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# Some algebraic properties of weakly compact and compact cardinals

by

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**Abstract.** A combinatorial property $[\varkappa, \lambda, \varrho]$ of cardinals is introduced and studied. Work of Jech shows that $\varkappa$ inaccessible and $\varkappa$ weakly compact implies $[\varkappa, \varkappa, 3]$. $[\varkappa, \varkappa, 3]$ is used to establish an algebraic embedding theorem for certain classes of universal algebras. One corollary of this embedding theorem is: if $\varkappa$ is inaccessible and weakly compact and $G$ is a group with $|G| = \varkappa$ and every subgroup of $G$ of smaller cardinality is free, then $G$ is free.

In 1949 R. Rado published the following: [9].

SELECTION LEMMA. *Let $A$ and $N$ be sets and let $A_\nu$ be a finite subset of $A$ for each $\nu \in N$. Suppose that for each finite $L \subseteq N$ we are given a function $f_L : L \to A$ such that $f_L(\nu) \in A_\nu$ for each $\nu \in L$. Then there is a function $f: N \to A$ such that given any finite $L \subseteq N$ there is a finite $M \subseteq N$ with $L \subseteq M$ and $f|L = f_M|L$.*

Through the years other have discovered versions of this lemma (see [4], [6], [7], [10]) and several have explored its connection with logical compactness (see [5], [7], [10]). It is natural to ask about possible generalizations of this lemma. Rado in [9] gave an example to show that "finite" could not be replaced by "denumerable." In [7] Jech defined "$\varkappa$ is $\lambda$-compact" for infinite cardinals $\varkappa \leqslant \lambda$ with $\varkappa$ regular, and in this same paper he gave a generalization of the Selection lemma for such $\varkappa$ and $\lambda$ which we denote by $[\varkappa, \lambda, 3]$ (we define this notation in § 0). Jech showed ([7], Theorem 2.2) that weakly compact inaccessible cardinals $\varkappa$ satisfy $[\varkappa, \varkappa, 3]$, and conversely that if $[\varkappa, \varkappa, 3]$ holds then $\varkappa$ is weakly compact. Further he in effect showed that $\varkappa$ is compact if and only if $[\varkappa, \lambda, 3]$ holds for all $\lambda \geqslant \varkappa$.

In this paper we study $[\varkappa, \varkappa, 3]$ and some related properties $[\varkappa, \lambda, \varrho]$. We assume their validity and derive some of their consequences, both set theoretical (§ 1 and § 2) and algebraic (§ 3). In § 1 we show that $[\varkappa, \varkappa, 3]$ implies that $\varkappa$ is a regular limit cardinal without appealing to weak compactness. In § 2 we use inverse limit systems to give a measurability criterion. Our main result, Theorem 3.1, uses $[\varkappa, \varkappa, 3]$ to prove an algebraic embedding theorem. Because of Jech's work this gives an algebraic property of weakly compact inaccessible cardinals and of compact cardinals, special cases of which have been proved by Mekler and Gregory ([1]).

([1]) We wish to thank Paul Eklöf for informing us of the work of Mekler and Gregory.

Throughout the paper we work in Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC). Unless otherwise specified we do not assume the generalized continuum hypothesis (GCH).

**§ 0. Notation and basic definitions.** As usual we identify an *ordinal* with its set of predecessors and a *cardinal* with the smallest ordinal having that cardinality. The letters $\alpha, \beta, \nu, \xi$ are used for ordinals and the letters $\varkappa, \lambda, \mu, \varrho, \theta$ for cardinals. Of course $\omega$ has its usual meaning. For $S$ a set $|S|$ denotes the cardinality of $S$, $P(S) = \{x|\ x \subseteq S\}$, $P_\varkappa(S) = \{x|\ x \subseteq S\ \&\ |x| < \varkappa\}$. $\varkappa^+$ denotes the first cardinal greater than $\varkappa$. We say $\varkappa$ is a *limit cardinal* if $\varkappa \neq \lambda^+$ for all $\lambda$. $\mathrm{cf}(\alpha)$ is the smallest $\beta$ which can be mapped onto a cofinal subset of $\alpha$. In the following definitions $\varkappa$ is infinite. $\varkappa$ is *regular* if $\mathrm{cf}(\varkappa) = \varkappa$. $\varkappa$ is *weakly inaccessible* if it is both a limit cardinal and regular. $\varkappa$ is *inaccessible* if it is regular and for each $\lambda < \varkappa$, $\mu < \varkappa$ we have $\lambda^\mu < \varkappa$. $\varkappa$ is *weakly compact* if the following holds: if $\Sigma$ is a set of sentences of $\mathscr{L}_\varkappa$ with $|\Sigma| = \varkappa$ and every $\Sigma' \in P_\varkappa(\Sigma)$ has a model, then $\Sigma$ has a model. $\varkappa$ is *compact* if it satisfies the above condition for weak compactness with the cardinality restriction on $\Sigma$ removed. $\varkappa$ is *measurable* if there is a nonprincipal $\varkappa$-complete ultrafilter over $\varkappa$.

In ZFC the following is known: if $\varkappa$ is weakly compact then $\varkappa$ is weakly inaccessible; if $\varkappa$ is compact then $\varkappa$ is measurable; if $\varkappa$ is measurable then $\varkappa$ is inaccessible. We refer the reader to [1], [2] for proofs and further details on the above definitions.

DEFINITION. $L \subseteq P(S)$. $L$ is a *$\varkappa$-cover* of $S$ if for each $X \in P_\varkappa(S)$ there is an $A \in L$ with $X \subseteq A$.

DEFINITION. $L$ a $\varkappa$-cover of $S$. $R$ a set. A collection of functions $\mathscr{F} = \{f_A|\ A \in L\}$ where $f_A\colon A \to R$ is called an *$L$-$R$ valuation*.

We now present two generalizations of the Selection lemma.

We assume $\omega \leqslant \varkappa \leqslant \lambda$.

I. $\varkappa, \lambda, \varrho$ have the Rado property (written $[\varkappa, \lambda, \varrho]$) if whenever $S$ is a set with $|S| \leqslant \lambda$, $L$ is a $\varkappa$-cover of $S$, and $\mathscr{F}$ is an $L$-$R$ valuation with $|R| < \varrho$ there exists a function $f\colon S \to R$ such that for each $X \in P_\varkappa(S)$ there is an $A \in L$ with $X \subseteq A$ and $f|X = f_A|X$.

Such an $f$ need not be unique. But abusing notation we write $f = \lim_{A \in L} f_A$.

II. $\varkappa, \lambda, \varrho$ have the $*$ Rado property (written $[\varkappa, \lambda, \varrho]^*$) if all the above holds with the removal of the cardinality restriction on $R$ and with the addition of: for each $t \in S$ $|R_t| < \varrho$ where $R_t = \{f_A(t)|\ A \in L\}$.

Remarks. Rado's Selection lemma is the statement "for all $\lambda [\omega, \lambda, \omega]^*$". In the terminology of Jech [7] $[\varkappa, \lambda, 3]$ is the assertion that every $\varkappa - \lambda$ mess is solvable. Clearly if $\varkappa \leqslant \lambda \leqslant \lambda'$ then $[\varkappa, \lambda', \varrho]$ implies $[\varkappa, \lambda, \varrho]$ and $[\varkappa, \lambda', \varrho]^*$ implies $[\varkappa, \lambda, \varrho]^*$.



**§ 1. Elementary consequences of the Rado property.** First we prove that it is easy to increase the size of $\varrho$.

THEOREM 1.1. *If* $[\varkappa, \lambda, 3]$ *then* $[\varkappa, \lambda, \varkappa]^*$.

Proof. Let $S$ be a set with $|S| \leqslant \lambda$, $L$ a $\varkappa$-cover of $S$ and $\mathscr{F}$ an $L$-$R$ valuation with $|R_t| < \varkappa$. For $X \in P(S)$ let $X^* = \{(t, r)|\ t \in X,\ r \in R_t\}$. $|S^*| \leqslant \lambda$. Clearly $L^* = \{A^*|\ A \in L\}$ is a $\varkappa$-cover of $S^*$. For $A^* \in L^*$ define $g_{A^*}\colon A^* \to \{0, 1\}$ by

$$g_{A^*}((t, r)) = \begin{cases} 1 & \text{for} \quad f_A(t) = r, \\ 0 & \text{otherwise}. \end{cases}$$

By $[\varkappa, \lambda, 3]$ there is a $g\colon S^* \to \{0, 1\}$, $g = \lim_{A^* \in L^*} g_{A^*}$. For $t \in S$ let $Y_t = \{t\} \times R_t$. $Y_t \subseteq S^*$ and $|Y_t| < \varkappa$. Hence there is an $A^* \supseteq Y_t$ with $g_{A^*}|Y_t = g|Y_t$. But $g_{A^*}((t, r)) = 1$ if and only if $f_A(t) = r$. So there is exactly one $r$ such that $g((t, r)) = 1$. Define $f\colon S \to R$ by

$$f(t) = r \quad \text{if and only if} \quad g((t, r)) = 1 .$$

We claim $f = \lim_{A \in L} f_A$. Let $X \in P_\varkappa(S)$. $|X^*| < \varkappa$. Hence there is an $A^* \in L^*$ with $X^* \subseteq A^*$ and $g|X^* = g_{A^*}|X^*$. Now $X \subseteq A$ and for all $t \in X$

$$f(t) = r \leftrightarrow g((t, r)) = 1 \leftrightarrow g_{A^*}((t, r)) = 1 \leftrightarrow f_A(t) = r .$$

Thus $f_A|X = f|X$.

From this point on whenever we assume $[\varkappa, \lambda, 3]$ we will use $[\varkappa, \lambda, \varkappa]^*$ or $[\varkappa, \lambda, \varkappa]$ without reference to Theorem 1.1.

We now study the effect of $[\varkappa, \varkappa, 3]$ on $\varkappa$.

THEOREM 1.2. *If* $[\varkappa, \varkappa, 3]$ *then* $\varkappa$ *is a limit cardinal.*

Proof. Suppose $\varkappa = \mu^+$. Let $S = \varkappa$, $L = P_\varkappa(\varkappa)$. For $A \in L$ we have $|A| \leqslant \mu$ since $|A| < \varkappa$. Let $f_A\colon A \to \mu$ be some injection. $\mathscr{F} = \{f_A|\ A \in L\}$ is an $L$-$\mu$ valuation. Now $\mu < \varkappa$ so by $[\varkappa, \varkappa, \varkappa]$ there is an $f = \lim_{A \in L} f_A$. $f\colon \varkappa \to \mu$. Since each $f_A$ is an injection, it is easy to verify that $f$ is an injection. Hence $\varkappa \leqslant \mu$. But $\varkappa = \mu^+$ and we have a contradiction. Thus $\varkappa$ is a limit cardinal.

THEOREM 1.3. *If* $[\varkappa, \varkappa, 3]$ *then* $\varkappa$ *is regular.*

Proof. Suppose $\varkappa$ is not regular. Then $\varkappa = \bigcup_{\alpha < \mu} X_\alpha$ where $\mu = \mathrm{cf}(\varkappa) < \varkappa$ and $X_\alpha \subset X_\beta$ if $\alpha < \beta < \mu$; $\omega \leqslant |X_\alpha| = \lambda_\alpha$ with $\lambda_\alpha < \varkappa$ and $\lambda_\alpha < \lambda_\beta$ for $\alpha < \beta < \mu$ and $\varkappa = \sup_{\alpha < \mu} \lambda_\alpha$.

Let $S = \varkappa$, $L = P_\varkappa(\varkappa)$. For each $\alpha$ consider $X_{\alpha+1} - X_\alpha$. $|X_{\alpha+1} - X_\alpha| = \lambda_{\alpha+1}$. Let $\{Y_{\alpha+1}, W_{\alpha+1}\}$ be a partition of $X_{\alpha+1} - X_\alpha$ with $|Y_{\alpha+1}| = |W_{\alpha+1}| = \lambda_{\alpha+1}$. Let $A \in L$. Since $|A| < \varkappa$ and $\{\lambda_\alpha|\ \alpha < \mu\}$ is an increasing sequence of cardinals with $\sup = \varkappa$, there is a smallest $\alpha$ such that $|A| \leqslant \lambda_\alpha$. Call this index $\alpha_0$. Consider $X_{\alpha_0+1}$.

Let $g$ be an injection from $A \cap (X_{\alpha_0+1} - X_{\alpha_0})$ into $Y_{\alpha_0+1}$ and let $h$ be an injection from $A \cap (S - X_{\alpha_0+1})$ into $W_{\alpha_0+1}$. Let us define $f_A\colon A \to S$ by

$$f_A(t) = \begin{cases} t & \text{for} \quad t \in X_{\alpha_0}, \\ g(t) & \text{for} \quad t \in X_{\alpha_0+1} - X_{\alpha_0}, \\ h(t) & \text{for} \quad t \in S - X_{\alpha_0+1}. \end{cases}$$

Each $f_A$ is an injection. Note that if $t \in X_\beta$ and $f_A(t)$ is defined then $f_A(t) \in X_\beta$. Thus

$$|\{f_A(t)|\ A \in L\}| \leqslant |X_\beta| = \lambda_\beta < \varkappa .$$

By $[\varkappa, \varkappa, \varkappa]^*$ there is an $f = \lim\limits_{A \in L} f_A$. As before it is easy to verify that $f\colon \varkappa \to \varkappa$ is an injection. We claim there is an $\alpha < \mu$ with $f[\varkappa] \subseteq X_\alpha$. If not, for each $\alpha < \mu$ there is an $y_\alpha$ with $f(y_\alpha) \notin X_\alpha$. Let $Y = \{y_\alpha|\ \alpha < \mu\}$. $|Y| < \varkappa$. Hence for some $A \in L$ $f_A| Y = f| Y$. But $f_A[Y] \subseteq X_{\alpha_0+1}$. Thus $f[Y] \subseteq X_{\alpha_0+1}$. And $f(y_{\alpha_0+1}) \in X_{\alpha_0+1}$ which contradicts the choice of $y_{\alpha_0+1}$. Hence $\varkappa$ is regular.

By Theorems 1.2, 1.3 we have: if $[\varkappa, \varkappa, 3]$ then $\varkappa$ is weakly inaccessible. Thus for uncountable $\varkappa$ $[\varkappa, \varkappa, 3]$ is a large cardinal assumption.

§ 2. **$\varkappa$-inverse limits.** In [6] the relationship between the Selection lemma (called theorem H in [6]) and inverse limit systems was explored. We now examine inverse limit systems using $[\varkappa, \lambda, 3]$.

Let $\langle I, \leqslant\rangle$ be an upper directed partially ordered set. A collection of sets $\{B_i|\ i \in I\}$ and functions $\{\varphi_{ij}|\ i \leqslant j, i, j \in I\}$ is called an inverse limit system if

i) $\varphi_{ij}\colon B_j \to B_i$,

ii) $\varphi_{ii}$ is the identity on $B_i$,

iii) $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$, $i \leqslant j \leqslant k$.

DEFINITION. The *$\varkappa$-inverse limit* $B_\varkappa^\infty$ of such a system is the set of all $p \in \prod\limits_{i \in I} B_i$ such that for each $X \in P_\varkappa(I)$ there is an $i_0 \in I$ and a $b \in B_{i_0}$ with $p(j) = \varphi_{ji_0}(b)$, $j \leqslant i_0$ for all $j \in X$.

DEFINITION. A partially ordered set $\langle I, \leqslant\rangle$ is *(upper) $\varkappa$-directed* if for each $X \in P_\varkappa(I)$ there is an $i_0 \in I$ with $j \leqslant i_0$ for all $j \in X$.

THEOREM 2.1. *Assume* $[\varkappa, \lambda, 3]$. *Let $\langle I, \leqslant\rangle$ be a $\varkappa$-directed partially ordered set with $|I| \leqslant \lambda$. Let $\{B_i\}$, $\{\varphi_{ij}\}$ be an inverse limit system with $B_i \neq \emptyset$ and $|B_i| < \varkappa$ and each $\varphi_{ij}$ a surjection. Let $N \subseteq I$ be such that $\langle N, \leqslant|\ N \times N\rangle$ is $\varkappa$-directed. Suppose there is a $g \in \prod\limits_{i \in N} B_i$ such that for each $X \in P_\varkappa(N)$ there is an $n_0 \in N$ and a $b \in B_{n_0}$ with $g(j) = \varphi_{jn_0}(b)$, $j \leqslant n_0$ for all $j \in X$. Then there is an $f \in B_\varkappa^\infty$ with $f(n) = g(n)$ for all $n \in N$.*

Proof. Let $S = I$, $L = P_\varkappa(I)$. For $A \in L$ let $A_N = A \cap N$. Since $|A_N| < \varkappa$ there is an $n_0 \in N$, $b_{n_0} \in B_{n_0}$ with $i \leqslant n_0$ and $g(i) = \varphi_{in_0}(b_{n_0})$ for all $i \in A_N$. Consider $A \cup \{n_0\}$. $|A \cup \{n_0\}| < \varkappa$. Hence there is an $i_A \in I$ with $i \leqslant i_A$, $n_0 \leqslant i_A$ for all $i \in A$.



Since $\varphi_{n_0 i_A}$ is a surjection, there is a $b_A \in B_{i_A}$ with $\varphi_{n_0 i_A}(b_A) = b_{n_0}$. Assume we have chosen such an $i_A$, $b_A$ for each $A \in L$. Define $f_A\colon A \to \bigcup\limits_{i \in I} B_i$ by

$$f_A(i) = \varphi_{ii_A}(b_A) .$$

Note $f_A(i) \in B_i$. Thus

$$|\{f_A(i)|\ A \in L\}| \leqslant |B_i| < \varkappa .$$

By $[\varkappa, \lambda, \varkappa]^*$ there is an $f = \lim\limits_{A \in L} f_A$. Clearly $f \in \prod\limits_{i \in I} B_i$. We claim $f$ is the desired function. Let $X \in P_\varkappa(I)$. There is an $A \in L$ with $X \subseteq A$ and $f|X = f_A|X$. So $f(i) = \varphi_{ii_A}(b_A)$ for $i \in X$ and $f \in B_\varkappa^\infty$. Let $n \in N$ there is an $A \in L$ with $n \in A$ and $f_A(n) = f(n)$. But

$$f_A(n) = \varphi_{ni_A}(b_A) = \varphi_{nn_0} \circ \varphi_{n_0 i_A}(b_A) = \varphi_{nn_0}(b_{n_0}) = g(n) .$$

Thus $f(n) = g(n)$ for all $n \in N$.

COROLLARY. *With the assumptions of the theorem and $N = \emptyset$ we have $B_\varkappa^\infty \neq \emptyset$.*

Remark. With $N = \emptyset$ the requirement that each $\varphi_{ij}$ be a surjection is no longer necessary. See Kurosh [8], pp. 168–169 for the proper modification.

The next theorem is just an observation on how much compactness the infinitary language $\mathscr{L}_\varkappa$ should have for $\varkappa$ to be measurable. See [1] for a language proof.

THEOREM 2.2. *Assume that either $\varkappa$ is inaccessible or that* GCH *holds. Then* $[\varkappa, 2^\varkappa, 3]$ *implies $\varkappa$ is measurable.*

Proof. If $[\varkappa, 2^\varkappa, 3]$ then $\varkappa$ is weakly inaccessible and if GCH holds we have $\varkappa$ is inaccessible. We proceed under the assumption that $\varkappa$ is inaccessible. Let $I = \{\sigma|\ \sigma$ partitions $\varkappa$ into $< \varkappa$ nonempty subsets$\}$; that is $\sigma = \{X_{\xi,\sigma}|\ \xi < \mu < \varkappa\}$ with $X_{\xi,\sigma} \neq \emptyset$ and $X_{\xi_1,\sigma} \cap X_{\xi_2,\sigma} = \emptyset$ if $\xi_1 \neq \xi_2$, $\varkappa = \bigcup\limits_{\xi < \mu} X_{\xi,\sigma}$. If $X \in P(\varkappa)$ and $\varkappa - X \in P_\varkappa(\varkappa)$ we say $X$ is large. Let $N = \{\sigma|\ \sigma \in I$ and some $X_{\xi,\sigma}$ is large$\}$. We partially order $I$ and $N$ by refinement. $\sigma \leqslant \tau$ if and only if each element of $\tau$ is in a (unique) element of $\sigma$. We claim $\langle I, \leqslant\rangle$ $\langle N \leqslant\rangle$ are $\varkappa$-directed. Let $\{\sigma_\nu|\ \nu < \varrho < \varkappa\} \subseteq I$ where $\sigma_\nu = \{X_{\xi,\sigma_\nu}|\ \xi < \varrho_{\sigma_\nu} < \varkappa\}$. Since $\varkappa$ is regular, $|\bigcup\limits_{\nu < \sigma} \sigma_\nu| = \theta < \varkappa$. We obtain a common refinement by taking in every possible way one element from each partition, forming their intersection and omitting the empty intersections. The common refinement $\sigma$ has $\leqslant \theta^\varrho$ elements. Since $\varkappa$ is inaccessible $\theta^\varrho < \varkappa$ and $\sigma \in I$. If for each $\sigma_\nu$, $\sigma_\nu \in N$ then for each $\sigma_\nu$ some $X_{\xi_\nu,\sigma_\nu}$ is large. Clearly $\bigcap\limits_{\nu < \varrho} X_{\xi_\nu,\sigma_\nu}$ is large and $\sigma \in N$.

Let $B_\sigma = \sigma$. For $\sigma \leqslant \tau$ define $\varphi_{\sigma\tau}\colon B_\tau \to B_\sigma$ by $\varphi_{\sigma\tau}(X_{\xi,\tau}) =$ unique $X_{\alpha,\sigma}$ such that $X_{\xi,\tau} \subseteq X_{\alpha,\sigma}$ Each $\varphi_{\sigma\tau}$ is a surjection. $\{B_\sigma\}$, $\{\varphi_{\sigma\tau}\}$ is an inverse limit system. Let $g \in \prod\limits_{\sigma \in N} B_\sigma$ be such that $g(\sigma)$ is large for all $\sigma \in N$. Note $|I| = 2^\varkappa$. Since $[\varkappa, 2^\varkappa, 3]$, by Theorem 2.1. there is an $f \in B_\varkappa^\infty$ with $f(\sigma) = g(\sigma)$ for all $\sigma \in N$. Let $U = \{X|\ \exists \sigma \in I, f(\sigma) = X\}$. Every large subset of $\varkappa$ is in $U$. For each $X \in P(\varkappa)$ either

$X\in U$ or $\varkappa-X\in U$. We claim $U$ has the $\varkappa$ intersection property; that is if $\{X_\xi|\ \xi<\theta<\varkappa\}\subseteq U$ then $\bigcap_{\xi<\theta} X_\xi\neq\emptyset$. Consider such an $\{X_\xi|\ \xi<\theta<\varkappa\}\subseteq U$. Let $\sigma_\xi$ be such that $f(\sigma_\xi)=X_\xi$. $|\{\sigma_\xi|\ \xi<\theta\}|<\varkappa$. Hence there is a $\sigma\in I$ and $Y_\sigma\in B_\sigma$ with $\sigma_\xi\leqslant\sigma$ and $f(\sigma_\xi)=\varphi_{\sigma_\xi,\sigma}(Y_\sigma)$ for all $\xi$. By the definition of $\varphi_{\sigma_\xi,\sigma}$, $Y_\sigma\neq\emptyset\subseteq f(\sigma_\xi)$. Thus $Y_\sigma\subseteq\bigcap_{\xi<\theta}X_\xi\neq\emptyset$. Let

$$U^*=\{X\in P(\varkappa)|\ \exists\{X_\xi|\ \xi<\theta<\varkappa\}\subseteq U,\ \bigcap_{\xi<\theta}X_\xi\subseteq X\}.$$

It is easy to verify that $U^*$ is a $\varkappa$-complete ultrafilter over $\varkappa$. $U^*$ is nonprincipal since every large subset of $\varkappa$ is in $U^*$. Hence $\varkappa$ is measurable.

## § 3. Some algebraic properties of weakly compact and compact cardinals.

Unless otherwise stated we assume $\varkappa$ is uncountable, weakly compact, and inaccessible. By Jech's work we have $[\varkappa,\varkappa,3]$ and hence $[\varkappa,\varkappa,\varkappa]$. $\tau$ denotes a type for a universal algebra, having finitary operation symbols and constants but no relation symbols. We assume $|\tau|<\varkappa$.

$\Gamma$ denotes a class of pairs $(A,X)$ where $A$ is a $\tau$-algebra and $X$ is a generating set for $A$. We write this as $A=\langle X\rangle$. $\Gamma$ is assumed to satisfy

(i) (*isomorphism closure*) if $(A,X)\in\Gamma$ and $\varphi$ is an isomorphism on $A$ then $(\varphi[A],\varphi[X])\in\Gamma$.

(ii) (*s-closure*) if $(A,X)\in\Gamma$ and $Y\subseteq X$ then $(\langle Y\rangle,Y)\in\Gamma$.

(iii) (*weak $\varkappa$-local property*) if $A$ is a $\tau$-algebra and $A=\langle X\rangle$ where $|X|=\varkappa$ and $(\langle Y\rangle,Y)\in\Gamma$ for each $Y\in P_\varkappa(X)$ then $(A,X)\in\Gamma$.

If $(A,X)\in\Gamma$ we say $X$ is a *$\Gamma$-basis* for $A$. We say a $\tau$-algebra $B$ is *$\Gamma$-embeddable* if for some $(A,X)\in\Gamma$ we have $B\subseteq A$.

THEOREM 3.1. *Let $D$ be a $\tau$-algebra with $|D|=\varkappa$. If every subalgebra $S\in P_\varkappa(D)$ is $\Gamma$-embeddable then $D$ is $\Gamma$-embeddable.*

Proof. Let $F$ be the free algebra of type $\tau$ with countable free basis $Z=\{z_1,\dots,z_n,\dots\}$. $|F|<\varkappa$. If $n<m$ are natural numbers, let $Z[n,m]=(z_n,z_{n+1},\dots,z_m)$. For any set $X$ let $X^*$ denote the set of non-empty finite ordered subsets of $X$.

Let $\mathcal{S}=\{S|\ S$ subalgebra of $D$, $S\in P_\varkappa(D)\}$. Since $|\tau|<\varkappa$ and $\varkappa>\omega$, $\mathcal{S}$ is a $\varkappa$-cover of D. Hence $\mathcal{S}^*=\{S^*|\ S\in\mathcal{S}\}$ is a $\varkappa$-cover of $D^*$. Let $S\in\mathcal{S}$. Since $S$ is $\Gamma$-embeddable we can choose a pair $(A(S),X(S))\in\Gamma$ with $S\subseteq A(S)$.

Construction of $f_{S^*}$. Fix $S\in\mathcal{S}$. Let $s\in S$. Since $s\in\langle X(S)\rangle$, there is some $\tau$ polynomial $p_s[Z[1,n(s)]]\in F$ and some $X[s]=(x_1,\dots,x_n(s))\in X(S)^*$ with $s=p\,[X[s]]$. We define $f_{S^*}$ on $S^*$ as follows:

Let $J=(s_1,\dots,s_m)\in S^*$. Let $u_0=0$ and $u_i=n(s_1)+\dots+n(s_i)$ for $1\leqslant i\leqslant m$.

$$f_{S^*}(J)=(\hat p_{s_1},\dots,\hat p_{s_m},R)$$

where $\hat p_{s_i}=p_{s_i}[Z[u_{i-1}+1,u_i]]\in F$ and $R$ is the set of all equational relations



involving $z_j$, $1\leqslant j\leqslant u_m$ induced by the homomorphism (from $F$ into $A(S)$) which maps $Z[u_{i-1}+1,u_i]$ to $X[s_i]$, $1\leqslant i\leqslant m$. We view $R$ as a subset of $F\times F$.

Thus $\{f_{S^*}|\ S^*\in\mathcal{S}^*\}$ is an $\mathcal{S}^*-F^*\times P(F\times F)$ valuation. Since $|F|<\varkappa$ and $\varkappa$ is inaccessible, $|F^*\times P(F\times F)|<\varkappa$. By $[\varkappa,\varkappa,\varkappa]$ we obtain an $f=\lim f_{S^*}$ and domain $f=D^*$.

Let $s\in D$. Then $f((s))=(\hat p_s[Z[1,n(s)]],R_s)$ since $f((s))=f_{S^*}((s))$ for some $S^*\in\mathcal{S}^*$. We now define ordered sets of symbols

$$\mathfrak{A}_s^0=\big(a(s,1),\dots,a(s,n(s))\big)\quad\text{for each } s\in D.$$

Let $\mathfrak{A}_s$ denote the corresponding unordered set of symbols. All such symbols are regarded as formally distinct. Let $\mathfrak{A}=\bigcup\{\mathfrak{A}_s|\ s\in D\}$. $\langle\mathfrak{A}\rangle$ denotes the free $\tau$-algebra on the symbols of $\mathfrak{A}$. We place equational relations on the generators $\mathfrak{A}$ in accordance with the information coded by $f$. In particular let $J=(s_1,\dots,s_m)\in D^*$. There is an $S^*$ such that $f|\{(s_1),\dots,(s_m),J\}=f_{S^*}|\{(s_1),\dots,(s_m),J\}$. Hence $f(J)=(\hat p_{s_1},\dots,\hat p_{s_m},R(J))$. The relations of $R(J)$ induce relations $\bar R(J)$ on the generators $\mathfrak{A}_{s_1}\cup\dots\cup\mathfrak{A}_{s_m}$ via the 1-1 correspondence $Z[u_{i-1}+1,u_i]\mapsto\mathfrak{A}^0_{s_i}$, $1\leqslant i\leqslant m$. Let $\bar R=\bigcup\{\bar R(J)|\ J\in D^*\}$. We claim i) $\mathfrak{A}/\bar R$ is a $\Gamma$-basis of $\langle\mathfrak{A}\rangle/\bar R$ and ii) $D$ is embedded in $\langle\mathfrak{A}\rangle/\bar R$ by $s\mapsto p_s[\mathfrak{A}^0_s]$.

Proof of (i). Let $T\in P_\varkappa(\mathfrak{A})$. Then

$$D_T=\{s\in D|\ T\cap\mathfrak{A}_s\neq\emptyset\}\in P_\varkappa(D).$$

By the properties of $f$ there is some $S\in\mathcal{S}$ with $D_T\subseteq S$ and $f|D_T^*=f_{S^*}|D_T^*$. Let $g\colon\bigcup\{\mathfrak{A}_s|\ s\in S\}\to X(S)$ be the mapping induced by the mapping $\mathfrak{A}^0_s\to X[s]$, $s\in S$. $g$ is well defined since all the symbols are distinct. Let $\bar R_T$ be those relations in $\bar R$ involving only members of $T$. Since $f$ and $f_{S^*}$ agree on $D_T^*$, the relations $\bar R_T$ are precisely those induced by $g|T$. Hence $g$ induces an isomorphism between $\langle T\rangle/\bar R$ and $\langle g(T)\rangle\subseteq A(S)$. By the $s$-closure and isomorphism closure of $\Gamma$ we conclude that $T/\bar R$ is a $\Gamma$-basis of $\langle T\rangle/\bar R$. Now i) follows from the weak $\varkappa$-local property of $\Gamma$.

Proof of (ii). Let $J=(s_1,\dots,s_m)\in D^*$.

Let $S^*$ and $f_{S^*}$ be as in the proof of i) and

$$f(J)=(\hat p_{s_1},\dots,\hat p_{s_m},R(J)).$$

Let $\bar R(J)$, be, as before, the relations induced on $\mathfrak{A}_{s_1}\cup\dots\cup\mathfrak{A}_{s_m}$ by $R(J)$ under the mapping induced by $\mathfrak{A}^0_{s_i}\to X[s_i]$, $1\leqslant i\leqslant m$. This induced mapping extends to an isomorphism between $\langle\mathfrak{A}_{s_1}\cup\dots\cup\mathfrak{A}_{s_m}\rangle/\bar R$ and $\langle X[s_i]|\ 1\leqslant i\leqslant m\rangle\subseteq A(S)$. The image of each $p_{s_i}[\mathfrak{A}^0_{s_i}]$ under this isomorphism is $p_{s_i}[X[s_i]]=s_i$. We conclude that the mapping $p_s[\mathfrak{A}^0_s]\to s$ for $s\in D$ is an isomorphism of a subalgebra of $\langle\mathfrak{A}\rangle/\bar R$ with $D$. This proves the theorem.

Before we can apply this theorem we need some additional definitions.

DEFINITION. Let $\Sigma$ be a class of $\tau$ algebras. $\Sigma$ is *weak $\varkappa$-local* if whenever $A$ is a $\tau$-algebra with $|A|=\varkappa$ and every subalgebra $S\in P_\varkappa(A)$ is in $\Sigma$, then $A\in\Sigma$.

DEFINITION. $\Sigma$ as above. $s\Sigma$ = class of subalgebras of algebras in $\Sigma$. We say $\Sigma$ is *s-closed* if $s\Sigma = \Sigma$.

COROLLARY 1. *Let $V$ be any quasivariety (universal Horn class) of $\tau$-algebras. Let $\Phi$ be the class of free $V$-algebras. Then $s\Phi$ is weak $\varkappa$-local.*

Proof. First by [3], pp. 236, $\Phi \neq \emptyset$. Let $\Gamma$ consist of all pairs $(A, X)$ where $A \in \Phi$ and $X$ is a $V$-free basis for $A$. Since $V$ is a universal Horn class, $\Gamma$ is $s$-closed and isomorphism closed. $\Gamma$ has the weak $\varkappa$-local property. In fact $\Gamma$ has the following stronger property: if $A \in V$ and $A = \langle X\rangle$ and for each $Y \in P_\omega(X)$ $(\langle Y\rangle, Y) \in \Gamma$, then $(A, X) \in \Gamma$. Later we call this the $\omega$-local property. Let $B \in V$ and let $\varphi\colon X \to B$ be any mapping. We must show there is a $\tau$-algebra homomorphism $\psi\colon A \to B$ such that $\psi|X = \varphi$. Now for each $Y \in P_\omega(X)$ there is a unique homomorphism $\varphi_Y\colon \langle Y\rangle \to B$ such that $\varphi_Y|Y = \varphi|Y$. This is because $(\langle Y\rangle, Y) \in \Gamma$. Using this one can easily define the required homomorphism $\psi$. Thus $(A, X) \in \Gamma$. $s\Phi$ is just the class of all $\Gamma$-embeddable algebras. The corollary now follows from Theorem 3.1.

COROLLARY 2 (Mekler, Gregory). *The classes of free groups and of free abelian groups are weak $\varkappa$-local.*

Proof. For each class $\Phi$ mentioned we have $s\Phi = \Phi$ and Corollary 1 applies.

The next two corollaries concern free and direct products of groups. $\Sigma$ is any class of groups which is $s$-closed and weak $\varkappa$-local. For example we can take $\Sigma$ to be any universal class of groups.

COROLLARY 3. *Let $\Sigma^*$ be the class of free products of $\Sigma$ groups. Then*

(i) *$s\Sigma^*$ is weak $\varkappa$-local,*

(ii) *if the infinite cyclic group is in $\Sigma$ then $\Sigma^*$ is weak $\varkappa$-local.*

Proof of (i). We define $\Gamma$ as follows. $(G, X) \in \Gamma$ if and only if $G$ is a group, $G = \langle X\rangle$ and there is a partition $\pi$ of $X$ such that a) each $Y \in \pi$ generates a $\Sigma$-subgroup of $G$ and b) $G$ is the free product of the subgroups $\langle Y\rangle$, $Y \in \pi$. $\Gamma$ is clearly $s$-closed and isomorphism closed. We claim $\Gamma$ has the weak $\varkappa$-local property. Suppose $|X| = \varkappa$ and for every $W \in P_\varkappa(X)$ $(\langle W\rangle, W) \in \Gamma$. Let $\pi_W$ be a partition of $W$ as in the definition of $\Gamma$. We view $\pi_W$ as a function $f_W\colon W \times W \to \{0, 1\}$ where $f_W(x, z) = 1 \leftrightarrow x \sim z \bmod \pi_W$. By $[\varkappa, \varkappa, 3]$ there is an $f\colon X \times X \to \{0, 1\}$, $f = \lim f_W$, defining a partition $\pi$ of $X$. It is easy to verify that $\pi$ satisfies a) and b). Hence $(\langle X\rangle, X) \in \Gamma$. Since $s\Sigma^*$ is the class of $\Gamma$-embeddable algebras, i) is proved.

Proof of (ii). By the Kurosh subgroup theorem for free products ([8], pp. 17–26) any group $G \in s\Sigma^*$ satisfies $G = F * H$ where $F$ is a free group and $H \in \Sigma^*$. Since $F$ is a free product of infinite cyclic groups, ii) is proved.

COROLLARY 4. *The class of subdirect products of $\Sigma$ groups is weak $\varkappa$-local.*

By subdirect product, we mean a subgroup of a (restricted) direct product. The proof is analogous to that of (i) in Corollary 3.

We now assume that $\varkappa$ is uncountable and compact. We alter the definition of the class $\Gamma$ by replacing (iii) with (iii′) (*$\varkappa$-local property*) if $A$ is a $\tau$-algebra and $A = \langle X\rangle$ and $(\langle Y\rangle, Y) \in \Gamma$ for each $Y \in P_\varkappa(X)$ then $(A, X) \in \Gamma$. Now Theorem 3.1 can be proven without any restriction on $|D|$. We just use $[\varkappa, \lambda, \varkappa]$ for the appropriate $\lambda$.



DEFINITION. Let $\Sigma$ be a class of $\tau$-algebras. $\Sigma$ is *$\varkappa$-local* if whenever $A$ is a $\tau$-algebra and every subalgebra $S \in P_\varkappa(A)$ is in $\Sigma$, then $A \in \Sigma$.

If $\varkappa$ is uncountable and compact Corollaries 1, 2, 3, and 4 hold when the conclusion that the relevant class is weak $\varkappa$-local is strengthened to $\varkappa$-local. Of course in 3 and 4 the hypothesis that "$\Sigma$ is weak $\varkappa$-local" must be changed to "$\Sigma$ is $\varkappa$-local." Corollary 2 in this form is also due to Mekler and Gregory.

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