Remarks on generic models

by

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Abstract. Two forcing free characterizations are given for generic structures. The height and Scott height of generic models are also discussed.

Intuitively, a generic structure is one which is conceived of as being, in some sense, representative. This notion of being representative may in turn be loosely thought of as comprised of both a positive and negative component. The positive component, which appears to be the simpler of the two, requires that the structure satisfy some condition, e.g. that it be a model for some theory. The negative component requires that the structure not be unusual or peculiar in certain respects. In short, a generic structure is envisioned as being a "typical" structure satisfying certain properties.

In this paper we use the term generic in a very precise technical way which is fairly standard by now. One of our principal objectives here is to show that these generic structures are indeed "generic" in an intuitive sense. The first task is, of course, to capture the essential content and flavor of the intuitive notion within the formal framework. In practice this proved to be the main difficulty. Once the correct conjecture was formulated, the proof was found directly.

It is natural to expect that, in some sense, "typical" structures be quite similar, and perhaps even indistinguishable. However, "typical" structures ought not, in general, to be necessarily of the same isomorphism type. Then, of course, assuming they are countable, such structures would always be distinguishable from one another, for example, by their Scott sentences. This observation is in no way surprising since one only expects generic structures to be typical within a certain limited context. More precisely, one usually discusses D-generic structures, for some set D, and the Scott sentences of D-generic structures need not be in D.

In turns out that generic structures are those structures which contain no "unusual" elements nor tuples of elements. Hence, within the given context, all generic structures will have the same "local" structure, and thus will be indistinguishable in that context.

Having concluded our very informal and sketchy, but hopefully somewhat heuristic discussion, we proceed to our formal investigation, culminating in Theorem 1.4 below.

§ **1.** We assume the reader is familiar with some of the basic properties of infinitary logic and admissible sets as presented, for example, in [7].

By the *height of a structure* we shall mean $o(A)$, the least ordinal not in A, where A is the smallest admissible set containing the structure (and hence the alphabet for the structure etc.). For uniformity we may assume that the elements of the structure are urelements, so that the height of a structure depends only on its isomorphism type. For any set x, x^+ denotes the smallest admissible set containing x.

Given a structure $\mathfrak{M}$, and tuples $\vec{m}$ and $\vec{n}$ of elements of $\mathfrak{M}$, we write $\vec{m}\sim^{\alpha}\vec{n}$ iff $\vec{m}$ and $\vec{n}$ satisfy in $\mathfrak{M}$ the same formulas of quantifier rank $\leqslant\alpha$. By the *Scott height of* $\mathfrak{M}$ we mean the least ordinal α such that for any tuples $\vec{m}$ and $\vec{n}$ of elements of $\mathfrak{M}$, $\vec{m}\sim^{\alpha}\vec{n}$ implies $\vec{m}\sim^{\alpha+1}\vec{n}$. Hence, if $\mathfrak{M}$ has Scott height α, the canonical Scott sentence of M has quantifier rank $\alpha+\omega$. It is shown in [7] that the Scott height of a structure is less than or equal to its height.

Familiarity is also assumed with the basic notions of forcing in infinitary logic, as presented in [4] or [5]. For simplicity we consider forcing properties of the form $\boldsymbol{P}(\mathscr{L}_B, T)$, where T is a theory in the fragment $\mathscr{L}_B$ and $P(\mathscr{L}_B, T)$ is the set of all finite sets of sentences of $\mathscr{K}_B$, the language obtained from $\mathscr{L}_B$ by finite substitutions of constants from some set C of new constants, holding in models $(\mathfrak{M}, m_c)_{c\in C}$ of T. Later, we will discuss the case in which we replace the fragment $\mathscr{L}_B$ by a more general set of sentences. In particular we will not repeat the definitions of $p \Vdash \varphi$, read "p forces φ" or of $p \Vdash^w \varphi$, read "p weakly forces φ". Given a theory $T\subseteq\mathscr{L}_B$, *a structure* $\mathfrak{M}$ *is said to be an A-generic model of* T iff there is a mapping of the structure onto some set of new constants C, such that for each sentence φ of $\mathscr{K}_A$, $(\mathfrak{M}, m_c)_{c\in C} \models \varphi$ iff there is some finite $p\subseteq \mathrm{Th}_B((\mathfrak{M}, m_c)_{c\in C})$, the complete theory of $(\mathfrak{M}, m_c)_{c\in C}$ in $\mathscr{K}_B$, such that $p \Vdash \varphi$ in the forcing property $\boldsymbol{P}(\mathscr{L}_B, T)$ formed with respect to C. For the purposes of this paper, it is not necessary to require that generic structures be countable. Then, for any condition p, and sentence φ of $\mathscr{K}_A$, for $\mathscr{L}_A$ countable, $p \Vdash^w \varphi$ iff φ holds in every A-generic structure $(\mathfrak{M}, m_c)_{c\in C}$ such that $p\subseteq \mathrm{Th}_B((\mathfrak{M}, m_c)_{c\in C})$. From now on, we confuse $\mathfrak{M}$ and $(\mathfrak{M}, m_c)_{c\in C}$, and always assume we have some mapping of the forcing constants onto the structure in mind.

We use $\vec{x}$ and $\vec{y}$ to represent arbitrary finite sequences of variables and $\vec{c}$ to represent an arbitrary finite sequence of forcing constants. We write $\varphi(\vec{x})$, resp. $\varphi(\vec{c})$; to mean that the free variables, resp. forcing constants of $\varphi(\vec{x})$, resp. $\varphi(\vec{c})$, are among the elements of $\vec{x}$, resp. $\vec{c}$. We assume the existence of some conveniently definable association between the variables and the forcing constants. In particular, $\vec{x}$ is associated with $\vec{c}$, and $\vec{y}$ with $\vec{d}$.

By an $\mathscr{L}_B$-type with respect to T we mean a set Φ of formulas of $\mathscr{L}_B$ whose free variables are among some finite set of variables, which is satisfied in some

model of T. If a type is denoted by $\Phi(\vec{x})$, it is assumed that the free variables of the formulas in $\Phi(\vec{x})$ are among those in $\vec{x}$. $\Phi(\vec{x})$ is said to be a *k-type* if $\vec{x}$ is a sequence of k variables. For convenience we assume $\vec{x}$ has length k for some arbitrary positive number k. An $\mathscr{L}_B$-type $\Phi(x)$ is said to be *principal* if there is some formula $\varphi_0(\vec{x})$ of $\mathscr{L}_B$ consistent with T such that $T\models \varphi_0(\vec{x})\to\varphi(\vec{x})$ for every $\varphi(\vec{x})\in\Phi(\vec{x})$. Otherwise a type is said to be *non-principal* with respect to T. A complete $\mathscr{L}_B$-type $\Phi(\vec{x})$ with respect to T is one which contains, for each formula $\varphi(\vec{x})$ of $\mathscr{L}_B$, either $\varphi(\vec{x})$ or $\neg\varphi(\vec{x})$, i.e. is the set of all formulas satisfied by some k-tuple of elements of a model of T. Since T remains fixed we omit mention of it, and say simply non-principal type, or complete type, etc.

The result below, for the case of finite forcing, was already mentioned by Robinson [8], and stated by MacIntyre [6] and Simmons [9].

THEOREM 1.1. *A structure* $\mathfrak{M}$ *is an A-generic model of T iff*

$$\mathfrak{M}\models(\forall\vec{x})[\varphi(\vec{x})\to\bigvee\{(\exists\vec{y})p(\vec{x},\vec{y})\colon\ p(\vec{c},\vec{d})\Vdash\varphi(c)\ \textit{for}\ \vec{d}\ \textit{arbitrary}\}],$$

for every formula $\varphi(\vec{x})$ *of* $\mathscr{L}_A$.

The proof of Theorem 1.1 is immediate from the definition of generic. Moreover one can replace "$\Vdash$" by "$\Vdash^w$" in Theorem 1.1.

Remark 1.2. If A is admissible and $\boldsymbol{P}\in A$, then, since "$\Vdash$" is defined by a Δ-recursion on A, it follows that for each $\varphi\in\mathscr{L}_A$, the sentence mentioned in Theorem 1.1 is again in $\mathscr{L}_A$. Hence, the A-generic models over $\boldsymbol{P}$ are axiomatizable by a set of sentences of $\mathscr{L}_A$. This is in sharp contrast to the non-axiomatizability result we shall obtain in the next section.

Our next result gives a forcing free characterization of A-generic models of T in the special case that T is a complete theory in $\mathscr{L}_B$. We remind the reader of a simple lemma [cf. 5] which states that, without assuming T complete, $p \Vdash^w\varphi$ iff $T\cup p\models\varphi$, for any $p\in\boldsymbol{P}$ and $\varphi\in\mathscr{K}_B$. We also need to observe that if, given $p(\vec{c})$ and $\varphi(\vec{c})$, there is some condition $q\supseteq p(\vec{c})$, such that $q\Vdash^w\varphi(\vec{c})$, then there is a condition $r(\vec{c})\supseteq p(\vec{c})$ such that $r(\vec{c})\Vdash^w\varphi(\vec{c})$. This is easily established by an induction on $\varphi(\vec{c})$, which we omit. Consequently, for any condition $p(\vec{c})$ and sentence $\varphi(\vec{c})$, $p(\vec{c})\Vdash^w\neg\varphi(\vec{c})$ iff for every condition $q(\vec{c})\supseteq p(\vec{c})$, not $q(\vec{c})\Vdash^w\varphi(\vec{c})$. Similar phenomena occur for disjunctions and existential quantifiers. It should be noted that in Theorem 1.3, the theory $\mathrm{Th}_A(\mathfrak{M})$ is in the language $\mathscr{L}_A$, rather than $\mathscr{K}_A$.

THEOREM 1.3. *Let T be complete for* $\mathscr{L}_B$. *Let* $\mathscr{L}_A$ *be any fragment extending* $\mathscr{L}_B$. *Then* $\mathfrak{M}$ *is an A-generic model of T iff* $\mathfrak{M}\models T$ *and for each formula* $\varphi(\vec{x})$ *of* $\mathscr{L}_A$

$$\mathfrak{M}\models(\forall\vec{x})\big(\varphi(\vec{x})\leftrightarrow\bigvee\{\theta(\vec{x})\in\mathscr{L}_B\colon\ \mathrm{Th}_A(\mathfrak{M})\models\theta(\vec{x})\to\varphi(\vec{x})\}\big).$$

Proof. One direction follows easily from Theorem 1.1 since if $\mathfrak{M}$ is A-generic, and $p(\vec{c},\vec{d})\Vdash\varphi(\vec{c})$, then $\mathrm{Th}_A(\mathfrak{M})\models(\exists\vec{y})\bigwedge p(\vec{x},\vec{y})\to\varphi(\vec{x})$.

To establish the reverse implication it suffices from Theorem 1.1 with "$\Vdash^w$", to show that for every $p(\vec{c}) \in \boldsymbol{P}$ and $\varphi(\vec{c}) \in \mathscr{K}_A$,

$$(*) \qquad T^* \cup p(\vec{c}) \models \varphi(\vec{c}) \quad \text{iff} \quad p(\vec{c}) \Vdash^w \varphi(\vec{c}),$$

where we use T^* as an abbreviation for $\mathrm{Th}_A(\mathfrak{M})$.

We first show that for any condition $p(\vec{c})$ and sentence $\varphi(\vec{c})$ of $\mathscr{K}_A$,

$$(**) \quad T^* \cup p(\vec{c}) \models \neg\varphi(\vec{c}) \text{ iff for all conditions } q(\vec{c}) \supseteq p(\vec{c}), \text{ not } T^* \cup q(\vec{c}) \models \varphi(\vec{c}).$$

One direction is trivial, i.e. if $T^* \cup p(\vec{c}) \models \neg\varphi(\vec{c})$, then not $T^* \cup q(\vec{c}) \models \varphi(\vec{c})$ for all conditions $q(\vec{c}) \supseteq p(\vec{c})$, since T is complete.

For the reverse direction, assume that not $T^* \cup q(\vec{c}) \models \varphi(\vec{c})$, for every condition $q(\vec{c}) \supseteq p(\vec{c})$. If it is not true that $T^* \cup p(\vec{c}) \models \neg\varphi(\vec{c})$ then

$$T^* \models (\exists \vec{x})\big(\bigwedge p(\vec{x}) \mathbin{\&} \varphi(\vec{x})\big).$$

Then under the general hypothesis, there is some $\theta(\vec{x}) \in \mathscr{L}_B$ such that

$$\mathfrak{M} \models \theta(\vec{c}) \quad \text{and} \quad T^* \models \theta(\vec{x}) \to \big(\bigwedge p(\vec{x}) \mathbin{\&} \varphi(\vec{x})\big).$$

Now, let $q(\vec{c}) = p(\vec{c}) \cup \{\theta(\vec{c})\}$. Clearly $q(\vec{c})$ is a condition, and we thus have,

$$T^* \cup q(\vec{c}) \models \varphi(\vec{c})$$

for some condition $q(\vec{c}) \supseteq p(\vec{c})$, a contradiction.

By substituting $\neg\varphi$ for φ we obtain

$$(***) \quad T^* \cup p(\vec{c}) \models \varphi(\vec{c}) \text{ iff for all conditions } q(\vec{c}) \supseteq p(\vec{c}), \text{ not } T^* \cup q(\vec{c}) \models \neg\varphi(\vec{c}).$$

Now we proceed to prove $(*)$ by induction on the formation of φ.

The step for φ atomic does not require the use of $(**)$ or $(***)$, but does require the hypothesis that T is complete. This is the only other step in which the completeness of T is needed. For φ atomic,

$$\begin{aligned} T^* \cup p(\vec{c}) \models \varphi(\vec{c}) \quad &\text{iff} \quad T^* \models (\forall \vec{x})[\bigwedge p(\vec{x}) \to \varphi(\vec{x})] \\ &\text{iff} \quad T \models (\forall \vec{x})[\bigwedge p(\vec{x}) \to \varphi(\vec{x})], \text{ since } \\ &\phantom{\text{iff}} \quad T \text{ is complete for } \mathscr{L}_B, \\ &\text{iff} \quad T \cup p(\vec{c}) \models \varphi(\vec{c}) \\ &\text{iff} \quad p(\vec{c}) \Vdash^w \varphi(\vec{c}), \text{ by our observation} \end{aligned}$$

above, since $\varphi \in \mathscr{K}_B$.

If $\varphi(\vec{c})$ is $\neg\psi(\vec{c})$, then, using $(**)$

$$\begin{aligned} T^* \cup p(\vec{c}) \models \neg\psi(\vec{c}) \quad &\text{iff} \quad \text{for all conditions } q(\vec{c}) \supseteq p(\vec{c}) \\ &\phantom{\text{iff}} \quad \text{not } T \cup q(\vec{c}) \models \psi(\vec{c}) \\ &\text{iff} \quad \text{(by induction hypothesis) for all} \\ &\phantom{\text{iff}} \quad \text{conditions } q(\vec{c}) \supseteq p(\vec{c}), \text{ not } q(\vec{c}) \Vdash^w \psi(\vec{c}) \\ &\text{iff} \quad p(\vec{c}) \Vdash^w \neg\psi(\vec{c}) \text{ by the observation} \end{aligned}$$

preceding the theorem.

If φ is $\bigvee \Psi(\vec{c})$, then using $(***)$,

$$\begin{aligned} T^* \cup p(\vec{c}) \models \bigvee \Psi \quad &\text{iff} \quad \text{for all conditions } q(\vec{c}) \supseteq p(\vec{c}) \\ &\phantom{\text{iff}} \quad \text{not } T^* \cup q(\vec{c}) \models \neg\bigvee \Psi(\vec{c}). \\ &\text{iff} \quad \text{for all conditions } q(\vec{c}) \supseteq p(\vec{c}) \text{ there} \\ &\phantom{\text{iff}} \quad \text{is a } \psi \in \Psi \text{ and a condition } r(\vec{c}) \supseteq q(\vec{c}) \\ &\phantom{\text{iff}} \quad \text{such that } T^* \cup r(\vec{c}) \models \psi(\vec{c}) \\ &\text{iff} \quad \text{(by induction hypothesis) for all conditions} \\ &\phantom{\text{iff}} \quad q(\vec{c}) \supseteq p(\vec{c}) \text{ there is a } \psi \in \Psi \text{ and a condition} \\ &\phantom{\text{iff}} \quad r(\vec{c}) \supseteq q(\vec{c}) \text{ such that } r(\vec{c}) \Vdash^w \psi(\vec{c}) \\ &\text{iff} \quad p \Vdash^w \bigvee \Phi, \text{ by the observation} \end{aligned}$$

preceeding the theorem.

The step for the existential quantifier is similar and we omit it. ■

We will see in Remark 2.2 below, that the hypothesis that T is complete cannot be omitted in Theorem 1.3. There is, however, a more general result, namely, the analog of Theorem 1.3 in which $\mathrm{Th}_A(\mathfrak{M})$ is replaced by

$$\begin{aligned} T_A^f &= \{\varphi\colon \varphi \text{ is a sentence of } \mathscr{L}_A \text{ with } 0 \Vdash^w \varphi\} \\ &= \{\varphi\colon \varphi \text{ is a sentence of } \mathscr{L}_A \text{ true in every } A\text{-generic model of } T\}. \end{aligned}$$

The proof of this result is more direct than the proof of Theorem 1.3 since in this case, $(*)$ follows immediately. Perhaps more interesting is Theorem 1.3′, and its corollary which we state next.

THEOREM 1.3′. *Let T be a theory in $\mathscr{L}_B$. Let $\mathscr{L}_A$ be any fragment extending $\mathscr{L}_B$. Then T_A^f is the unique theory T' in $\mathscr{L}_A$ satisfying*

(i) *For each sentence θ of $\mathscr{L}_B$, $T \models \theta$ iff $T' \models \theta$.*

(ii) *For each sentence θ of $\mathscr{L}_A$, if $T' \models \theta$, then $\theta \in T'$.*

(iii) *For each formula $\varphi(\vec{x})$ of $\mathscr{L}_A$ consistent with T', there is a formula $\theta(\vec{x})$ of $\mathscr{L}_B$ consistent with T such that*

$$T' \models \theta(\vec{x}) \to \varphi(\vec{x}).$$

Proof. The verifications that T_A^f satisfy (i), (ii) and (iii) are straightforward. For example, (iii) can be established as follows.

Suppose the formula $\varphi(\vec{x})$ of $\mathscr{L}_A$ is consistent with T_A^f. Then, since not $T_A^f \models \neg(\exists \vec{x})\varphi(\vec{x})$, $(\exists \vec{x})\varphi(\vec{x})$ holds in some A-generic structure $\mathfrak{M}$. Hence, for some condition $p(\vec{c}, \vec{d})$, satisfied in $\mathfrak{M}$, $p(\vec{c}, \vec{d}) \Vdash^w \varphi(\vec{c})$. Thus, in every A-generic model, if $\bigwedge p(\vec{c}, \vec{d})$ holds, $\varphi(\vec{c})$ holds. Then, we must have $T_A^f \models (\exists \vec{y}) \bigwedge p(\vec{x}, \vec{y}) \to \varphi(\vec{x})$.

The proof that T_A^f is the unique theory on $\mathscr{L}_A$ satisfying (i), (ii) and (iii), proceeds by again showing $(*)$ as in the proof of Theorem 1.3, but with T' replacing T^*. The argument is almost the same as in that proof, except that (i) is used instead of the completeness of T, while (iii) is used rather than the hypothesis in Theorem 1.3.

COROLLARY. *Let T be a theory in $\mathscr{L}_B$. Let $\mathscr{L}_A$ be any fragment extending $\mathscr{L}_B$. Then T_A^f is the unique theory T' in $\mathscr{L}_A$ satisfying* (i) *and* (ii) *above and such that $\mathfrak{M}$ is an A-generic model of T iff for each formula $\varphi(\vec{x})$ of $\mathscr{L}_A$,*

$$\mathfrak{M} \models (\forall \vec{x})\big(\varphi(\vec{x}) \leftrightarrow \bigvee \{\theta(\vec{x}) \in \mathscr{L}_B \colon T' \models \theta(\vec{x}) \to \varphi(\vec{x})\}\big).$$

Now limiting our attention to admissible A, we state our primary result, which may be viewed as a formalization of our intuitive introductory remark.

THEOREM 1.4. *Let A be admissible, and suppose A contains the forcing property $P = P(T, \mathscr{L}_B)$, where T is a theory in the fragment $\mathscr{L}_B$. Then a structure is an A-generic model of T iff it is a model of T and omits every non-principal $\mathscr{L}_B$-type in A.*

Proof. That an A-generic model of T is a model of T and omits every non-principal $\mathscr{L}_B$-type in A is already apparent from [4] and to this we refer the reader. We will prove the converse.

Suppose $\mathfrak{M} \models T$ and $\mathfrak{M}$ omits every non-principal $\mathscr{L}_B$-type in A. We fix some one to one assignment of the elements of $\mathfrak{M}$ onto a set C of new constant symbols, and show that for any sentence $\varphi \in \mathscr{K}_A$,

$$\mathfrak{M} \models \varphi \quad \text{iff} \quad (\exists p \in G)[p \Vdash \varphi],$$

where $G = \mathrm{Th}_B(\mathfrak{M}, m_c)_{c \in C}$. Before proceeding to prove this by induction on the formation of φ, we make one crucial observation.

Let $\varphi(\vec{c})$ be a sentence of $\mathscr{K}_A$ and define

$$\Theta(\vec{x}) = \{\neg(\exists \vec{y}) \wedge p(\vec{x}, \vec{y}) \colon p(\vec{c}, \vec{d}) \Vdash \varphi(\vec{c}) \text{ or } p(\vec{c}, \vec{d}) \Vdash \neg\varphi(\vec{c}), \text{ where } \vec{d} \text{ is arbitrary.}\}$$

We claim that Θ is not realized in $\mathfrak{M}$. By our hypothesis it is sufficient to assume that Θ is a type and show that it is in A and non-principal.

Since $P \in A$, and A is admissible, Θ is easily seen to be an element of A, as in Remark 1.2.

We assume next that, on the contrary, Θ is principal, and derive a contradiction. Suppose there is a formula $\theta(\vec{x})$, consistent with T, such that $T \models \theta(\vec{x}) \to \neg(\exists \vec{y}) \wedge p(\vec{x}, \vec{y})$, for each formula $\neg(\exists \vec{y}) \wedge p(\vec{x}, \vec{y}) \in \Theta$. Let $p = \{\theta(\vec{c})\}$. Then $p \in P$. Now, directly from the definition of forcing, there is some $q(\vec{c}, \vec{d}) \in P$ such that $q(\vec{c}, \vec{d}) \supseteq p$ and either $q(\vec{c}, \vec{d}) \Vdash \varphi(\vec{c})$ or $q(\vec{c}, \vec{d}) \models \neg\varphi(\vec{c})$. However, this means that $\neg(\exists \vec{y}) \wedge q(\vec{x}, \vec{y}) \in \Theta$. Hence, we arrive at the desired contradiction, viz. $T \cup p(\vec{c}) \models \neg q(\vec{c}, \vec{d})$, and the claim is established. Stated slightly differently, we have shown

$$(*) \qquad \mathfrak{M} \models (\forall \vec{x}) \vee \{(\exists \vec{y}) \wedge p(\vec{x}, \vec{y}) \colon p(\vec{c}, \vec{d}) \Vdash \varphi \quad \text{or} \quad p(\vec{c}, \vec{d}) \Vdash \neg\varphi, \text{ for } \vec{d} \text{ arbitrary.}\}$$

It is fairly well-know that $(*)$ is really sufficient since it implies that for each $\varphi \in \mathscr{K}_A$, there is a p in $\mathrm{Th}_B(\mathfrak{M}, m_c)_{c \in C}$ which "decides" φ. However we continue anyway and proceed to the induction.

The initial step for φ an atomic sentence of $\mathscr{K}_A$ is immediate, without appealing to the claim.

Next, suppose φ is $\neg\psi$, and we have shown that $\mathfrak{M} \models \psi$ iff there is a $p \in G$ such that $p \Vdash \psi$. Then

$$\begin{aligned} \mathfrak{M} \models \varphi \quad & \text{iff not } \mathfrak{M} \models \psi \\ & \text{iff } (\forall p \in G)[\text{not } p \Vdash \psi], \text{ hence by } (*) \\ & \text{iff } (\exists p \in G)[p \Vdash \neg\psi]. \end{aligned}$$

The induction steps for φ of the form $\bigvee \Psi$ or $(\exists x)\psi(x)$ are straightforward. We verify only the second case.

$$\begin{aligned} \mathfrak{M} \models (\exists x)\psi(x) \quad & \text{iff } \mathfrak{M} \models \psi(c) \text{ for some } c \in C \\ & \text{iff (by induction hypothesis) } (\exists p \in G)[p \Vdash \psi(c) \text{ for some } c] \\ & \text{iff } (\exists p \in G)[p \Vdash (\exists x)\psi(x)]. \end{aligned}$$

This completes the induction and the proof of the Theorem.

Remark 1.5. From the preceeding proof it is clear that the $\mathscr{L}_A$ sentences given by $(*)$ are a set of axioms for the A-generic models of T. These axioms are in a simpler form than those in Remark 1.2 since we have eliminated the formula φ itself.

Remark 1.6. Some of the above results admit certain obvious generalizations. For example, in Theorems 1.1 and 1.3, A need not be a fragment, but only closed under subformulas. In addition, in Theorems 1.1 and 1.4 one need require only that $\mathscr{L}_B$ be closed under subformulas. In that case, in Theorem 1.4, rather than considering $\mathscr{L}_B$-types, one considers sets of $\forall \vee \exists$ formulas over $\mathscr{L}_B$ in the sense of [4]. In particular, then, this generalization of Theorem 1.4 includes the original finite forcing of Robinson.

§ 2. This section is devoted to the consideration of two examples which will serve to settle certain questions which arose in our investigations. In order to sharpen our conclusions for our first example we consider a complete theory T. We select T to be a theory without a prime model, for otherwise, as we observed in [5], we find ourselves in the "degenerate" situation of having a unique generic model.

The complete finitary theory T of the additive integers $\langle Z, + \rangle$, as considered in [1], will serve our purpose. We assume the reader has at least some very basic acquaintance with group theory, especially concerning torsion free abelian groups. An excellent reference for our purposes is [2].

For n a positive integer, and x a group element, we write nx to represent the sum of x added to itself $n-1$ times. If n is a negative integer then nx denotes $(-n)(-x)$ where $-x$ is the inverse of x in the group. An element x of a group G is said to be *divisible by the integer* n if there is some $y \in G$ such that $ny = x$. We abbreviate this as $n|x$, and the negation as $n \nmid x$. A group G is said to be of *rank* 1 if for any two elements x, y of G, there are integers n, m, not both zero, such that $nx+my = 0$. It is easy to see that $\langle Z, +\rangle$ has rank 1. However, the property of having rank 1 is not expressible in $\mathscr{L}_{\omega\omega}$, and there are models of T which are not of rank 1. In $\mathscr{L}_{\omega_1\omega}$, this property is easily captured by the sentence

$$(\sim)\qquad (\forall x)(\forall y) \bigvee \{nx+my = 0\colon n, m \text{ not both zero}\}\,.$$

We will be working with the forcing property $\boldsymbol{P} = \boldsymbol{P}(T, \mathscr{L}_{\omega\omega})$. We assume that A is admissible and contains $\boldsymbol{P}$ and the above sentence $(\sim)$. The smallest admissible set containing ω will serve. The reader will observe that admissibility is not really required for certain of our observations. It is to be understood that when we refer to an A-generic group below we are more precisely referring to an A-generic model of T. Our first observation is:

Every A-generic group is a torsion free Abelian group of rank 1.

The theory T itself directly insures the first two conditions. We simply show that $0 \Vdash^{w} (\sim)$. Using Theorem 2.1 of [4], it is sufficient to show that for each formula $\varphi(x, y)$ consistent with T, there are m, n not both zero such that $T \cup \{\varphi(x, y)\} \cup \cup \{nx+my = 0\}$ is consistent. However, if $\varphi(x, y)$ is consistent with T, then there are integers x_0, y_0 such that $\langle \boldsymbol{Z}, +\rangle \models \varphi(x_0, y_0)$. Furthermore, there are integers n_0, m_0, not both zero, such that $n_0x_0+m_0y_0 = 0$, whence $T \cup \{\varphi(x, y)\} \cup \cup \{n_0x+m_0y = 0\}$ is consistent, and we have completed the proof. Of course, this same result would hold if we began with any complete theory of groups admitting a group of rank 1.

It is our good fortune that the structure of torsion free Abelian groups of rank 1 is well understood, and that such groups are easily representable. With each element x of such a group G we may associate an infinite sequence S^x of length ω taking values in $\omega \cup \{\infty\}$ where $\infty \notin \omega$. The sequence S^x is defined so that the value S^x_n at the nth place in S^x is m if m is the highest power of p_n, the nth prime, which divides x, and ∞ if x is divisible by all powers of p_n. Then, using the fact that G is of rank 1, it is easy to show that for any $x, y \in G$, S^x_n, S^y_n can differ at only finitely many n, and then only if $S^x_n \neq \infty$. Consequently, each torsion free Abelian group of rank 1 is completely determined by such a sequence. Conversely, such a sequence in turn defines a torsion free Abelian group of rank 1 in a canonical way as a subgroup of the additive rationals. Of course, two sequences which differ at only finitely many places, and never with only one taking value ∞ at such a place, give rise to the same group.

Given elements x, y of the torsion free Abelian group G of rank 1, it is easy to show that x can be taken to y by an automorphism of G iff $S^x = S^y$. Equivalently, x can be taken to y by an automorphism of G iff x and y satisfy the same complete $\mathscr{L}_{\omega\omega}$ type. Similarly, using the fact that G is of rank 1, we see that a k-tuple $\vec{x}$ of elements of G can be carried coordinatewise onto the k-tuple $\vec{y}$ by an automorphism of G provided that $\vec{x}$ and $\vec{y}$ satisfy the same complete $\mathscr{L}_{\omega\omega}$ types. Given pairs $\langle x_0, x_1\rangle$ and $\langle y_0, y_1\rangle$ for example, there is an automorphism of G taking x_0 to y_0 and x_1 to y_1 iff x_0 and y_0 satisfy the same complete $\mathscr{L}_{\omega\omega}$ type and n_0y_0+

$+n_1y_1 = 0$ for some integers n_0, n_1, not both zero, such that $n_0x_0+n_1x_1 = 0$. Consequently since any group of rank 1 is countable, we see that any torsion free Abelian group of rank 1 has Scott height at most ω.

It is an easy exercise to show that any type of the form $\{x \neq 0, n_0|x, n_1|x, \ldots\}$, for infinitely many distinct n_i is non-principal with respect to T. Hence A-generic groups will never have an ∞ in their representations. Furthermore, each type of the form $\{p^{n_i}_{k_i} \nmid x\}$ for infinitely many k_i's is also seen to be non-principal with respect to T. Hence, any sequence representing an A-generic group cannot have any infinite subsequence in A.

Next, we note that from the preceeding discussion, it is clear that if the torsion free Abelian group G of rank 1 is represented by some sequence not in A, then every non-principal complete type realized by a tuple of elements of G is not in A.

In addition we observe that any torsion free Abelian group G of rank 1 represented by a sequence without ∞ is a model of T. This is seen by noting that for any prime p, G/pG is isomorphic to the integers modulo p, and appealing to the result of Szmielew [10].

Remark 2.1. Finally, we conclude that, since there are sequences without ∞ which are not in A, but which have infinite subsequences in A, the statement of Theorem 1.4 cannot in general be weakened by replacing "types" by "complete types".

Our second example will serve a dual purpose. The alphabet $\mathscr{L}$ involved will consist, in addition to the equality symbol, of the single unary relation symbol $\mathscr{U}$, and the constant symbols $\underline{0}, \underline{1}, \underline{2}, \ldots$ For the theory T we choose the following sentences of $\mathscr{L}_{\omega\omega}$:

$$\underline{n} \neq \underline{m} \quad \text{for} \quad n, m \in \omega,\ n \neq m\,,$$

$$(\exists x_0) \ldots (\exists x_{n-1})[\bigwedge_{i\neq j<n} x_i \neq x_j \mathbin{\&} \bigwedge_{i<n} \mathscr{U}(x_i)] \quad \text{for} \quad n \in \omega\,,$$

$$(\exists x_1) \ldots (\exists x_{n-1})[\bigwedge_{i\neq n<j} x_i \neq x_j \mathbin{\&} \bigwedge_{i<n} \neg\mathscr{U}(x_i)] \quad \text{for} \quad n \in \omega\,.$$

Hence, models of T consist of distinct interpretations of the constants, and perhaps other elements, such that the interpretation of $\mathscr{U}$ is infinite, as well as the set of elements of the model not in the interpretation of $\mathscr{U}$.

It is clear that every completion of T in $\mathscr{L}_{\omega\omega}$ is $\aleph_0$-categorical. Furthermore, suppose $\mathscr{L}_A$ is a fragment containing the sentence $\psi = (\forall x) \bigvee_{n\in\omega} x = \underline{n}$, $\varphi(x_0, \ldots, x_{k-1})$ is in $\mathscr{L}_A$, and $\mathfrak{M} = \langle M, U, 0, 1, \ldots\rangle$ is a model of $T \cup \{\psi\}$, i.e. an ω-model of T.

For each finite sequence s of k natural numbers, let $\theta_s(x_0, \ldots, x_{k-1}) = \bigwedge_{i<k} x_i = \underline{s_i}$. Then, in $\mathfrak{M}$, θ_s completely determines the isomorphism type of $\langle x_0, \ldots, x_{k-1}\rangle$, whence

$$\mathfrak{M} \models (\forall x_0) \ldots (\forall x_{k-1})(\varphi \leftrightarrow \bigvee \{\theta \in \mathscr{L}_{\omega\omega}\colon \mathrm{Th}_A(\mathfrak{M}) \models \theta \to \varphi\})\,.$$

However, it is clear that not every such $\mathfrak{M}$ is A-generic. One need only select U in A and appeal to Theorem 1.4.

Remark 2.2. It is now clear that in Theorem 1.3, in general we cannot drop the hypotheses that T is a complete theory.

Now, if A is any admissible set containing ω, and $\mathfrak{M} = \langle M, U, 0, 1, 2, \ldots\rangle$ is A-generic, then $\mathfrak{M}$ will be an ω-model, since the type $\{x_0 \neq \underline{0}, x_0 \neq \underline{1}, \ldots\} \in A$ is clearly non-principal with respect to T.

Conversely, from Theorem 1.4, in order that $\mathfrak{M}$ be A-generic, it is sufficient that $\mathfrak{M}$ be an ω-model, both U and $M \setminus U$ be infinite, and that neither U, nor $M \setminus U$ have any infinite subset in A.

For countable A it is quite simple to construct such models, and simultaneously incorporate extra information, via a simple diagonal argument.

Let f be any function from ω to ω, and let $S_0, S_1, \ldots$ be a listing of all subsets of ω in A which are neither finite nor cofinite. We proceed as follows:

0th Stage: (i) Put into U all numbers through first number n_0 missing from S_0.

(ii) Continue in increasing order beyond n_0 putting numbers in U until reaching the next number in S_0. We omit this number from U, as well as the next $f(0)$ numbers. Let m_0 be the last number omitted.

$(k+1)$th Stage: (i) Put into U all numbers begining with m_k through next number n_{k+1} missing from S_{k+1}.

(ii) Continue putting numbers in U until reaching the next number in S_{k+1}. We omit this number from U, as well as the next $f(k+1)$ numbers. Let m_{k+1} be the last number omitted.

It is clear that such a process can be carried out since each S_k is neither finite nor cofinite. Moreover, any U constructed in this manner can have no infinite subset in A, nor can $\omega \setminus U$. Finally, it is clear that f is recursive in U.

Remark 2.3. Using the above construction, one can, for countable admissible A, find A-generic models of arbitrarily large countable height by choosing f accordingly. In particular then, it is clear that not every A-generic model need be A-^{s}generic in the sense of [5]. Moreover, it shows that there may be no sentence φ of $\mathscr{L}_{\omega_1\omega}$ axiomatizing the set of A-^{s}generic, or A-generic models of height $\leqslant o(A)$. Let φ be such a sentence, and suppose $\varphi \in \mathscr{L}_D$, for D countable admissible. Then since every A-^{s}generic model, or every A-generic model of height $\leqslant o(A)$ would satisfy φ, it follows that $0 \Vdash^w \varphi$. In that case, all $\mathscr{L}_D$-generic models would have to satisfy φ. However, the above shows that $\mathscr{L}_D$-generic models need not have height $\leqslant o(A)$.

Finally, there may be A-generic models of height $\leqslant o(A)$ which are not A-^{s}generic. Regarding the universe of these models as ω, then the $\mathscr{U}$ of an A-^{s}generic model is seen to be a Cohen generic real over A. However, if $\mathscr{U}$ corresponds to a Solovay random real over A, then the corresponding model will also be A-generic and have height $\leqslant o(A)$. Those two classes of reals are well-known to be disjoint.

§ 3. In this section we consider the Scott height of generic structures. We assume that our forcing property $\boldsymbol{P}$ is an element of some countable admissible set A, containing ω which may just as well be thought of as the smallest admissible set containing $\boldsymbol{P}$.

In [5] we used the fact that A-^{s}generic structures have height $\leqslant o(A)$ to ascertain that they also have Scott height $\leqslant o(A)$. As we have seen earlier, A-generic structures need not have height $\leqslant o(A)$.

We assume for convenience that all structures considered are structures for the same arbitrary but fixed countable alphabet. Suppose for each countable α there were a sentence ϱ_α of $\mathscr{L}_{\omega_1\omega}$ such that for any structure $\mathfrak{M}$, $\mathfrak{M} \models \varrho_\alpha$ iff $\mathfrak{M}$ has Scott height $\leqslant \alpha$. Then, since for every A-^{s}generic structure $\mathfrak{M}$, we have $\mathfrak{M} \models \varrho_{o(A)}$, it follows that $0 \Vdash^w \varrho_{o(A)}$. Whence if $A' \supseteq A \cup \{\varrho_{o(A)}\}$ every A'-generic structure satisfies $\varrho_{o(A)}$, thus having Scott height $\leqslant o(A)$.

We now show how to construct the sentence ϱ_α. The essential idea involved is already in [3], though we noticed the fact without having Vaught sentences in mind.

First, for each $k \in \omega$, and ordinal α we construct a formula

$$\psi_k^\alpha(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1})$$

such that for any elements $m_0, \ldots, m_{k-1}, n_0, \ldots, n_{k-1}$ of the model $\mathfrak{M}$,

$$\mathfrak{M} \models \psi_k^\alpha[m_0, \ldots, m_{k-1}, n_0, \ldots, n_{k-1}] \quad \text{iff} \quad \langle m_0, \ldots, m_{k-1}\rangle \sim^\alpha \langle n_0, \ldots, n_{k-1}\rangle.$$

The formula $\psi_k^\alpha(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1})$ will say nothing about the specific type realized by $\langle x_0, \ldots, x_{k-1}\rangle$ or $\langle y_0, \ldots, y_{k-1}\rangle$, but simply that they are the same. It is for this reason that these formulas are constructible directly from the alphabet, unlike the more common canonical Scott types for which one needs a structure or a reasonable fascimile. We define recursively

$$\psi_k^0(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}) = \bigwedge \{\theta(x_0, \ldots, x_{k-1}) \leftrightarrow \theta(y_0, \ldots, y_{k-1}) \colon \theta \text{ is an atomic formula}\}$$

$$\begin{aligned}\psi_k^{\alpha'}&(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}) \\ &= (\forall x_k)(\exists y_k)\psi_{k+1}^\alpha(x_0, \ldots, x_{k-1}, x_k, y_0, \ldots, y_{k-1}, y_k) \\ &\quad \& (\forall y_k)(\exists x_k)\psi_{k+1}^\alpha(x_0, \ldots, x_{k-1}, x_k, y_0, \ldots, y_{k-1}, y_k),\end{aligned}$$

$$\psi_k^\delta(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}) = \bigwedge_{\alpha<\delta} \psi_k^\alpha(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}) \quad \text{for } \delta \text{ a limit ordinal}.$$

It is quite clear that ψ_k^α behaves as advertised.

We can now define ϱ_α to be

$$\bigwedge_{k\in\omega} (\forall x_0) \ldots (\forall x_{k-1})(\forall y_0) \ldots (\forall y_{k-1})[\psi_k^\alpha \leftrightarrow \psi_k^{\alpha'}].$$

It is clear that ϱ_α will be in any admissible set containing ω, the alphabet under consideration, and α.

In particular, we have now established

THEOREM 3.1. *Suppose $\mathfrak{M}$ is $(P^+)^+$-generic. Then $\mathfrak{M}$ has Scott height at most $o(P^+)$.*

An absoluteness argument shows that it is not really necessary to assume that A is countable.

The example of the previous section shows that a $(P^+)^+$-generic structure need not be P^+-^{s}generic, nor even of height $\leqslant o(P^+)$. If we assume that our original theory T is complete, it is then clear that all $(P^+)^+$-generic structures have the same Scott height.

References

[1] J. T. Baldwin, A. R. Blass, A.M.W. Glass and D. W. Kueker, *A 'natural' theory without a prime model*, Algebra Universalis, vol. 3/2 (1973), pp. 152–155.

[2] L. Fuchs, *Infinite Abelian Groups*, New York 1970.

[3] V. Harnik and M. Makkai, *Application of Vaught sentences and the covering theorem*, Centre de Reserche Mathématique, Université de Montréal, technical report, 1974.

[4] H. J. Keisler, *Forcing and the omitting types theorem*, Studies in Model Theory, MAA Studies, Vol. 8.

[5] V. W. Lee and M. E. Nadel, *On the number of generic models*, Fund. Math. 90 (1975) pp. 105–114.

[6] A. Macintyre, *Omitting quantifier-free types in generic structures*, J. Symb. Logic 37 (1972), pp. 512–520.

[7] M. E. Nadel, *Scott sentences and admissible sets*, Ann. of Math. Logic 7 (1974), pp. 267-294.

[8] A. Robinson, *Forcing in model theory*, 1st. Nat. Alta Math. Symposia Math. 5 (1970), pp. 69–82.

[9] H. Simmons, *An omitting types theorem, with an application to the construction of generic structures*, Math. Scand. 33 (1973), pp. 46–54.

[10] W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. 41 (1955), pp. 203–271.

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Homogeneity, universality and saturatedness of limit reduced powers III

by

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Abstract. Let $\mathscr{F}$ be an ultrafilter on I and $\mathscr{G}$ a filter over $I\times I$. The paper gives a characterization of those pairs $(\mathscr{F}, \mathscr{G})$ which have the property that for every relational structure $\mathfrak{A}$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}$ is $\varkappa^+$-saturated. The notion used to obtain this characterization is a natural extension of Keisler's notion of a $\varkappa$-good filter.

A property **P** of a relational structure $\mathfrak{A}$ is a compactness type property if there is a definition of **P** which is of the form: for every set Σ of formulae (of some language connected with $\mathfrak{A}$), Σ can be satisfied in $\mathfrak{A}$ if and only if every finite subset of Σ can be satisfied in $\mathfrak{A}$. The saturatedness, universality and homogeneity of relational structures can be considered as properties of the compactness type. Various other properties of the compactness type have been investigated by several authors (e.g. atomic compactness [6], [11], positive compactness [11]). Here we restrict ourselves to saturatedness, homogeneity and universality.

By the classical results of Keisler ([3], [4]) ultraproducts can be used to obtain structures with a given compactness type property. For example, if a filter $\mathscr{F}$ is $(\omega, \varkappa)$-regular, then for every relational structure $\mathfrak{A}$ with $|L(\mathfrak{A})|\leqslant\varkappa$ the ultrapower $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa^+$-universal. If $\mathscr{F}$ is $\varkappa$-good, then for every family $\{\mathfrak{A}_i\colon i\in I\}$ of similar relational structures with $|L(\mathfrak{A}_i)|\leqslant\varkappa$ the ultraproduct $\prod_{i\in I}\mathfrak{A}_i/\mathscr{F}$ is $\varkappa^+$-saturated. The results of Keisler have been extended by Shelah and the present author to the case of products which are not necessarily maximal (see [7] and [10]). Another application of reduced products to compactness can be found in [8]. For the generalization of Keisler's results to Boolean ultrapowers see [5].

The problem of homogeneity of reduced products had not been extensively investigated. By a recent result of Wierzejewski [13] if the ultrapower $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa^+$-homogeneous for every structure $\mathfrak{A}$, then for every $\mathfrak{A}$ the ultrapower $\mathfrak{A}^I_{\mathscr{F}}$ is $\varkappa^+$-saturated.

In the present paper we investigate the problem of compactness of limit ultrapowers. We give a characterization of pairs $(\mathscr{F}, \mathscr{G})$ which have the property that for every relational structure $\mathfrak{A}$ such that $|L(\mathfrak{A})|\leqslant\varkappa$ the limit ultrapower $\mathfrak{A}^I_{\mathscr{F}}|\mathscr{G}$ is