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Accepté par la Rédaction le 4. 2. 1975

Note on decompositions of metrizable spaces I

by

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Abstract. In this note we investigate, in the class of metrizable spaces, the property of being σ-locally of weight $<\mathfrak{t}$, introduced by A. H. Stone in his theory of non-separable absolutely Borel spaces [8], and we prove some facts related to the questions raised in [8].

In this note we investigate, in the class of metrizable spaces, the property of being σ-locally of weight $<\mathfrak{t}$, introduced by A. H. Stone in his theory of non-separable absolutely Borel spaces [8], and we prove some facts related to the questions raised in [8].

Our topological terminology and notation is from [2] and [5]; our set-theoretical terminology will follow [4]. *All of our spaces are assumed to be metrizable.* For a given space X we say that ϱ is a metric on X if ϱ is any metric compatible with the topology of X. For a metric ϱ, a set $A\subset X$ and $\varepsilon>0$, we write $B(A,\varepsilon)=\{x\in X\colon \varrho(x,A)<\varepsilon\}$. The symbol $w(X)$ denotes the weight of a space X and $|S|$ the cardinality of a set S. The set of all ordinals less than a given ordinal λ is denoted by $W(\lambda)$. For an initial ordinal λ of a regular cardinality $\mathfrak{t}$ we call a set $\Gamma\subset W(\lambda)$ *stationary* if and only if for every function $\Phi\colon \Gamma\to W(\lambda)$ with $\Phi(\xi)<\xi$, there exists $\alpha<\lambda$ such that $|\Phi^{-1}(\alpha)|=\mathfrak{t}$. The successor of a cardinal number $\mathfrak{t}$ is denoted by $\mathfrak{t}^+$.

We say that a space X is $\mathfrak{h}$-*locally of weight* $<\mathfrak{t}$ (in symbols, $X\in\mathfrak{h}-\mathrm{Lw}(<\mathfrak{t})$; see [8], 2.1) provided $X=\bigcup\{X_a\colon a\in A\}$, where $|A|\leqslant\mathfrak{h}$ and each X_a is locally of weight $<\mathfrak{t}$. It is easy to verify (cf. [8], 2.1) that for a metric ϱ on X this is equivalent to the following condition: there are families $\mathscr{F}_s$ of subsets of X of weight $<\mathfrak{t}$ and $\varepsilon_s>0$ for $s\in S$, where $|S|\leqslant\mathfrak{h}$, such that

(1) $$X=\bigcup\{\bigcup\mathscr{F}_s\colon s\in S\}\quad\text{and}\quad \varrho(F',F'')\geqslant\varepsilon_s\quad\text{for different } F',F''\in\mathscr{F}_s.$$

For $\mathfrak{h}=\aleph_0$ we write $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$; if $X\in\mathfrak{h}-\mathrm{Lw}(<\aleph_0)$ we say that X is $\mathfrak{h}$-*discrete*.

PROPOSITION (cf. [8], Theorem 3). *Suppose that $\mathfrak{t}$ is a regular or sequential cardinal and $\mathfrak{h}<\mathfrak{t}$. If $X\in\mathfrak{h}-\mathrm{Lw}(<\mathfrak{t})$, then $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$.*

Proof. Let $\mathcal{F}_s$, for $s\in S$, be families satisfying (1). For each $n\in N$ put $S_n = \{s\in S\colon \varepsilon_s \geqslant 1/n\}$. Then $S=\bigcup_n S_n$ and for each open ball B of radius $1/2n$ we have

$$(2)\qquad |\{F\in\mathcal{F}_s\colon F\cap B\neq\emptyset\}|\leqslant 1 \quad \text{for each } s\in S_n .$$

Assume that $\mathfrak{t}$ *is regular*. Write $E_n=\bigcup\{\bigcup\mathcal{F}_s\colon s\in S_n\}$. Since $X=\bigcup_n E_n$ it suffices to verify that the local weight of $E_n<\mathfrak{t}$. Let B be any open ball of radius $1/2n$. Then we have

$$B\cap E_n\subset\bigcup\{F\in\mathcal{F}_s\colon F\cap B\neq\emptyset,\ s\in S_n\}=\bigcup_{s\in S_n}\bigcup\{F\in\mathcal{F}_s\colon F\cap B\neq\emptyset\},$$

and, by (2), we obtain

$$w(B\cap E_n)\leqslant\sum_{s\in S_n}w(F_s), \quad \text{where } F_s\in\mathcal{F}_s ,$$

thus $w(B\cap E_n)<\mathfrak{t}$, because $\mathfrak{t}$ is a regular cardinal.

Assume that $\mathfrak{t}=\lim\mathfrak{t}_k$, *where* $\mathfrak{t}_k<\mathfrak{t}$. We can assume moreover that $\mathfrak{t}_k=\mathfrak{m}_k^+>\mathfrak{h}$. Put $\mathcal{F}_{sk}=\{F\in\mathcal{F}_s\colon w(F)<\mathfrak{t}_k\}$. Since $\bigcup_k\mathcal{F}_{sk}=\mathcal{F}_s$, for $E_{sk}=\bigcup\mathcal{F}_{sk}$ and $X_k=\bigcup_{s\in S}E_{sk}$, we have $X=\bigcup_k X_k$. Each E_{sk} has the local weight $<\mathfrak{t}_k$, thus each $X_k\in\sigma\mathrm{Lw}(<\mathfrak{t}_k)$, therefore $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$.

The proposition was proved in the case of X absolutely Borel and $\mathfrak{t}$ non-limit by A. H. Stone [8] (Theorem 3). It gives an answer to the question raised in [8] (see Remark on page 261). The assumptions about $\mathfrak{t}$ cannot be omitted, as was shown by a simple example in [8].

A disjoint covering $\mathcal{E}$ of a space X we call a decomposition of X; a selector for a decomposition $\mathcal{E}$ is a set S intersecting each non-empty member of $\mathcal{E}$ in exactly one point.

Further we shall deal with the following natural decomposition $\mathcal{P}$ of a space X. Let λ be an initial ordinal with $\mathrm{cf}(\lambda)>\omega_0$. Let $\{X_\xi\}_{\xi<\lambda}$ be a sequence of subsets of X such that

$$(3)\qquad X_1\subset\ldots\subset X_\xi\subset\ldots\subset X, \quad \overline{X_\xi}=X_\xi, \quad w(X_\xi)<w(X), \quad \text{for } \xi<\lambda,$$

$$(4)\qquad X=\bigcup_{\xi<\lambda}X_\xi \quad\text{and}\quad X_\xi=\overline{\bigcup_{\alpha<\xi}X_\alpha} \quad \text{for limit } \xi<\lambda .$$

Let us put

$$(5)\qquad \mathcal{P}=\{P_\xi\colon \xi<\lambda\}, \quad \text{where} \quad P_\xi=X_\xi\setminus\bigcup_{\alpha<\xi}X_\alpha ,$$

$$(6)\qquad \Gamma(\mathcal{P})=\{\xi\colon P_\xi\neq\emptyset,\ \xi \text{ is a limit ordinal } <\lambda\}, \quad \mathcal{P}^*=\{P_\xi\colon \xi\in\Gamma(\mathcal{P})\}.$$

If $w(X)=\mathfrak{t}$ is a non-sequential cardinal, λ the initial ordinal of cardinality $\mathfrak{t}$ and $\{x_\xi\colon \xi<\lambda\}$ a dense set in X, then the sets $X_\xi=\overline{\{x_\alpha\colon \alpha<\xi\}}$ satisfy (3) and (4) and hence define a decomposition $\mathcal{P}$.

THEOREM 1. *Let X be a space of regular weight $w(X)=\mathfrak{t}$ and let $\mathcal{P}$ be any decomposition of the space X satisfying* (5). *Then the following conditions are equivalent*:

(i) $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$,

(ii) *there exists a selector for $\mathcal{P}^*$ (see* (6)) *which is* $\sigma\mathrm{Lw}(<\mathfrak{t})$,

(iii) $\Gamma(\mathcal{P})$ *is not stationary.*

Proof. Let ϱ be a metric on X. Note that for a decomposition $\mathcal{P}$ satisfying (5) we have $|\lambda|=\mathfrak{t}$, which is an easy consequence of regularity of $\mathfrak{t}$. The implication (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii) Write $\Gamma=\Gamma(\mathcal{P})$ and let $a_\xi\in P_\xi$ be points such that the space $A=\{a_\xi\colon \xi\in\Gamma\}\in\sigma\mathrm{Lw}(<\mathfrak{t})$. By (1) there is a decomposition $\{A_i\}_{i\in N}$ of A such that $A_i=\bigcup\{A_s\colon s\in S_i\}$, where $S_i\cap S_j=\emptyset$ for $i\neq j$ and

$$(7)\qquad w(A_s)<\mathfrak{t} \ \text{ for } s\in\bigcup_i S_i \quad\text{and}\quad \varrho(A_{s'},A_{s''})\geqslant\varepsilon_i \ \text{for different } s',s''\in S_i .$$

Put $\Gamma_i=\{\xi\in\Gamma\colon a_\xi\in A_i\}$, $\Gamma_s=\{\xi\in\Gamma\colon a_\xi\in A_s\}$. Thus $\{\Gamma_i\}_{i\in N}$ is a decomposition of Γ and $\{\Gamma_s\}_{s\in S_i}$ is a decomposition of Γ_i. Note that

$$(8)\qquad |\Gamma_s|<\mathfrak{t} \quad\text{for}\quad s\in\bigcup_i S_i ,$$

since, by (7), it follows that for some $\xi<\lambda$ we have $A_s\subset X_\xi$ and hence $A_s\subset\{a_\alpha\colon \alpha\leqslant\xi\}$. Let

$$\varphi(\xi)=i\ \equiv\ \xi\in\Gamma_i .$$

For each $\xi\in\Gamma$, by (4), we can choose $\alpha<\xi$ and $b_\xi\in B(a_\xi,\tfrac{1}{3}\varepsilon_{\varphi(\xi)})\cap X_\alpha$. Put $\Phi(\xi)=\alpha$. Thus

$$(9)\qquad \Phi\colon \Gamma\to W(\lambda), \quad \Phi(\xi)<\xi \quad\text{for}\quad \xi\in\Gamma,$$

$$(10)\qquad \varrho(a_\xi,b_\xi)<\tfrac{1}{3}\varepsilon_{\varphi(\xi)} \quad\text{and}\quad b_\xi\in X_{\Phi(\xi)} .$$

It is enough to show that

$$(11)\qquad |\Phi^{-1}(\alpha)|<\mathfrak{t} \quad\text{for}\quad \alpha<\lambda .$$

Suppose contrary, that there exists $\alpha<\lambda$ with $|\Phi^{-1}(\alpha)|=\mathfrak{t}$. Then there exists $i\in N$ such that $|\Phi^{-1}(\alpha)\cap\Gamma_i|=\mathfrak{t}$ and since by the regularity of $\mathfrak{t}$ and (8) the set $\{s\colon \Gamma_s\cap(\Phi^{-1}(\alpha)\cap\Gamma_i)\neq\emptyset\}$ is of cardinality $\mathfrak{t}$, we can choose a set $\Gamma'\subset\Phi^{-1}(\alpha)\cap\Gamma_i$ of cardinality $\mathfrak{t}$ intersecting each Γ_s in at most one point. For each $\xi\in\Gamma'$ we have $\Phi(\xi)=\alpha$, $\varphi(\xi)=i$ and thus, by (10), $\{b_\xi\colon \xi\in\Gamma'\}\subset X_\alpha$ and $\varrho(a_\xi,b_\xi)<\frac{1}{3}\varepsilon_i$, for $\xi\in\Gamma'$. By (3) we have $w(X_\alpha)<\mathfrak{t}$, therefore we can find two different ordinals $\xi',\xi''\in\Gamma'$ such that $\varrho(b_{\xi'},b_{\xi''})<\frac{1}{3}\varepsilon_i$, whence $\varrho(a_{\xi'},a_{\xi''})<\varepsilon_i$. But this contradicts (7), as $\xi'\in\Gamma_{s'}$, $\xi''\in\Gamma_{s''}$ for different indices $s',s''\in S_i$ and thus $a_{\xi'}\in A_{s'}$, $a_{\xi''}\in A_{s''}$.

(iii)$\Rightarrow$(ii) Write $\Gamma=\Gamma(\mathcal{P})$ and let

$$\Phi\colon \Gamma\to W(\lambda), \quad \Phi(\xi)<\xi \text{ for } \xi\in\Gamma, \quad |\Phi^{-1}(\alpha)|<\mathfrak{t} \text{ for } \alpha<\lambda .$$

Extend Φ over the whole $W(\lambda)$ (not changing the notation) taking

$$\Phi(\xi) = \xi \text{ for a limit ordinal } \xi \notin \Gamma \quad \text{and} \quad \Phi(\xi+1) = \xi \text{ for } \xi < \lambda .$$

Let us notice that

(12) $\Phi(\xi) < \xi$ for all $\xi \in \Gamma$ and non-limit $\xi < \lambda$ and $|\Phi^{-1}(\alpha)| < \mathfrak{t}$ for $\alpha < \lambda$.

Put (cf. [5], § 30, X, (8))

$$X_{\xi n} = X_\xi \setminus B(X_{\Phi(\xi)}, 1/n) \quad \text{for} \quad 1 \leqslant \xi < \lambda,\ n \in N \text{ (we take } X_0 = \emptyset),$$

$$X_n = \bigcup \{X_{\xi n}: \xi < \lambda\} \quad \text{for} \quad n \in N .$$

First we shall show that each X_n is locally of weight $< \mathfrak{t}$. Take $x \in X$ and let ξ_0 be the first $\xi < \lambda$ with $x \in X_\xi$. By (12) and the regularity of $\mathfrak{t}$ it follows that there exists $\xi_1 < \lambda$ such that for $\xi \geqslant \xi_1$ we have $\Phi(\xi) > \xi_0$. Thus

$$B(x, 1/n) \cap X_{\xi n} \subset B(X_{\xi_0}, 1/n) \cap X_{\xi n} = \emptyset \quad \text{for} \quad \xi \geqslant \xi_1 ,$$

so

$$B(x, 1/n) \cap X_n \subset \bigcup \{X_{\xi n}: \xi < \xi_1\} \subset X_{\xi_1},$$

and, by (3),

$$w(B(x, 1/n) \cap X_n) < \mathfrak{t} .$$

It is enough to show that $X = \bigcup_n X_n$. Take $x \in X$ and let ξ_0 be chosen as before. Since $x \in X_{\xi_0} \setminus \bigcup_{\alpha < \xi_0} X_\alpha$, if ξ_0 is a limit ordinal, then $\xi_0 \in \Gamma$. By (12) we infer that $\Phi(\xi_0) < \xi_0$, thus $x \notin X_{\Phi(\xi_0)}$ and for some $n \in N$ also $x \notin B(X_{\Phi(\xi_0)}, 1/n)$, whence $x \in X_{\xi_0 n} \subset X_n$.

Corollary 1. *Let $\mathfrak{m} = \aleph_\alpha$ be a regular cardinal. For every space Z of weight $< \aleph_{\alpha+\omega_0}$ with $Z \notin \sigma\mathrm{Lw}(<\mathfrak{m})$ there exists a subspace $E \subset Z$ such that $E \notin \sigma\mathrm{Lw}(<\mathfrak{m})$ and $w(M) = |M|$ for every $M \subset E$.*

Notice that, by Proposition, for $\mathfrak{m} = \mathfrak{n}^+$ the condition $E \notin \sigma\mathrm{Lw}(<\mathfrak{m})$ implies that E is not $\mathfrak{n}$-discrete.

Proof. Let $\mathfrak{t} = \min\{w(Y): Y \subset Z,\ Y \notin \sigma\mathrm{Lw}(<\mathfrak{m})\}$. Since $\aleph_\alpha \leqslant \mathfrak{t} < \aleph_{\alpha+\omega_0}$, $\mathfrak{t}$ is a regular cardinal. Take $X \subset Z$ such that $w(X) = \mathfrak{t}$ and $X \notin \sigma\mathrm{Lw}(<\mathfrak{m})$. Since for every $Y \subset X$ with $w(Y) < \mathfrak{t}$ we have $Y \in \sigma\mathrm{Lw}(<\mathfrak{m})$, we obtain $X \notin \sigma\mathrm{Lw}(<\mathfrak{t})$. Let $\mathscr{P}$ be a decomposition of X satisfying (5) and let $\{x_\xi: \xi \in \Gamma(\mathscr{P}) = \Gamma\} = E$ be a selector for $\mathscr{P}^*$. By Theorem 1 we infer that $E \notin \sigma\mathrm{Lw}(<\mathfrak{t})$. Let $M = \{x_\xi: \xi \in C \subset \Gamma\}$ be any subspace of E. Using the arguments given in [5], § 24, II (see Remark 1), we shall show that $w(M) = |M|$. Let $\mathscr{B}$ be a base of M with $|\mathscr{B}| = w(M)$. For each $\xi \in C$ denote by ξ^* the successor of ξ in C and choose $U_\xi \in \mathscr{B}$ such that $X_\xi \cap U_\xi = \emptyset$ (cf. (3)) and $x_{\xi^*} \in U_\xi$. Then for different ordinals $\xi', \xi'' \in C$ we have $U_{\xi'} \neq U_{\xi''}$ and thus $|M| \leqslant |\mathscr{B}| = w(M) \leqslant |M|$.

A. H. Stone proved the above statement for absolutely analytic spaces without the restrictions on the weight of a space Z (see [8], Theorem 2 and Section 3.5).

Example. Let $B(\xi) = W(\xi)^{\aleph_0}$, for an ordinal ξ, be the Baire space, i.e. the product of $\aleph_0$ copies of the set $W(\xi)$ with the discrete topology. Let $X = B(\lambda)$, where λ is an initial ordinal of regular cardinality $\mathfrak{t}$ and $X_\xi = B(\xi)$, for $\xi < \lambda$. Then the sequence $X_1 \subset \ldots \subset X_\xi \subset \ldots$ satisfies (3) and (4) and thus we obtain the decomposition $\mathscr{P}$ defined by (5). It is easy to verify that $\Gamma(\mathscr{P}) = \{\xi < \lambda: \mathrm{cf}(\xi) = \omega_0\}$ (see (6)). Choosing a selector $E(\mathfrak{t})$ for $\mathscr{P}^*$ we obtain the space defined by A. H. Stone (cf. [7], 5; [8], 3.5). Since $X \notin \sigma\mathrm{Lw}(<\mathfrak{t})$ (by Baire's theorem; cf. [8], 2.1), by Theorem 1 we receive Stone's result (cf. [8], Lemma 2) that $E(\mathfrak{t})$ is not $\sigma\mathrm{Lw}(<\mathfrak{t})$ (for $\mathfrak{t} = \mathfrak{h}^+$ — equivalently, by Proposition — $E(\mathfrak{h}^+)$ is not $\mathfrak{h}$-discrete).

In the sequel we shall use the following fact kindly communicated to the author by K. Alster.

Lemma (K. Alster). *Let $\mathscr{E} = \{E_s: s \in S\}$ be a decomposition of a space X such that $w(E_s) < \mathfrak{t}$, for $s \in S$, and each selector for E is $\mathfrak{t}$-discrete. Let λ be the initial ordinal of cardinality $\mathfrak{t}$. Then there are sets $W_{\xi s}$, for $\xi < \lambda$ and $s \in S$, such that $E_s = \bigcup \{W_{\xi s}: \xi < \lambda\}$ and each family $\mathscr{W}_\xi = \{W_{\xi s}: s \in S\}$ is $\mathfrak{t}$-discrete* ([1]).

Proof. Let $\mathfrak{A}$ be the class of all $\mathfrak{t}$-discrete families $\mathscr{W} = \{W_s: s \in S\}$ such that every W_s is an open subset of the space E_s. Since for each selector A for E there exists $\mathscr{W} \in \mathfrak{A}$ which covers A, we can define by the transfinite induction a sequence of families $\mathscr{W}_\xi = \{W_{\xi s}: s \in S\} \in \mathfrak{A}$ for $\xi < \lambda$, such that for every $s \in S$ and $\xi < \lambda$

$$\text{if} \quad E_s \setminus \bigcup \{W_{\alpha s}: \alpha < \xi\} \neq \emptyset, \quad \text{then} \quad W_{\xi s} \cap (E_s \setminus \bigcup \{W_{\alpha s}: \alpha < \xi\}) \neq \emptyset .$$

It remains to verify that for every $s \in S$ we have $E_s = \bigcup_{\xi < \lambda} W_{\xi s}$. In the opposite case we would obtain a strictly increasing sequence $U_\xi = \bigcup_{\alpha < \xi} W_{\alpha s}$ of open subsets of E_s of type λ, contrary to the assumption $w(E_s) < \mathfrak{t}$.

Theorem 2. *Let $\mathscr{E}$ be a decomposition of a space X such that for a regular, or sequential cardinal $\mathfrak{t}$ each member $E \in \mathscr{E}$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$ and for some $\mathfrak{m} < \mathfrak{t}$ each selector for $\mathscr{E}$ is $\mathfrak{m}$-discrete. Then X is $\sigma\mathrm{Lw}(<\mathfrak{t})$.*

Proof. We shall consider only the case of $\mathfrak{t}$ regular. The case of $\mathfrak{t}$ sequential can be derived then easily by the same reasons as in the second part of the proof of Proposition. We shall prove our theorem in three steps.

Suppose first that $w(E) < \mathfrak{t}$, for $E \in \mathscr{E}$, and $w(X) = \mathfrak{t}$. Let λ be the initial ordinal of cardinality $\mathfrak{t}$. We shall define a sequence $X_1 \subset \ldots \subset X_\xi \subset \ldots \subset X$ of type λ satisfying (3), (4) and

(13) $$\mathrm{St}(X_\xi, \mathscr{E}) \subset X_{\xi+1} \quad \text{for} \quad \xi < \lambda \text{ ([2])}.$$

Let $\{x_\xi: \xi < \lambda\}$ be a dense set in X. Put $X_1 = \{x_1\}$, $X_{\xi+1} = \overline{\mathrm{St}(X_\xi, \mathscr{E}) \cup \{x_\alpha: \alpha < \xi\}}$ and $X_\xi = \overline{\bigcup_{\alpha < \xi} X_\alpha}$, for limit ordinals. It suffices to show that $w(X_\alpha) < \mathfrak{t}$ for $\alpha < \lambda$.

([1]) It means that $\mathscr{W}_\xi$ is the union of $\mathfrak{t}$ discrete subfamilies.
([2]) $\mathrm{St}(A, \mathscr{E}) = \bigcup \{E \in \mathscr{E}: E \cap A \neq \emptyset\}$.

Suppose that we have done it for $\alpha<\xi$. If ξ is a limit ordinal it follows immediately from the regularity of $\mathfrak{t}$ that $w(\overline{\bigcup_{\alpha<\xi} X_\alpha})<\mathfrak{t}$. Let $\xi=\eta+1$. Since each selector for $\mathscr{E}$ is $\mathfrak{m}$-discrete, we have

$$|\{E\colon E\in\mathscr{E}, E\cap X_\eta\neq\emptyset\}|\leqslant\mathfrak{m}\cdot w(X_\eta)<\mathfrak{t}$$

and, by the regularity of $\mathfrak{t}$, also $w(\mathrm{St}(X_\eta,\mathscr{E}))<\mathfrak{t}$, therefore $w(X_\xi)<\mathfrak{t}$. Let $\mathscr{P}^*=\{P_\xi\colon \xi\in\Gamma(\mathscr{P})=\Gamma\}$ be the decomposition satisfying (6). By Theorem 1 it is enough to show that choosing points $p_\xi\in P_\xi$ we obtain a selector $M=\{p_\xi\colon \xi\in\Gamma\}\in\sigma\mathrm{Lw}(<\mathfrak{t})$. We shall show that M is $\mathfrak{m}$-discrete (our conclusion will follow then by Proposition); to see this it suffices to verify that for each $E\in\mathscr{E}$ we have $|M\cap E|\leqslant 1$. Indeed, if $\xi,\eta\in\Gamma$, with $\xi<\eta$, we have $\xi+1<\eta$ (since η is limit), and, by (13), none of $E\in\mathscr{E}$ intersects both P_ξ and P_η.

Next suppose that $w(E)<\mathfrak{t}$, *for* $E\in\mathscr{E}$. Let $\mathscr{E}=\{E_s\colon s\in S\}$. Using the Lemma we infer that X is the union $X=\bigcup\{\bigcup\mathscr{W}_\xi\colon \xi<\lambda\}$ and since each $\mathscr{W}_\xi$ is a $\mathfrak{t}$-discrete family consisting of sets of weight $<\mathfrak{t}$, we have $X\in\mathfrak{t}-\mathrm{Lw}(<\mathfrak{t})$. Thus $X\in\mathfrak{t}-\mathrm{Lw}(<\mathfrak{t}^+)$ and, by Proposition, $X\in\sigma\mathrm{Lw}(<\mathfrak{t}^+)$; therefore we have a decomposition (1) of X (with $|S|=\aleph_0$). Since each $F\in\mathscr{F}_s$ is of weight $\leqslant\mathfrak{t}$, by the case considered before (where $\mathscr{E}$ is changed by $\mathscr{E}|F=\{E\cap F\colon E\in\mathscr{E}\}$), we infer that $F\in\sigma\mathrm{Lw}(<\mathfrak{t})$ and it follows easily that $X\in\sigma\mathrm{Lw}(<\mathfrak{t})$.

Finally, *assume that* $E\in\sigma\mathrm{Lw}(<\mathfrak{t})$, *for* $E\in\mathscr{E}$. Write $\mathscr{E}=\{E_s\colon s\in S\}$. By (1), for each $s\in S$, we have $E_s=\bigcup\{\bigcup\mathscr{F}_i^s\colon i\in N\}$, where $\mathscr{F}_i^s$ consists of sets of weight $<\mathfrak{t}$ and $\varrho(F',F'')\geqslant\varepsilon_{si}>0$, for different $F',F''\in\mathscr{F}_i^s$. Put $\mathscr{F}_{ik}=\bigcup\{\mathscr{F}_i^s\colon \varepsilon_{si}\geqslant 1/k\}$. Then $\mathscr{F}_{ik}$ is a decomposition of $X_{ik}=\bigcup\mathscr{F}_{ik}$ and each member of $\mathscr{F}_{ik}$ is of weight $<\mathfrak{t}$. We shall show that each selector for $\mathscr{F}_{ik}$ is $\mathfrak{m}$-discrete and hence $X_{ik}\in\sigma\mathrm{Lw}(<\mathfrak{t})$. Let M be any selector for $\mathscr{F}_{ik}$; write $M=\bigcup_{s\in S} M_s$, where $M_s=M\cap E_s$. Each open ball B of radius $1/2k$ contains at most one point of M_s, for every $s\in S$; thus $B\cap M$ is a subset of some selector for $\mathscr{E}$ and thus it is $\mathfrak{m}$-discrete. The set M is therefore locally $\mathfrak{m}$-discrete and so is $\mathfrak{m}$-discrete.

We shall define now a class of mappings preserving the property $\sigma\mathrm{Lw}(<\mathfrak{t})$ for some cardinals $\mathfrak{t}$. Namely, call a one-to-one function $f\colon X\xrightarrow[\text{onto}]{} Y$ *d-isomorphism* if both f and f^{-1} take σ-discrete sets to σ-discrete sets.

COROLLARY 2. *Let $f\colon X\to Y$ be a d-isomorphism. For a regular or sequential cardinal $\mathfrak{t}$, if X is $\sigma\mathrm{Lw}(<\mathfrak{t})$, then so is Y.*

Proof. By (1) we have $X=\bigcup\{\bigcup\mathscr{F}_i\colon i\in N\}$, where each $\mathscr{F}_i$ is a discrete family consisting of sets of weight $<\mathfrak{t}$. Put $\mathscr{E}_i=f\mathscr{F}_i$. Then $\mathscr{E}_i$ is a decomposition of a space $Y_i=\bigcup\mathscr{E}_i$, consisting of sets of weight $<\mathfrak{t}$ (because, as is easy to verify, d-isomorphism preserves the weight) and each selector for $\mathscr{E}_i$ is σ-discrete (as the image of some selector for $\mathscr{F}_i$). From Theorem 2 it follows that $Y_i\in\sigma\mathrm{Lw}(<\mathfrak{t})$ and thus also $Y=\bigcup_i Y_i\in\sigma\mathrm{Lw}(<\mathfrak{t})$.

COROLLARY 3. *Let $f\colon X\xrightarrow[\text{onto}]{} Y$ be a one-to-one function such that both f and f^{-1} take absolutely Borel (resp. absolutely analytic) spaces to absolutely Borel (resp. absolutely analytic) spaces. If $\mathfrak{t}$ is a regular or sequential cardinal and X is $\sigma\mathrm{Lw}(<\mathfrak{t})$, then so is Y.*

Proof. It suffices to show, in virtue of Corollary 2, that f is a d-isomorphism. But this follows immediately from A. H. Stone's (resp. A. G. El'kin's) theorem that an absolutely Borel (resp. absolutely analytic) space, each subspace of which is a Borel (resp. analytic) set, is σ-discrete ([7], Theorem 2; [1]).

A. H. Stone [8] proved that the property $\sigma\mathrm{Lw}(<\mathfrak{t})$ is invariant under analytic isomorphisms in the class of absolutely analytic spaces for all cardinals $\mathfrak{t}$ (Theorem 1′). Our Corollaries 2 and 3 are related to the question raised in [8] (Remark 3.3): to what extent the above fact apply to spaces, which need not be absolutely analytic?

Note that the mappings considered in Corollary 3 differ in general from Borel or analytic isomorphisms.

Let us finish with a few remarks.

Remark 1. It would be interesting to know answers to the following questions.

QUESTION 1. *Can we omit the assumption on weight of Z in Corollary 1; is it true for $\mathfrak{m}=\aleph_1$ and $w(Z)=\aleph_{\omega_0}$?*

QUESTION 2. *Can we omit the assumption that $\mathfrak{t}$ is a regular or sequential cardinal in Theorem 2 and Corollaries 2 and 3; are these statements true for $\mathfrak{t}=\aleph_{\omega_1}$?*

Remark 2. We can improve a special case of Theorem 2. We call a decomposition $\mathscr{E}=\{E_s\colon s\in S\}$ of a space X *$\mathfrak{t}$-discretely decomposable* if there exist sets E_{st}, for $t\in T$ with $|T|=\mathfrak{t}$, such that $E_s=\bigcup_{t\in T} E_{st}$ and each of the family $\{E_{st}\colon s\in S\}$ is discrete (see [3], 1). It is easy to verify (cf. [3], Lemma 3) that if $\mathscr{E}|F=\{E_s\cap F\colon s\in S\}$ is $\mathfrak{t}$-discretely decomposable for each $F\in\mathscr{F}$, where $\mathscr{F}$ is a σ-discrete covering of a space, then $\mathscr{E}$ is $\mathfrak{t}$-discretely decomposable.

Let $\mathscr{E}$ be a decomposition of a space X such that each selector for $\mathscr{E}$ is $\mathfrak{t}$-discrete and $E\in\sigma\mathrm{Lw}(<\mathfrak{t}^{++})$ for $E\in\mathscr{E}$. Then $\mathscr{E}$ is $\mathfrak{t}$-discretely decomposable.

Proof. By Theorem 2 our proof reduces to the case $w(X)\leqslant\mathfrak{t}^+$. Thus we can assume also that $\mathscr{E}=\{E_\xi\colon \xi<\lambda\}$ where λ is the initial ordinal of cardinality $\mathfrak{t}^+$. Let $X_1\subset\ldots\subset X_\xi\subset\ldots\subset X$ be a sequence of type λ satisfying (3). Put

$$F_\xi=E_\xi\cap X_\xi \quad\text{and}\quad G_{\xi n}=E_\xi\setminus B(X_\xi,1/n)\,.$$

Since $E_\xi=F_\xi\cup\bigcup_n G_{\xi n}$ it suffices to show that families $\mathscr{F}=\{F_\xi\colon \xi<\lambda\}$ and $\mathscr{G}_n=\{G_{\xi n}\colon \xi<\lambda\}$ are $\mathfrak{t}$-discretely decomposable.

Let $F=\bigcup\mathscr{F}$. Since $w(F_\xi)<\mathfrak{t}^+$ we infer, by Theorem 2, that F is $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ and this implies easily that $\mathscr{F}$ is $\mathfrak{t}$-discretely decomposable.

Put $U_{\xi n} = B(X_\xi, 1/n)$ and $\mathscr{U}_n = \{U_{\xi n}: \xi<\lambda\}$. Then $\mathscr{U}_n$ is an open covering of X and for each $\alpha>\xi$ we have $U_{\xi n} \cap G_{\alpha n} = \emptyset$. Thus $\mathscr{G}_n$ is locally $\mathfrak{t}$-discretely decomposable and hence, by paracompactness of X, it is $\mathfrak{t}$-discretely decomposable.

QUESTION 3. *Let $\mathscr{E}$ be a decomposition of a space X such that each selector for $\mathscr{E}$ is σ-discrete. Is then $\mathscr{E}$ $\aleph_0$-discretely decomposable (equivalently: is every d-isomorphism a σ-discrete mapping in the sense of* [3], 3.1)?

By the above proposition, if $\mathscr{E}$ consists of sets which are $\sigma \mathrm{Lw}(<\aleph_2)$, the answer to Question 3 is positive. Notice, that the negative answer to Question 2 would give the negative answer to Question 3.

Remark 3. Corollary 3 can be reformulated in the following manner. Assign to a space X the lattice $\widetilde{\mathscr{B}}(X)(\mathscr{A}(X))$ of its absolutely Borel (absolutely analytic) subspaces. Then for every regular or sequential cardinal $\mathfrak{t}$ and spaces X and Y, if the lattices $\widetilde{\mathscr{B}}(X)(\mathscr{A}(X))$ and $\widetilde{\mathscr{B}}(Y)(\mathscr{A}(Y))$ are isomorphic, and X is $\sigma \mathrm{Lw}(<\mathfrak{t})$, then so is Y.

Remark 4. A space X of weight $\aleph_1$ is d-isomorphic to the Baire space $B(\omega_1)$ (see Example) if and only if for every (equivalently — for some) decomposition $\mathscr{P}$ of the space X satisfying (5) the set $\{\xi<\omega_1: |P_\xi|<2^{\aleph_0}\}$ is not stationary.

This result is closely related to the following statement: a subset $C\subset W(\omega_1)$ contains a closed cofinal subset of $W(\omega_1)$ if and only if the lattice of all non-stationary subsets of C is isomorphic to the lattice of all non-stationary subsets of $W(\omega_1)$.

Remark 5. Adopt the notation of Example. Write $\Gamma = \Gamma(\mathscr{P})$ and let $\{x_\xi: \xi \in \Gamma\} = E(\mathfrak{t})$ be a selector for $\mathscr{P}^*$. Put $f(\xi) = x_\xi$. Using the arguments given in the proof of Theorem 1 we can show that for every $C\subset\Gamma$ the set $f(C)$ is $\sigma \mathrm{Lw}(<\mathfrak{t})$ if and only if C is not stationary. Thus f induces an isomorphism of the lattices of all non-stationary subsets of Γ and all $\sigma \mathrm{Lw}(<\mathfrak{t})$ subsets of $E(\mathfrak{t})$.

Remark 6. Using A. H. Stone's characterization of absolute F_σ-spaces [6] it is easy to prove, by Theorem 2 (for $\mathfrak{t} = \aleph_1$), the following statement: *a space X is an absolute F_σ-space if and only if it is a continuous image of the free union of Cantor's sets under a mapping which takes discrete sets to σ-discrete sets.*

Added in proof. W. G. Fleissner has kindly informed the author (letter, June 1976) that he constructed a model of set theory in which the answer to Question 3, and — *a fortiori* — to Question 2 is positive. On the other hand, the author proved in the second part of this paper (*Note on decompositions of metrizable spaces II,* Fund. Math.) that in a model of set theory the answer to Question 3 is negative.

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Accepté par la Rédaction le 24. 3. 1975