{r+1}, \ldots, h_u\},$$

where $a'_i = a_i - \sum_{k>r} h_k x_{jk} - \sum_l p_l x'_{jl}$, $i = r+1, \ldots, t$. By (11), the handles belong to $\operatorname{Im}\partial_{q+1}$. Hence we can use Lemma 3 to replace them by $(q+2)$-handles. The proof is complete.

COROLLARY. *Let M^n be a closed manifold. If $3<q$ or $q<n-3$, then the Morse numbers $\mu_q M^n$ depend only on the homotopy type of M^n. In particular, if $n>6$, then each $\mu_q M^n$ is homotopy invariant.*

4. Algebraic decompositions. By an algebraic decomposition we shall mean a chain complex with the properties which guarantee good relations to decompositions of closed manifolds. In particular, each chain associated with a decomposition of a closed manifold will be an algebraic decomposition. Somewhat more precisely, if we have a chain complex associated with a decomposition of a manifold M^n, $n \geqslant 6$, and an algebraic decomposition which is simply homotopy equivalent to the former, then there exists a decomposition of M^n for which the latter is an associated chain complex (see Corollary A in the next section). In this section we shall prove a result analogous to Theorem 1.

Given a homomorphism ε: $\pi \to Z_2 = \{+1, -1\}$, we have the involution $*$: $Z\pi \to Z\pi$ defined by $g^* = \varepsilon(g) g^{-1}$ for $g \in \pi$. For any right $Z\pi$-module R, the involution provides a right $Z\pi$-module structure on $\operatorname{Hom}_{Z\pi}(R, Z\pi)$ by the formula $(\varphi\lambda)(x) = \lambda^*\varphi(x)$. For a $Z\pi$-homomorphism f: $R \to S$ we have the dual homomorphism f^*: $S^* \to R^*$ and for a base of R we have the dual base of R^*. In the sequel we shall assume that ε is the orientation homomorphism of the fundamental group π (i.e., the first Stiefel–Whitney class).

An algebraic decomposition of dimension n is a chain complex $D = \{D_i, \delta_i\}$ of free $Z\pi$-modules with a class of preferred bases such that:

(I) for any preferred base $\{a_1, \ldots, a_k\}$ of D_i, a base $\{b_1, \ldots, b_k\}$ of D_i is preferred if and only if there exist $g_1, \ldots, g_k \in \pi$ such that $b_j = \pm a_j g_j$ for $j = 1, \ldots, k$,

(II) if $D_i \neq 0$, then $0 \leqslant i \leqslant n$,

(III) $D_0 \simeq Z\pi \simeq D_n$ (it is assumed that $\pm\pi \subset Z\pi$ corresponds to preferred generators of D_0 and D_n),

(IV) there is a preferred base A of D_1 such that $\omega(a) = 1 - \delta_1 a \in \pi$ for $a \in A$ and if $\hat{A}$ is the group freely generated by A, then the induced homomorphism ω: $\hat{A} \to \pi$ is an epimorphism,

(V) the dual complex $D = \{D^*_{n-i}, (-1)^i \delta^*_{n-i}\}$ also satisfies (IV).

THEOREM 2. *Let M^n be a closed, connected manifold with the fundamental group π, $C'M$ a complex associated to a decomposition of M, and $D = \{D_i, \delta_i\}$ an algebraic decomposition of $Z\pi$-modules. If f: $C'M \to D$ is a chain homotopy equivalence and $T \leqslant n-4$, then there exist a decomposition of M with the associated complex CM and a homomorphism g: $CM \to D$ such that:*

(a) *g_i is a base-preserving isomorphism for $i \leqslant T$,*

(b) *there exists a homomorphism θ: $CM \to C'M$ induced by id_M and such that the diagram*

$$\begin{tikzcd} CM \arrow[rr, "g"] \arrow[dr, "\theta"'] & & D \\ & C'M \arrow[ur, "f"'] & \end{tikzcd}$$

commutes up to chain homotopy.

Proof. As in Theorem 1, we can assume $n>4$. By (IV), the coset $1 + \operatorname{Im}\delta_1$ is a generator of $H_0 D \simeq Z\pi/\operatorname{Im}\delta_1 \simeq Z$. We can assume (changing, if necessary, the orientations of handles with indices 0 and 1) that f carries a preferred generator of $C'_0 M$ onto $1 + u \in D_0$ with $u \in \operatorname{Im}\delta_1$. Hence f is homotopic to a map sending the generator of $C'_0 M$ to 1.

Consider the presentation $\omega\colon \hat{A}\to\pi$ of the group π determined by D and a decomposition of M giving that presentation. Let CM be the complex associated with that decomposition, and let $\theta\colon CM\to C'M$ be a homomorphism induced by id_M and preserving generators with index 0.

If the generator $\tilde{a}\in C_1M$ corresponds to $a\in A$, then $\delta_1(f\theta\tilde{a}-a) = f\theta(\delta_1\tilde{a}-\delta_1 a) = 0$, since $(f\theta)_0$ preserves generators and $\partial\tilde{a} = 1-\omega(a)$. Hence $a\equiv f\theta(\tilde{a}) \bmod \ker\delta_1$ and it follows that $f\theta$ is homotopic to a homomorphism whose restriction to C_1M is a base-preserving isomorphism. The rest of the proof is identical to that of the case $2\leqslant q\leqslant n-4$ in Theorem 1.

Remark. The considerations in the proof of Theorem 2 are valid also for manifolds with boundary, provided we consider dual decompositions to those considered above, i.e., provided we consider decompositions of the triad $(M;\emptyset,\partial M)$ on $\emptyset$. Since Theorem 2 is concerned with the stable situation, the condition (V) as well as the isomorphism $D_n\simeq Z\pi$ are superfluous.

5. Nonstable theorem in the closed case. The results of the previous section will now be used to prove our main theorem. The proof is roughly as follows. Consider dual complexes C^*X and C^*M, which are homotopy equivalent, and apply Theorem 2. We get, for $T>3$, a decomposition of X such that the associated chain complexes CX and CM are isomorphic in dimensions $\geqslant T$. By Theorem 1 we may consider C_qX and C_qM as isomorphic for $q\leqslant T-2$. There remain two dimensions only in which the decompositions of X and M may differ. The addition of handles in those dimensions completes the proof.

We start with some definitions. Let $f=\{f_i\}\colon \{C_i, d_i\}\to\{C_i', d_i'\}$ be a homomorphism of based chain complexes of free $Z\pi$-modules. The cone of f is the complex $Cf = \{C_{i-1}\oplus C_i', d_i - d_{i-1}' + f_{i-1}\}$. If f is a homotopy equivalence, then Cf is acyclic and the torsion $\tau(Cf)\in \mathrm{Wh}(\pi)$ is defined (cf. [9], part III). By definition, the torsion τf of f is equal to $\tau(Cf)$ and a homotopy equivalence is called simple if $\tau f = 0$.

Consider a split exact sequence of based free $Z\pi$-modules

$$0\to K\to L\to M\to 0\,. \tag{12}$$

If $\underline{k} = \{k_1,\ldots,k_i\}\subset K$, $\underline{m} = \{m_1,\ldots,m_j\}\subset M$ and $\underline{l} = \{l_1,\ldots,l_{i+j}\}\subset L$ are preferred bases, then lifting every element m_t to $\tilde{m}_t\in L$ gives a base

$$\underline{km} = \{k_1,\ldots,k_i,\tilde{m}_1,\ldots,\tilde{m}_j\}$$

of L. We say that the sequence (12) *splits in the category of based modules* if the matrix determined by passing from $\underline{l}$ to $\underline{km}$ represents the zero element in $\mathrm{Wh}(\pi)$. The definition does not depend on the choice of liftings (see [7]).

A split exact sequence of chain complexes of based free $Z\pi$-modules $0\to A\to B\to C\to 0$ is said to *split in the category of based complexes* if the sequence

$$0\to A_i\to B_i\to C_i\to 0$$

splits in the category of based modules for any i.

LEMMA 4. *If a homotopy equivalence $F=\{F_i\}\colon C=\{C_i,\partial_i\}\to D=\{D_i,\partial_i\}$ is an isomorphism for $i\geqslant T$, then there exists a homomorphism homotopically reverse to F and equal to F_i^{-1} for $i\geqslant T$.*

Proof. Choose a homomorphism $f\colon D\to C$ which is homotopically reverse to F. There exists a $w\colon D\to D$ homotopic to $1-Ff$ such that $w_i=0$ for $i<T-1$ and $w_i = 1-F_if_i$ for $i>T-1$, and there is a homotopy $\varDelta\colon w\sim 0$ such that $\varDelta_i=0$ for $i<T-1$. Since $(F_i^{-1})^*$ carries $\mathrm{Im}\,\partial^*$ into $\mathrm{Im}\,\partial^*$, there exists a homomorphism φ making the following diagram commute:

$$\begin{tikzcd} C_{T-1}^* \arrow[r, "\varphi"] \arrow[d, "\partial^*"'] & D_{T-1}^* \arrow[d, "\partial^*"] \\ C_T^* \arrow[r, "(F_T^{-1})^*"'] & D_T^* \end{tikzcd} \tag{13}$$

Define $G\colon D\to C$ as F_i^{-1} for $i\geqslant T$, φ^* for $i=T-1$, and the zero homomorphism for $i<T-1$. Since F_i and ∂ commute, (13) yields

$$\partial_iG_i = G_{i-1}\partial_i \quad \text{for} \quad i>T-1\,. \tag{14}$$

Define $g\colon D\to C$ by:

$$g_i = \begin{cases} G_iw_i & \text{for} \quad i\geqslant T-1\,,\\ 0 & \text{for} \quad i<T-1\,.\end{cases}$$

We shall verify that

$$\partial_ig_i = g_{i-1}\partial_i \quad \text{for each } i\,. \tag{15}$$

If $i\geqslant T$, this follows from (14) and the commuting of w and ∂. By definition, $\varDelta_i=0$ for $i<T-1$ and $w_i = \partial_{i+1}\varDelta_i + \varDelta_{i-1}\partial_i$; hence $w_{T-1} = \partial\varDelta$. Since $G_{T-1}=\varphi^*$ preserves $\mathrm{Im}\,\partial$, we have $\partial g_{T-1}=0$, which proves (15).

Now (14) implies that

$$g_i = \partial_{i+1}G_{i+1}\varDelta_i + G_i\varDelta_{i-1}\partial_i \qquad \text{for} \quad i\geqslant T-1\,,$$

$$g_i = 0 = \partial_{i+1}G_{i+1}\varDelta_i + G_i\varDelta_{i-1}\partial_i \quad \text{for} \quad i<T-1\,.$$

Thus the map $G_{i+1}\varDelta_i$ gives a homotopy $G\varDelta\colon g\sim 0$. Since $g_i = F_i^{-1}-f_i$ for $i\geqslant T$, the homomorphism $f+g$ satisfies the lemma.

THEOREM 3. *Let X^n be a closed connected manifold and let $D=\{D_i,\delta_i\}$ be an algebraic decomposition of dimension $n>4$. If there is a homotopy equivalence $f^1\colon C^1X\to D$, where C^1X is the chain complex associated with a decomposition of X, then there exist a decomposition of X with the associated complex CX and a homomorphism $g\colon CX\to D$ such that*

(a) *there exists a simple homotopy equivalence φ such that the diagram*

$$\begin{tikzcd} CX \arrow[r, "g"] \arrow[dr, "\varphi"'] & D \\ & C^1X \arrow[u, "f^1"'] \end{tikzcd}$$

commutes,

(b) *the sequence* $0\to\ker g\to CX\xrightarrow{g}D\to 0$ *is exact and splits in the category of based complexes* (*the base of* $\ker g$ *is inherited from* CX),

(c) $\ker g$ *is acyclic,*

(d) $\tau(\ker g)=\tau f$.

Proof. If $f\colon C\to D$ is a homotopy equivalence, then $\tau f^* = (-1)^{\dim C+1}(\tau f)^*$ (see [7], § 10). In particular, a homotopy equivalence dual to a simple homotopy equivalence is simple.

Let F^1 be a homomorphism homotopically reverse to f and $k+1>3$. By Theorem 2 and the duality theorem [7] there is a decomposition of X with the associated complex C^2X and a homomorphism $(F^2)^*\colon (C^2X)^*\to D^*$ which is an isomorphism for $i\leqslant n-k-1$. In fact, there is even a homotopy commutative diagram

$$\begin{tikzcd}
(C^1X)^* \arrow[rr, "(F^1)^*"] \arrow[dr, "A"'] & & D^* \\
& (C^2X)^* \arrow[ur, "(F^2)^*"'] &
\end{tikzcd}$$

where A is a simple homotopy equivalence. By dualizing this diagram and passing to homotopically reverse homomorphisms we get a homotopy commutative diagram

$$\begin{tikzcd}
C^1X \arrow[rr, "f^1"] \arrow[dr, "\varphi^1"'] & & D \\
& C^2X \arrow[ur, "f^2"'] &
\end{tikzcd}$$

with φ^1 a simple homotopy equivalence. By Lemma 4 one can assume f_i^2 to be isomorphism for $i\geqslant k+1$. Applying Theorem 2 for $T=k-2$, we get a decomposition of X equal to the preceeding one in dimensions exceeding k and a homomorphism $f_i^3\colon C^3X\to D$ equal to f^2 on C_iX for $i>k$; f_i^3 are isomorphisms for $i\neq k, k-1$.

Add pairs of complementary handles of indices $k-1$ and k, and, in the same way as in the proof of Theorem 1, change the decomposition in such a way that the homomorphism $f^4\colon C^4X\to D$ determines a 1-1 correspondence between new $(k-1)$-handles and generators of D_{k-1}. By further changes old handles are replaced by handles from $\ker f_{k-1}^4$. They form a base of $\ker f_{k-1}^4$, for f_{k-1}^4 now carries generators either to generators or to zero. Call the resulting chain complex C^5X and the new homomorphism f^5. By construction, f_i^5 are epimorphisms for $i\neq k$ and isomorphisms for $i\neq k-1, k$. We shall show that $H^*(\operatorname{coker} f^5)=0$. Since f^5 induces epimorphisms $\psi_i\colon H_i(C^5X/\ker f^5)\to H_iD$ for all i, we infer, by considering the long exact sequence induced by the exact sequence $0\to C^5X/\ker f^5\to D\to\operatorname{coker} f^5\to 0$, that it remains to prove that $\psi_{k-1}\colon H_{k-1}(C^5X/\ker f^5)\to H_{k-1}D$ is a monomorphism. If a coset $a+\ker f_{k-1}^5$ represents an element of $\ker\psi_{k-1}$, then $f^5a\in\operatorname{Im}\partial$ and $\partial a\in\ker f_{k-1}^5$. Since f_{k-2}^5 is an isomorphism, $\partial a=0$ and therefore $a\in\operatorname{Im}\partial$, for f^5 is a homotopy equivalence. This implies that $a+\ker f_{k-1}^5$ belongs to the image of the differential of $C^5X/\ker f^5$. Note that also the homology of $\ker f^5$ is trivial. If f_i^5 are epimorphisms for $i\neq k$ and $H_*(\operatorname{coker} f^5)=0$, then also f_k^5 must be an epimorphism. Hence we get the exact sequence $0\to\ker f^5\to C^5X\to D\to 0$. The module $\ker f_k^5$ is free, for $\ker f_{k-1}^5$ is free and $\partial|\ker f_k^5=\partial^0\colon \ker f_k^5\to\ker f_{k-1}^5$ is an isomorphism.

Consider the base of $\ker f_k^5$ carried from $\ker f_{k-1}^5$ via ∂^0. That base and the base of D_k determine a new base of C_k^5X. Let A be the matrix which transforms a preferred base to that new one and let t denote the number of k-handles. Add to X_{k-1} t complementary pairs of handles, i.e., add to C_k^5X the complex

$$\ldots\to 0\to F_k\xrightarrow{d}F_{k-1}\to 0\to\ldots$$

where $\operatorname{rank}F_k=\operatorname{rank}F_{k-1}=t$ and d preserves bases.

Now change the preferred base of $C_k^5X\oplus F_k$ by the matrix

$$B=\begin{bmatrix}A^{-1} & 0\\ 0 & A\end{bmatrix}$$

and denote the resulting complex by C^6X. The composition of the embedding $C^5X\to C^5X\oplus F$ and the isomorphism $C^5X\oplus F\to C^6X$ is a homotopy equivalence; call it η. The torsion of η is represented by B, and hence is zero. The homomorphism $f^6=f^5\eta^{-1}\colon C^6X\to D$ carries any generator to a generator or to zero. Since the matrix of $\partial^0\colon \ker f_k^6\to f_{k-1}^6$ is non-degenerate, one can put the matrix of ∂_k^6, by the addition of appropriate linear combinations of generators of $\ker f_k^6$ to the remaining generators, in the form

$$\begin{bmatrix}\partial^0 & 0\\ 0 & N\end{bmatrix}.$$

In this way we obtain a chain complex CX associated to with a decomposition of X and a homomorphism $g\colon CX\to D$ satisfying conditions (a), (b), (c). We shall check that it satisfies also (d). If $\varphi\colon \ker g\to 0$ is the zero homomorphism and $\psi=g\colon CX/\ker g\to D$, then $\tau g=\tau\varphi+\tau\psi$ by the additivity of torsion [7]. Since $\tau\psi=0$ and $\tau\varphi=\tau(\ker g)$, we get $\tau(\ker g)=\tau g=\tau f$.

The proof is complete.

Corollary A. *Under the assumptions of Theorem* 3, *if moreover* $\tau f=0$ *and* $\dim X\geqslant 6$, *then there is a base-preserving isomorphism* $g\colon CX\to D$.

In fact, if $a_1,\ldots,a_r$ are handles corresponding to the generators of $\ker g$, then the triad obtained by the addition of $a_1,\ldots,a_r$ to a collar on

$$\partial(X_{k-2}+h_1^{k-1}+\ldots+h_p^{k-1})$$

($h_1^{k-1},\ldots,h_p^{k-1}$ denote the remaining $(k-1)$-handles) is, by the acyclity of $\ker g$, an h-cobordism. Provided $\tau(\ker g)=\tau f=0$, it follows from the s-cobordism theorem (cf. [10]) that this h-cobordism is trivial. The handles $a_1,\ldots,a_r$ may be collected in such an h-cobordism (for they have indices equal to $k-1$ or to k and the decomposition is nice), and hence they may be removed.

Corollary B. *Let $f\colon X^n \to M^n$ be a homotopy equivalence of closed manifolds and $n>4$. Then for any decomposition of M there exist a decomposition of X and a homomorphism $g\colon CX \to CM$ such that:*

(a) *the sequence $0 \to \ker g \to CX \to CM \to 0$ is exact and splits in the category of based complexes,*

(b) $\ker g$ *is acyclic,*

(c) $\tau(\ker g) = \tau f$.

Corollary C. *Let X^n, M^n be closed manifolds. If $n>6$, X^n and M^n are homotopy equivalent, then $\mu_q X^n = \mu_q M^n$ for each q. If $n \geqslant 6$, X^n and M^n are simply homotopy equivalent, then $\mu X = \mu M$ and $\mu_q X = \mu_q M$ for each q.*

Remark. For orientable manifolds of dimension $n>6$ the second part of Corollary C can be derived also from a theorem of Mazur ([5], Theorem II or [6]).

By the results of Kirby and Siebenmann [4] we also have:

Corollary D. *If manifolds X^n and M^n are closed and (topologically) homeomorphic, $n \geqslant 6$, then $\mu X^n = \mu M^n$ and $\mu_q X^n = \mu_q M^n$.*

6. Nonstable theorem in the general case. Now we shall prove a general theorem, valid for compact manifolds, which, if the boundaries of manifolds are empty, reduces to Corollary B. The proof of that theorem is essentially that of Theorem 3 and differs from it only by some considerations concerning the boundary. We omit details which seem to be routine.

Theorem 4. *Let X^n and M^n be compact connected manifolds of dimension $n>4$ with connected boundaries and let $f\colon (X, \partial X) \to (M, \partial M)$ be a homotopy equivalence of pairs. For any decomposition of M there exist a decomposition of X with the associated chain complex $C(X, \partial X)$ and a homomorphism $g\colon C(X, \partial X) \to C(M, \partial M)$ such that:*

(a) *the sequence $0 \to \ker g \to C(X, \partial X) \to C(M, \partial M) \to 0$ is exact and splits in category of based complexes,*

(b) $\ker g$ *is acyclic,*

(c) $\tau(\ker g) = \tau f - i_* \tau(f|\partial X)$, *where $i_*\colon \mathrm{Wh}(\pi_1 \partial M) \to \mathrm{Wh}(\pi_1 M)$ is the homomorphism induced by inclusion.*

Proof. By Theorem 1 there exists a decomposition of X such that f induces a homomorphism $f_c\colon C^1(X, \partial X) \to C(M, \partial M)$, $(f_c)_i$ being isomorphisms up to $i = n-4$. By the remark at the end of Section 4, we can apply Theorem 2 to the dual decomposition. The same considerations as in the proof of Theorem 3 show that there exist a decomposition of X and a homomorphism $g\colon C(X, \partial X) \to C(M, \partial M)$ satisfying conditions (a) and (b) in Theorem 4 as well as the equality

$$\tau(\ker g) = \tau f_c .$$

In the PL category the CW-complex W associated with a decomposition of M (cf. [10]) contains ∂M as the subcomplex. In the smooth category one can fix a C^1-simplicial structure of ∂M and change the attaching maps in order to obtain

a CW-complex with ∂M as a subcomplex. Denote by W^q the qth skeleton of W. Let

$$C_q M = H_q(p_W^{-1} W^q, p_W^{-1} W^{q-1}) \quad \text{and} \quad C_q^0 M = H_q(p_W^{-1}(\partial M)^q, p_W^{-1}(\partial M)^{q-1});$$

p_W denotes the universal covering map. The modules $C_q M$, $C_q^0 M$ with connecting homomorphisms of the respective triple form chain complexes CM, $C^0 M$. There is an isomorphism

$$C_q M \simeq C_q(M, \partial M) \oplus C_q^0 M .$$

The map f induces a chain homotopy equivalence $\varphi\colon CX \to CM$ such that $\varphi|C(X, \partial X) = f$. Let $\varphi^0 = \varphi|C^0 M$. It is not hard to verify that

$$\tau f = \tau\varphi = \tau f_c + \tau\varphi^0,$$

$$\tau\varphi^0 = i_* \tau(f|\partial X) .$$

Hence (c) is satisfied.

Corollary. *Under the assumptions of Theorem 4 there is a base-preserving isomorphism $g\colon C(X, \partial X) \to C(M, \partial M)$ provided $\tau f = i_* \tau(f|\partial X)$, and $n \geqslant 6$.*

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