[13] S. Mardešić and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.
[14] — *Equivalence of the Borsuk and the* ANR-*system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.
[15] D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. 79 (1964), pp. 327–337.
[16] T. B. Rushing, *Topological Embeddings*, Academic Press, New York 1973.
[17] L. C. Siebenmann, *On detecting open collars*, Trans. Amer. Math., Soc. 142 (1969), pp. 201–227.
[18] J. R. Stallings, *The piecewise linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. 58 (1962), 481–488.
[19] C. T. C. Wall, *Finiteness conditions for* CW-*complexes*, Ann. of Math. 81 (2) (1965), pp. 56–59.
[20] J. H. C. Whitehead, *Combinatorial homotopy*, I, Bull. Amer. Math. Soc. 55 (1949), pp. 213–245.
[21] E. C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. 78 (1963), pp. 501–526.
[22] — *Twisting spun knots*, Trans. Amer. Math. Soc. 115 (1965), pp. 471–495.

UNIVERSITY OF UTAH

Accepté par la Rédaction le 10. 2. 1975

Adjoint groups and the Mal'cev correspondence (a tale of four functors)

by

Ian Stewart (Coventry)

Abstract. We make several observations on the connection between the structure of an associative algebra, its Lie algebra, and its adjoint group; with especial reference to the Mal'cev correspondence between Lie algebras and groups. We view in this light the construction, by Levič and Tokarenko, of a locally nilpotent non-Gruenberg Lie algebra.

1. Introduction. The Mal'cev correspondence (Mal'cev [11]) associates to each complete locally nilpotent torsion-free group (in the sense of Kuroš [8] pp. 233, 248) a locally nilpotent Lie algebra over the rational field Q. It defines a pair of mutually inverse *exact* functors

$$\mathscr{L}\colon \mathscr{C}_{\mathscr{G}}\to\mathscr{C}_{\mathscr{L}}\,,\qquad \mathscr{G}\colon \mathscr{C}_{\mathscr{L}}\to\mathscr{C}_{\mathscr{G}}$$

where $\mathscr{C}_{\mathscr{G}}$ and $\mathscr{C}_{\mathscr{L}}$ are the categories of complete locally nilpotent torsion-free groups and of locally nilpotent Lie algebras over Q. A treatment of these results in a manner appropriate to what follows may be found in [15] where the functors are first constructed in the finitely generated ("local") case and then extended to the "global" one.

There is a situation in which standard constructions give rise to groups and Lie algebras of this type. Let R be an associative ring. Under commutation R form a Lie ring $[R]$. Under the operation $\circ$ given by

$$a\circ b = a+b+ab \quad (a, b\in R)$$

R forms a semigroup with 0 as identity. The invertible elements form a group R^0 known as the *adjoint* (or *associated*) group of R. (Compare Kuroš [8] p. 38 where, however, a different definition of $\circ$ is used. This makes no difference since the map $a\to -a$ converts one into the other). If R is a nil ring then every element of R is invertible. Suppose now that $\mathscr{A}$ is the category of locally nilpotent associative algebras over Q, and that $R\in\mathscr{A}$. Then $[R]$ lies in $\mathscr{C}_{\mathscr{L}}$; and it may be shown that R^0 lies in $\mathscr{C}_{\mathscr{G}}$ (cf. Mal'cev [12]). We may therefore form $\mathscr{G}([R])$ and $\mathscr{L}(R^0)$. We shall exhibit isomorphisms

$$R^0\cong \mathscr{G}([R])\,,\qquad [R]\cong\mathscr{L}(R^0)\,.$$

Essentially these are due to Mal'cev [12] pp. 360–361 in the finite-dimensional case. We shall give a proof which ties in with the approach used in [15].

These isomorphisms are "natural" and the diagram of functors

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "\circ"'] \arrow[dr, "{[\,]}"] & \\ \mathscr{C}_{\mathscr{G}} \arrow[rr, shift left, "\mathscr{L}"] & & \mathscr{C}_{\mathscr{L}} \arrow[ll, shift left, "\mathscr{G}"] \end{tikzcd}$$

commutes "up to natural transformations" in a sense we make precise in Section 2.

Though the existence of these isomorphisms is easy to prove, it has a variety of applications. Some of Mal'cev's results in [12] that "generalised nilpotent" properties of R carry over to R^0 can easily be deduced from them when $R \in \mathscr{A}$. We shall not pursue the matter here. Instead we shall apply the results to a recent construction of Levich and Tokarenko, which yields a locally nilpotent torsion-free non-Gruenberg group. (A *Gruenberg group* is one generated by ascendant abelian subgroups, cf. Robinson [13] p. 100.) Levich and Tokarenko remark that one may now construct a locally nilpotent non-Gruenberg Lie algebra (answering question 1 of [16] p. 81): presumably what they have in mind consists of forming the completion of their group and applying the Mal'cev correspondence. The resulting object, however, is not presented in a very explicit form. Using our result we can give a somewhat simpler construction, although leaning heavily on the work of Levich and Tokarenko.

Another application (in Section 2) concerns the problem of characterising adjoint groups of algebras (or equivalently groups of units). We show that a group G with trivial centre embeds in the associated group of a locally nilpotent ring whose additive group is torsion-free provided G is a torsion-free Baer group. (A *Baer group* is one generated by abelian subnormal subgroups, cf. Robinson [13] p. 101).

Finally in Section 4 we sketch other possible applications of our results.

2. The Mal'cev correspondence. We summarise the construction of the functors $\mathscr{G}$ and $\mathscr{L}$ given in [15]. It boils down to this: for elements $x, y \in L$ ($L \in \mathscr{C}_{\mathscr{L}}$) we define a product

$$x \cdot y = x+y+\tfrac{1}{2}[x, y]+\tfrac{1}{12}[x, y, y]+\tfrac{1}{12}[y, x, x]+\ldots$$

where the expression on the right is that occurring in the Campbell–Hausdorff formula (Jacobson [6] p. 171). Under this operation, L becomes a group $\mathscr{G}(L)$ lying in $\mathscr{C}_{\mathscr{G}}$. Conversely for every group $G \in \mathscr{C}_{\mathscr{G}}$ we can define Lie operations

$$\lambda g = g^{\lambda},$$
$$g+h = gh(g, h)^{-1/2}(g, h, g)^{-1/12}(g, h, h)^{-1/12}\ldots,$$
$$[g, h] = (g, h)(g, h, g)^{-1/2}(g, h, h)^{-1/2}\ldots$$

($\lambda \in \mathbf{Q}$, $g, h \in G$), where $(g, h) = g^{-1}h^{-1}gh$ is the group commutator. These commutator expressions arise from "inverting" the Campbell–Hausdorff formula (cf.

Lazard [9]). Then G with these operations gives a Lie algebra in $\mathscr{C}_{\mathscr{L}}$, which we call $\mathscr{L}(G)$.

Next we turn to the adjoint group. If $R \in \mathscr{A}$ we can redefine R^0 in the usual way. Adjoin an identity 1 to R and consider the resulting ring S. The elements $1+r$ of S ($r \in R$) form a subgroup of the multiplicative semigroup of S, which is isomorphic to R^0. We may identify R^0 with the subgroup $1+R$ of S.

We define a function

$$\varphi\colon \mathscr{G}([R]) \to R^0$$

by

$$\varphi(x) = \exp(x) = 1+x+\frac{1}{2!}x^2+\ldots$$

for $x \in [R]$. This is well-defined and its image lies inside $1+R = R^0$. Further, φ is a group homomorphism, for

$$\begin{aligned}\varphi(x)\varphi(y) &= \exp(x)\exp(y)\\ &= \exp(x+y+\tfrac{1}{2}[x, y]+\ldots)\\ &= \varphi(xy)\end{aligned}$$

because the Campbell–Hausdorff formula holds in R. Now φ is injective, for if $\varphi(x) = 1$ then

$$x = \log\varphi(x) = \log 1 = 0$$

and is surjective because if $r \in R$ then

$$\begin{aligned}1+r &= \exp\log(1+r)\\ &= \exp(r-\tfrac{1}{2}r^2+\tfrac{1}{3}r^3-\ldots)\,.\end{aligned}$$

This shows that $\mathscr{G}([R]) \cong R^0$. Since $\mathscr{G}$ and $\mathscr{L}$ are mutual inverses we have $\mathscr{L}(R^0) \cong [R]$. We state these results as:

THEOREM 1. *Suppose R is a locally nilpotent associative algebra over $\mathbf{Q}$. Then*

$$R^0 \cong \mathscr{G}([R])\,, \qquad [R] \cong \mathscr{L}(R^0)\,.$$

Let $\mathscr{A}_{\mathscr{G}}$ and $\mathscr{A}_{\mathscr{L}}$ be the subcategories of $\mathscr{C}_{\mathscr{G}}$ and $\mathscr{C}_{\mathscr{L}}$ whose objects are the R^0, $[R]$ for $R \in \mathscr{A}$ and whose morphisms are induced from $\mathscr{A}$. We can redefine $\mathscr{G}$ and $\mathscr{L}$ as functors $\mathscr{A}_{\mathscr{L}} \to \mathscr{A}_{\mathscr{G}}$ and $\mathscr{A}_{\mathscr{G}} \to \mathscr{A}_{\mathscr{L}}$ using exp and log; this redefinition amounts to a natural transformation of functors. Clearly now the diagram

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "\circ"'] \arrow[dr, "{[\,]}"] & \\ \mathscr{A}_{\mathscr{G}} \arrow[rr, shift left, "\mathscr{L}"] & & \mathscr{A}_{\mathscr{L}} \arrow[ll, shift left, "\mathscr{G}"] \end{tikzcd}$$

commutes.

3. The Levich-Tokarenko construction. The theorem of Levich–Tokarenko [10] says that every periodic locally nilpotent group is an epimorphic image of a torsion-free locally nilpotent group. We sketch their construction, with a slightly different proof.

Periodic locally nilpotent groups are uniquely expressible as a direct product of locally finite p-groups for distinct primes p (Kuroš [8] p. 229), whence it suffices to prove the theorem for a locally finite p-group P. Form the group algebra $\boldsymbol{Z}_p P$ and let I be its augmentation ideal, spanned by the elements $g-1$ ($g \in P$). From standard results in the representation theory of finite p-groups (Curtis and Reiner [2] p. 189) it follows that I is a locally nilpotent algebra. Thus the multiplicative semigroup $\dot{I}$ of I is a locally nilpotent semigroup (with zero). We may form the semigroup rings $\boldsymbol{Z}\dot{I}$ and $\boldsymbol{Z}_p\dot{I}$ (identifying the zero of $\dot{I}$ with that of the coefficient ring) and these are also locally nilpotent. There are obvious ring homomorphisms

$$\boldsymbol{Z}\dot{I} \to \boldsymbol{Z}_p\dot{I} \to I$$

which induce group homomorphisms

$$(\boldsymbol{Z}\dot{I})^0 \to (\boldsymbol{Z}_p\dot{I})^0 \to I^0.$$

Now the map $g \to g-1$ gives a group monomorphism $P \to \dot{I}$.

Since $\boldsymbol{Z}\dot{I}$ is locally nilpotent and has torsion-free additive group it is easy to see that $(\boldsymbol{Z}\dot{I})^0$ is a locally nilpotent torsion-free group. (This also follows from our subsequent application of the Mal'cev correspondence.) We identify P with its image in I^0 and co-restrict, giving a subgroup $\mathscr{T}(P)$ of $(\boldsymbol{Z}\dot{I})^0$ and a canonical epimorphism

$$\mathscr{T}(P) \to P.$$

Let P be the Kovacs–Neumann–Kargapolov group, which is a non-Gruenberg locally finite p-group (Robinson [13] p. 108, Kargapolov [7]). Then $G = \mathscr{T}(P)$ is locally nilpotent torsion-free, and has a completion G^*. It follows from [15] p. 307 Lemma 2.4.5 that $\mathscr{L}(G^*)$ is a locally nilpotent non-Gruenberg Lie algebra.

However, as we said, the structure of $\mathscr{L}(G^*)$ is not very explicit. Instead, we take $\dot{I}$ as above and form the semigroup algebra $\boldsymbol{Q}\dot{I}$ (again identifying zeros) which lies in $\mathscr{A}$. Then $L = [\boldsymbol{Q}\dot{I}]$ is a locally nilpotent Lie algebra, and

$$(**) \qquad \mathscr{G}(L) \cong (\boldsymbol{Q}\dot{I})^0 \supseteq (\boldsymbol{Z}\dot{I})^0 \supseteq \mathscr{T}(P).$$

Thus $\mathscr{G}(L)$ has a section (subquotient) isomorphic to P, and is non-Gruenberg. So L is non-Gruenberg.

Further, $\boldsymbol{Q}\dot{I}$ is a non-Gruenberg associative algebra in the sense that it is not generated by ascendant zero subalgebras (with the obvious definitions). Thus, at a stroke, we have:

THEOREM 2. *Let $\dot{I}$ be the multiplicative semigroup of the augmentation ideal of $\boldsymbol{Z}_p P$, where P is any locally finite p-group which is non-Gruenberg (e.g. the Kovács–Neumann–Kargapolov group). If A is the semigroup algebra $\boldsymbol{Q}\dot{I}$ then:*

(i) *A is a locally nilpotent non-Gruenberg associative algebra over $\boldsymbol{Q}$,*
(ii) *A^0 is a locally nilpotent torsion-free non-Gruenberg group,*
(iii) *$[A]$ is a locally nilpotent non-Gruenberg Lie algebra over $\boldsymbol{Q}$.*

Notice that (**) above implies $(\boldsymbol{Z}\dot{I})^0$ locally nilpotent and torsion-free in the Levich–Tokarenko construction, as remarked at the start of this section.

4. Embedding in adjoint groups. The question arises of characterising the Lie algebras in $\mathscr{A}_{\mathscr{L}}$, and the groups in $\mathscr{A}_{\mathscr{G}}$. It is easy to see that every $L \in \mathscr{A}_{\mathscr{L}}$ has the following property: if X is any finite-dimensional subspace of L, then there exists an integer $n = n(X)$ such that

$$(*) \qquad [L, \underbrace{X, \ldots, X}_{n}] = 0.$$

This is a kind of Engel condition, and we shall refer to algebras satisfying it as $\mathfrak{E}$-algebras.

For any $x \in L$ we write, as usual, $\mathrm{ad}(x)$ for the map $L \to L$ defined by

$$y\,\mathrm{ad}(x) = [y, x] \quad (x \in L)$$

and we let $\mathrm{ad}(L)$ be the associative algebra generated by the $\mathrm{ad}(x)$ ($x \in L$). Clearly L is an $\mathfrak{E}$-algebra if and only if $\mathrm{ad}(L)$ is locally nilpotent as an associative algebra. If L has trivial centre then the adjoint representation $\mathrm{ad}\colon L \to \mathrm{ad}(L)$ is faithful, so we have:

PROPOSITION 3. *A Lie algebra L over $\boldsymbol{Q}$ with trivial centre can be embedded in a Lie algebra in $\mathscr{A}_{\mathscr{L}}$ if and only if L is an $\mathfrak{E}$-algebra.*

Proof. Note that a subalgebra of an $\mathfrak{E}$-algebra is an $\mathfrak{E}$-algebra.

From Theorem 1 we have the:

COROLLARY 4. *A group G with trivial centre can be embedded in the adjoint group of a ring with torsion-free additive group if and only if G is a locally nilpotent torsion-free group such that $\mathscr{L}(G^*)$ is an $\mathfrak{E}$-algebra, where G^* is the completion of G.*

In particular we deduce:

PROPOSITION 5. *Any torsion-free Baer group can be embedded in the adjoint group of a locally nilpotent ring (indeed $\boldsymbol{Q}$-algebra).*

Proof. If G is a Baer group then so is G^* and therefore $\mathscr{L}(G^*)$ is a Baer algebra (in the obvious sense) by [15] p. 307. But Baer algebras satisfy (*) because every finite-dimensional subspace lies inside a nilpotent subideal (Hartley [5] p. 259).

It is easy to see (cf. Robinson [13] pp. 11, 68) that a group G is a Baer group if and only if for every finitely generated subgroup H of G there exists $n = n(H)$ such that

$$(**) \qquad (G, \underbrace{H, \ldots, H}_{n}) = 1.$$

This is a reasonable analogue of the condition defining $\mathfrak{E}$-algebras. However, we have just constructed an $\mathfrak{E}$-algebra which is non-Baer, so certainly non-Baer. In consequence:

(a) $\mathfrak{E}$-algebras need not be Baer algebras.

(b) Subgroups of A^0, for $A \in \mathscr{A}$, need not be Baer groups.

(c) Condition (**) for G is not equivalent to condition (*) for $\mathscr{L}(G^*)$.

(d) Lemma 2.5.2 of [15] p. 310 is not always true for non-normal subgroups A, B.

Proposition 5 has some bearing on the question of characterising groups of units of rings (Fuchs [3] p. 299).

Finally we note that $\mathscr{A}_{\mathscr{L}}$ is strictly smaller than $\mathscr{C}_{\mathscr{L}}$. For let L be a Lie algebra with basis $x_0, x_1, \ldots, \sigma$ where $[x_i, x_j] = 0 = [x_0, \sigma]$, $[x_i, \sigma] = x_{i-1}$ $(i \geqslant 1)$. Then $L \in \mathscr{C}_{\mathscr{L}}$ but L is not an $\mathfrak{E}$-algebra (take $X = \langle \sigma \rangle$). By the same token $\mathscr{G}(L)$ is in $\mathscr{C}_{\mathfrak{g}}$ but not in $\mathscr{A}_{\mathfrak{g}}$.

5. Further remarks.

(1) If $\mathfrak{k}$ is any field of characteristic zero and A is as in Theorem 3, then $\mathfrak{k} \otimes_Q A$ and $[\mathfrak{k} \otimes_Q A]$ will be locally nilpotent non-Gruenberg associative and Lie algebras over $\mathfrak{k}$.

(2) Further, let L be any locally nilpotent non-Gruenberg Lie algebra over $\mathfrak{k}$ and let $\Gamma = \gamma(L)$ be its Gruenberg radical (Hartley [5] p. 258). By using exponentials (as in Hartley [5] p. 262] or by a more general theorem of Hartley (reported in [16] p. 85) we have Γ an ideal of L. Now Simonjan [14] has shown that a Lie algebra over a field of characteristic zero is Gruenberg if and only if

(a) it is locally nilpotent,

(b) it has an ascending series (cf. [13] p. 7) with abelian factors.

It follows that L/Γ has *trivial* Gruenberg radical. Hence over any $\mathfrak{k}$ there exists a non-zero locally nilpotent Lie algebra with trivial Gruenberg radical.

(3) With the notation of Section 3, $(Q\dot{I})^0$ is in fact (isomorphic to) the completion of $(Z\dot{I})^0$. More generally, using the characterisation of completions as isolators (Kuroš [8] p. 255) we can see that if R is a locally nilpotent ring with torsion-free additive group then the completion of R^0 is (isomorphic to) $(Q \otimes_Z R)^0$.

(4) The Levich-Tokarenko construction has an obvious functorial property. If P, P', P'' are periodic locally nilpotent groups then the diagram

(***)
$$\begin{tikzcd}
P \arrow[rr] \arrow[dr] & & P'' \\
& P' \arrow[ur] & \\
\mathscr{T}(P) \arrow[uu] \arrow[rr] \arrow[dr] & & \mathscr{T}(P'') \arrow[uu] \\
& \mathscr{T}(P') \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

commutes whenever the top triangle does, where the vertical maps are the canonical epimorphisms and those in the lower triangle are induced in the obvious way. Now even if P is small, $\mathscr{T}(P)$ may be very large. If P is a finite p-group of order p^r then it appears that $\mathscr{T}(P)$ has Hirsch-number

$$p^{p^r-1} - p^r + 1$$

(where the Hirsch-number is the number of infinite factors in a cyclic series; or in this case the dimension of the corresponding Lie algebra). One way to reduce the size of $\mathscr{T}(P)$ while keeping (***) commutative is to replace it by the subgroup of $(Z\dot{I})^0$ generated by the elements $g-1$ $(g \in P)$ of $\dot{I}$. This also maps onto P, but is smaller than the complete inverse image of P.

Both this functor and $\mathscr{T}$ may be studied using Theorem 3 to simplify computations. In particular it appears that $\mathscr{T}$ can strictly increase nilpotency classes.

(5) The methods of this paper can be used to give an integrated treatment of completions of locally nilpotent torsion-free groups and of the Mal'cev correspondence. Basic tools are the Campbell-Hausdorff formula, the theorem of Hall [4] p. 56 that any finitely generated torsion-free group embeds in a group of unitriangular matrices over (which has a very short proof due to Swan [17]) and the analogous theorem of Birkhoff [1] for finite dimensional nilpotent Lie algebras over Q using zero-triangular matrices.

Let A_n be the associative algebra of $n \times n$ zero-triangular matrices over Q. By Hall's theorem we can find n such that A_n^0 contains a given finitely generated nilpotent torsion-free group. Now A_n^0 is complete, which yields a completion G^* of G. The Mal'cev correspondence can be defined for G by means of logarithms. Similarly if L is a finite dimensional nilpotent Lie algebra over Q then $L \subseteq [A_n]$ for some n, and we may define the Mal'cev correspondence by exponentials.

This gives a local version of the theorems. For a global version, we argue exactly as in [15] for the Mal'cev correspondence, and follow the outline given in Hall [4] p. 46 for completions.

The advantages of this approach are the compact presentation, and the way that the underlying representation theory is brought to the fore.

References

[1] G. Birkhoff, *Representability of Lie algebras and Lie groups by matrices*, Ann. Math. 38 (2) (1937), pp. 526–532.

[2] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York 1962.

[3] L. Fuchs, *Abelian Groups*, 3rd ed. Oxford 1960.

[4] P. Hall, *Nilpotent groups*, Canad. Math. Congr. Summer Seminar, Alberta 1957. Reprinted as QMC mathematics notes 1972.

[5] B. Hartley, *Locally nilpotent ideals of a Lie algebra*, Proc. Cambridge Phil. Soc. 63 (1967), pp. 257–272.

[6] N. Jacobson, *Lie Algebras*, Interscience, New York 1962.
[7] M. I. Kargapolov, *Generalized soluble groups*, Algebra i Logika 2 (1963), pp. 19–28.
[8] A. G. Kuroš, *Theory of Groups*, vol. II, Chelsea, New York 1956.
[9] M. Lazard, *Problemes d'extension concernant les N-groupes: inversion de la formule de Hausdorff*, C. R. Acad. Sci. Paris 237 (1953), pp. 1377–1379.
[10] E. M. Levich and A. I. Tokarenko, *A note on locally nilpotent torsion-free groups*, Sibirsk. Mat. Ž. 11 (1970), pp. 1406–1408. Translated in Siberian Math. J. 11 (1970), pp. 1033–1034.
[11] A. I. Mal'cev, *Nilpotent torsion-free groups*, Izv. Akad. Nauk. SSSR Ser. Mat. 13 (1949), pp. 201–212.
[12] — *Generalized nilpotent algebras and their adjoint groups*, Mat. Sbornik 25 (1949), pp. 347–366.
[13] D. J. S. Robinson, *Infinite soluble and nilpotent groups*, QMC Mathematics Notes, London 1969.
[14] L. A. Simonjan, *Two radicals of Lie algebras*, Dokl. Akad. Nauk. SSSR 157 (1964, pp. 281–283. Translated Soviet Math. Doklady 5 (1964), pp. 941–944.
[15] I. N. Stewart, *An algebraic treatment of Mal'cev's theorems concerning nilpotent Lie groups and their Lie algebras*, Compositio Math. 22 (1970), pp. 289–312.
[16] — *Structure theorems for a class of locally finite Lie algebras*, Proc. London Math. Soc. 24 (3) (1972) 79–100.
[17] R. G. Swan, *Representations of polycyclic groups*, Proc. Amer. Math. Soc. 18 (1967), pp. 573–574.

MATHEMATICS INSTITUTE
UNIVERSITY OF WARWICK
Coventry

Accepté par la Rédaction le 23. 6. 1975

Properties of connected functions in terms of their levels *

by

K. M. Garg (Edmonton, Alberta)

Abstract. Let f be a connected real-valued function defined on a connected, locally connected, Hausdorff space X. In this paper we investigate necessary and sufficient conditions on the levels of f under which f is continuous, monotone or injective, and obtain some structural properties of f when it is nowhere monotone.

Among the main results, f is proved to be always continuous relative to the closure of the union S of connected levels of f. If f^{-1} preserves relatively compact sets, and either S is dense in X, or its image $f(S)$ is dense in the range of f, then f is proved to be continuous, monotone and proper. When f assumes a dense set of its values only once, it is found to be continuous and monotone, and when the singleton levels of f are dense in X, f is even injective. If X is second countable and f is nowhere monotone, it is proved that the level $f^{-1}(\alpha)$ is dense-in-itself for a residual set of values of α in R, and there further exists a residual set of points x in X such that x is a limit point of the level $f^{-1}\{f(x)\}$. Some earlier results on the distribution of closed, connected, singleton, dense-in-itself and perfect levels of f are also extended to the present setting.

1. Introduction. Before discussing the results of this paper, we first give some definitions that are used throughout the paper.

1.1. DEFINITION. The space of real numbers is denoted as usual by R. If $E \subset R$, a point $x \in R$ is said to be a *bilateral limit point* of E if it is a limit point of E from both the sides, and we call E *bilaterally closed* if it contains all of its bilateral limit points. Any set E is said to be *singleton* if it contains one and only one point, and E is called *countable* if it is finite or countably infinite. When E is a subset of a topological space X, E is called a *boundary set* [8] if its interior is empty, E is *meager* if it is a countable union of nowhere dense sets, and E is *residual* if its complement $X-E$ is meager.

Let X, Y be two topological spaces and f be a function mapping X into Y.

1.2. DEFINITION. For every $y \in Y$, the set

$$f^{-1}(y) = \{x \in X: f(x) = y\}$$

* This paper was presented at the Conference on Monotone Mappings and Open Mappings, State University of New York, Binghamton, 1970 under the title "Monotonicity of connected functions". The research was partially supported by the National Research Council of Canada under grant no. A-4826.