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PIECEWISE MARKOV PROCESSES IN DISCRETE TIME AND THEIR CERTAIN EXTENSIONS

The definition of a piecewise Markov process in discrete time is analogous to that of Kuczura [6] who considered the case of continuous time. Processes with homogeneous Markovian segments and their extensions to non-homogeneous segments are analyzed. The paper gives the relations between the stationary probability distributions of the state of the process and certain imbedded Markov chains.

1. INTRODUCTION

Numerous problems in queueing and reliability theory can be described by using processes which, starting from some initial condition, behave as Markovian in some time segment the length of which is defined at the beginning of the segment, then have a jump at the end of the segment and again behave as Markovian in the next segment with a new initial condition. Stochastic processes having such a structure are called *piecewise Markovian* (see Kuczura [6] and Çinlar [1]).

Kuczura considered in queueing theory such piecewise Markov processes with jumps, processes which were dependent upon a continuous time parameter. Using renewal theory methods, Kuczura investigated the relations between the limit probability distributions of the process and those of some imbedded Markov chains.

New proofs and some generalizations of these results may be found in [3] where methods published in [4] were used. This paper contains a modification of problems, methods and results for the case of discrete time. The paper contains also an analysis of piecewise Markov processes with non-homogeneous segments. The results have been applied to mass service systems, computer systems, teleinformation networks, etc. (see [7] and [8]).

2. PIECEWISE MARKOV PROCESSES IN DISCRETE TIME

2.1. Definitions and notation. Let S be a discrete state space. The stochastic process $\{Y(t), t = 0, 1, \dots\}$ valued on S is called *piecewise Markovian* if

(a) there exists a sequence $0 = w_0 < w_1 < w_2 < \dots$ of integer-valued random variables such that for every $m = 1, 2, \dots$ the process $\{Y(t), w_{m-1} < t < w_m\}$ is a homogeneous Markov chain with transition matrix $(P_{ij,k})$ depending upon the condition $\{Y(w_{m-1}) = k\}$;

(b) at the moments w_m ($m = 1, 2, \dots$) the change of the state of the process is the sum of two independent transitions: a Markovian one with transition probability matrix $(P_{ij,k})$ and a transition with transition probability matrix (p_{ij}) ;

(c) for every $m = 1, 2, \dots$ and $k \in S$, the probabilities

$$\Pr\{w_m - w_{m-1} = n \mid Y(w_{m-1}) = k\} = f_k(n) \quad (n = 1, 2, \dots)$$

form a distribution independent of m .

The moments w_m ($m = 1, 2, \dots$) are called *process regeneration moments*, and the intervals $w_{m-1} \leq t < w_m$ — *Markov segments*. The change of the state of the process in the interval $w_{m-1} < t < w_m$ is called a *Markovian transition*. At the moments of process regeneration the transition with probability matrix (p_{ij}) is called a *regeneration transition*.

2.2. Imbedded Markov chains and imbedded renewal streams. Assume, for simplicity, that the segment lengths $w_m - w_{m-1}$ ($m = 1, 2, \dots$) are independent and identically distributed, $f_k(n) = f(n)$ ($n = 1, 2, \dots$), and let $(P_{ij,k}) = (P_{ij})$ be independent of k .

Let $\{Y(t), t = 0, 1, \dots\}$ be a piecewise Markov process characterized by the triple $(\{f(n)\}, (p_{ij}), (P_{ij}))$. Using the split of the state change at a regeneration moment into a Markovian state change and a regenerative state change, denote by $Y^-(w_m)$ ($m = 1, 2, \dots$) the state of the process $\{Y(t), t = 0, 1, \dots\}$ at the moment w_m after the Markovian transition and before the regenerative transition. Consider now the following two sequences of random variables:

$$R_m = Y^-(w_m) \quad \text{and} \quad S_{m-1} = Y(w_{m-1}) \quad (m = 1, 2, \dots).$$

As may easily be seen, they form homogeneous Markov chains. Introducing for the transition probabilities the notation

$$r_{ij} = \Pr\{R_m = j \mid R_{m-1} = i\}, \quad s_{ij} = \Pr\{S_m = j \mid S_{m-1} = i\},$$

we have

$$(2.1) \quad r_{ij} = \sum_{k \in S} p_{ik} \sum_{n=1}^{\infty} f(n) P_{kj}(n), \quad s_{ij} = \sum_{n=1}^{\infty} f(n) \sum_{k \in S} P_{ik}(n) p_{kj} \quad (i, j \in S)$$

(further on, the summation region will be omitted in case of summation over S), where $(P_{ik}(n))$ is the n -th power of matrix (P_{ik}) .

Assume that the limit probability distributions exist (and that they are independent of the initial condition):

$$\lim_{m \rightarrow \infty} \Pr\{R_m = j | R_1 = i\} = \varrho_j, \quad \lim_{m \rightarrow \infty} \Pr\{S_m = j | S_0 = i\} = \sigma_j \quad (i, j \in S).$$

The limit probabilities satisfy the equation systems

$$(2.2) \quad \varrho_j = \sum_i \varrho_i r_{ij}, \quad \sigma_j = \sum_i \sigma_i s_{ij} \quad (j \in S),$$

and the entire probability conditions

$$\sum_j \varrho_j = \sum_j \sigma_j = 1.$$

Also, since the random variables R_m and S_m are related by the regenerative transition, the relation

$$(2.3) \quad \sigma_j = \sum_i \varrho_i p_{ij} \quad (j \in S)$$

between the distributions $\{\varrho_j\}$ and $\{\sigma_j\}$ holds.

Consider now the stream of regenerative transitions to the state j ($j \in S$) in the process $\{Y(t), t = 0, 1, \dots\}$. The time intervals between successive signals in this stream are independent identically distributed random variables, thus they form an imbedded renewal stream. Assume that the mean interval between signals of this stream is finite and denote it by $1/\mu_j$.

THEOREM 2.1. *In the piecewise Markov process $\{Y(t), t = 0, 1, \dots\}$, the sequences $\{\mu_j\}$ and $\{\sigma_j\}$, if they exist, are related by*

$$(2.4) \quad \mu_j = v\sigma_j \quad (j \in S),$$

where

$$\frac{1}{v} = \sum_{n=1}^{\infty} n f(n).$$

Proof. Let $f_{ij}(n)$ for $n = 1, 2, \dots$ ($i, j \in S$) be the distribution of the time interval from the moment of the regenerative transition to the state i to the moment of the regenerative transition to the state j . Further put

$$F_{ij}(n) = \sum_{k=n+1}^{\infty} f_{ij}(k) \quad (n = 0, 1, \dots),$$

$$\frac{1}{\mu_{ij}} = \sum_{n=0}^{\infty} F_{ij}(n) = \sum_{n=1}^{\infty} n f_{ij}(n), \quad \frac{1}{\mu_j} = \frac{1}{\mu_{jj}}.$$

The quantities $F_{ij}(n)$ satisfy the equation system

$$(2.5) \quad \begin{aligned} F_{ij}(0) &= F(0), \\ F_{ij}(n) &= F(n) + \sum_{m=1}^n f(m) \sum_k P_{ik}(m) \sum_{l, l \neq j} p_{kl} F_{lj}(n-m) \end{aligned} \quad (n = 1, 2, \dots),$$

where

$$F(n) = \sum_{k=n+1}^{\infty} f(k) \quad (n = 0, 1, \dots).$$

Summation of (2.5) over n from 0 to ∞ leads to

$$\frac{1}{\mu_{ij}} = \frac{1}{\nu} + \sum_{m=1}^{\infty} f(m) \sum_k P_{ik}(m) \sum_{l, l \neq j} p_{kl} \frac{1}{\mu_{lj}},$$

and because of (2.1) we have

$$\frac{1}{\mu_{ij}} = \frac{1}{\nu} + \sum_{l, l \neq j} s_{il} \frac{1}{\mu_{lj}}.$$

Let us multiply both sides of these equalities by σ_i and sum them over i . Using (2.2) we obtain

$$\sum_i \frac{\sigma_i}{\mu_{ij}} = \frac{1}{\nu} + \sum_{l, l \neq j} \sigma_l \frac{1}{\mu_{lj}} \quad \text{or} \quad \frac{\sigma_j}{\mu_j} = \frac{1}{\nu}.$$

This completes the proof of Theorem 2.1.

2.3. Limit distributions. Consider the probabilities

$$Q_j(t) = \Pr\{Y(t) = j\} \quad (t = 0, 1, \dots)$$

and the limit probabilities

$$q_j = \lim_{t \rightarrow \infty} Q_j(t) \quad (j \in S).$$

The form of the limit probability distribution of a piecewise Markov process and the relations between it and the limit probability distributions of the imbedded Markov chains $\{R_m\}$ and $\{S_m\}$ are given by the following two theorems:

THEOREM 2.2. *In the piecewise Markov process $\{Y(t), t = 0, 1, \dots\}$, if the probability distribution $\{f(n)\}$ is non-periodic, the limit distributions*

$\{q_j\}$ exist and are of the form

$$(2.6) \quad q_j = \sum_k \mu_k \sum_{n=0}^{\infty} F(n) P_{kj}(n) \quad (j \in S),$$

where

$$F(n) = \sum_{k=n+1}^{\infty} f(k) \quad (n = 0, 1, \dots).$$

THEOREM 2.3. *In the piecewise Markov process $\{Y(t), t = 0, 1, \dots\}$, the limit probability distributions $\{q_j\}$, $\{e_j\}$ and $\{\sigma_j\}$, if they exist, satisfy the system of equations*

$$(2.7) \quad q_j + \nu e_j = \sum_i q_i P_{ij} + \nu \sigma_j \quad (j \in S).$$

Proof of Theorem 2.2. Let $u_k(n)$ be the renewal probability at the moment n in the renewal stream of regenerative transitions to the state k . It is easy to see that for $Y(0) = i$ and $i, j \in S$ we have

$$(2.8) \quad Q_j(t) = F(t) P_{ij}(t) + \sum_{n=1}^t \sum_k u_k(n) F(t-n) P_{kj}(t-n) \\ (t = 1, 2, \dots).$$

The first right-hand side term of the sum in (2.8) tends to zero with $t \rightarrow \infty$. Using the discrete version of the fundamental renewal theorem we obtain (see also Çinlar [1]) convergency of the second right-hand side term of the sum in (2.8) to the expression being the right-hand side of (2.6). This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Let $X(t)$ be the time length from the moment t to the nearest regeneration moment. The stochastic process $\{Y(t), X(t), t = 0, 1, \dots\}$ is a Markov chain. Investigation of the limit properties of this chain can be restricted to stationary processes which are defined similarly to piecewise Markov processes but have a suitable modification in the first segment. Write, for $j \in S$,

$$P_j(x) = \Pr\{Y(t) = j, X(t) = x\} \quad (x = 1, 2, \dots).$$

Analyzing the state of the process at the moments t and $t+1$ we obtain, for $x = 1, 2, \dots$ and $j \in S$,

$$\Pr\{Y(t+1) = j, X(t+1) = x\} = \sum_i \Pr\{Y(t) = i, X(t) = x+1\} P_{ij} + \\ + \sum_i \Pr\{Y(t) = i, X(t) = 1\} \sum_l P_{il} p_{lj} f(x).$$

Hence

$$(2.9) \quad P_j(x) = \sum_i P_i(x+1)P_{ij} + \sum_i P_i(1) \sum_l P_{il}p_{lj}f(x).$$

Summation of (2.9) over x from 1 to ∞ gives

$$(2.10) \quad q_j = \sum_i [q_i - P_i(1)]P_{ij} + \sum_i P_i(1) \sum_l P_{il}p_{lj}.$$

Consider now the expressions $P_i(1)$. Let

$$P(x) = \Pr\{X(t) = x\} = \sum_j P_j(x);$$

summation of (2.9) over j gives

$$P(x) = P(x+1) + P(1)f(x).$$

Hence $P(k+1) = P(1)F(k)$ ($k = 0, 1, \dots$), and summation over k from 0 to ∞ leads to $P(1) = \nu$.

Notice that

$$(2.11) \quad \begin{aligned} P_i(1) &= \Pr\{Y(t) = i, X(t) = 1\} \\ &= \Pr\{Y(t) = i | X(t) = 1\} \Pr\{X(t) = 1\} = \nu P_i^*, \end{aligned}$$

where $P_i^* = \Pr\{Y(t) = i | X(t) = 1\}$. Insertion of (2.11) into (2.10) gives

$$q_j = \sum_i q_i P_{ij} - \nu \sum_i P_i^* P_{ij} + \nu \sum_i P_i^* \sum_l P_{il} p_{lj}.$$

Using the obvious relations

$$\sum_i P_i^* P_{ij} = q_j,$$

and also (2.3), we obtain Theorem 2.3.

2.4. Application. Let us consider the discrete-time queueing system $(M^B + GI^B)/M/1$ in which the input stream is the sum of two streams: M^B and GI^B . The first of them is a discrete, batched Poisson stream, thus the numbers of arrivals $A(t)$ at the moments $t = 1, 2, \dots$ are independent random variables having the same distribution

$$p_j = \Pr\{A(t) = j\} \quad (j = 0, 1, \dots).$$

The second stream is a discrete, batched GI -stream, thus the inter-arrival lengths $\{w_m\}$ are independent, have the same distribution

$$\Pr\{w_m - w_{m-1} = n\} = f(n) \quad (n = 1, 2, \dots),$$

and the numbers of arrivals $A^*(m)$ at the moments w_m are independent random variables with the same distribution

$$p_j^* = \Pr\{A^*(m) = j\} \quad (j = 0, 1, \dots).$$

In the considered queueing system, the distribution of service length is geometrical. Let $B(t)$ be the virtual number of services in unit time, thus let $B(t)$ ($t = 1, 2, \dots$) be independent random variables with the distribution

$$\Pr\{B(t) = 0\} = 1 - b, \quad \Pr\{B(t) = 1\} = b.$$

In particular, if $b = 1$, the service time is deterministic.

Let $Y(t)$ be the number of items in the system at the moment $t = 0, 1, \dots$. As can easily be seen, this is a piecewise Markov process $\{f(n)\}$, (p_{ij}) , (P_{ij}) , where the moments of arrivals in the stream GI^B are regeneration moments in this process.

To complete the description of the process $\{Y(t), t = 0, 1, \dots\}$ it is sufficient to find the matrices of regenerative transitions (p_{ij}) and of Markov transitions (P_{ij}) . It is easy to see that, for $m = 1, 2, \dots$,

$$(2.12) \quad \begin{cases} Y(t+1) = (Y(t) - B(t+1))^+ + A(t+1), & t+1 \neq w_m, \\ Y^-(w_m) = [(Y(t) - B(t+1))^+ + A(t+1)]|_{t=w_m-1}, \\ Y(w_m) = Y^-(w_m) + A^*(m), \end{cases}$$

where $(X)^+ = \max(0, X)$. Thus for $i, j \in S$ we have

$$(2.13) \quad p_{ij} = \Pr\{A^*(t) = j - i\} = \begin{cases} 0 & \text{for } j < i, \\ p_{j-i}^* & \text{for } j \geq i, \end{cases}$$

$$(2.14) \quad P_{ij} = \Pr\{(i - B(t+1))^+ + A(t+1) = j\} = \begin{cases} p_j & \text{for } i = 0, \\ bp_{j-i+1} + (1-b)p_{j-i} & \text{for } i > 0, j \geq i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Knowing the probability distribution $\{f(n)\}$ and the matrices (p_{ij}) and (P_{ij}) one can, using Theorem 2.3, find the relation between the stationary probability distributions of the system state at the moments $t = 0, 1, \dots$ and the state of the system at the arrival moments of the stream GI^B . This relation is formulated with the use of the generating functions of the appropriate probability distributions.

THEOREM 2.4. *In the queueing system $(M^B + GI^B)/M/1$, the generating functions $Q(z)$, $R(z)$ and $S(z)$ of the probability distributions $\{q_j\}$, $\{r_j\}$ and $\{s_j\}$, respectively, satisfy the equalities*

$$(2.15) \quad S(z) = R(z)P^*(z),$$

$$(2.16) \quad Q(z) + \nu R(z) = P(z) \left[q_0 + (Q(z) - q_0) \left(\frac{b}{z} + 1 - b \right) \right] + \nu S(z),$$

where $P^*(z)$ and $P(z)$ are the generating functions of the distributions $\{p_j^*\}$ and $\{p_j\}$, respectively.

Proof. Formula (2.15) is an immediate consequence of (2.3) and (2.13). Calculate now

$$(2.17) \quad \sum_{j=0}^{\infty} z^j \sum_{i=0}^{\infty} q_i P_{ij} = q_0 \sum_{j=0}^{\infty} z^j p_j + \sum_{i=1}^{\infty} q_i \sum_{j=i-1}^{\infty} z^j (b p_{j-i+1} + (1-b) p_{j-i}) \\ = P(z) \left[q_0 + (Q(z) - q_0) \left(\frac{b}{z} + 1 - b \right) \right].$$

Hence and from (2.7) we obtain formula (2.16). This completes the proof of Theorem 2.4.

Remark. The probability q_0 in (2.16) is expressed by the formula

$$q_0 = 1 - \frac{1}{b} [P'(1) + \nu S'(1) - \nu R'(1)],$$

where $P'(1)$, $S'(1)$ and $R'(1)$ are mean values in the probability distributions $\{p_j\}$, $\{\sigma_j\}$ and $\{\rho_j\}$, respectively.

Let $W(t)$ be the virtual waiting time at the moment $t = 0, 1, \dots$. It can easily be noticed that this is a piecewise Markov process having the same properties as the process $\{Y(t), t = 0, 1, \dots\}$.

Indeed, the regeneration moments of these processes are identical and the matrices of Markovian and regenerative transitions in the process $\{W(t), t = 0, 1, \dots\}$ are found from the equalities, being analogous to (2.12),

$$W(t+1) = (W(t) - 1)^+ + Z(t+1), \quad t+1 \neq w_m,$$

$$W^-(w_m) = [(W(t) - 1)^+ + Z(t+1)]|_{t=w_m-1}, \quad W(w_m) = W^-(w_m) + Z^*(m),$$

where $Z(t)$ is the sum of service times of all items in the batch with distribution of the number of items being equal to $\{p_j\}$, and $Z^*(m)$ is the sum of service times of all items in the batch with distribution of the number of items being equal to $\{p_j^*\}$.

3. GENERALIZED PIECEWISE MARKOV PROCESSES IN DISCRETE TIME

3.1. Definitions and notation. Let S be a discrete state space. The stochastic process $\{Y(t), t = 0, 1, \dots\}$ valued on S is called a *generalized piecewise Markov process* if

(a) there exists a sequence $0 = w_0 \leq w_1 < w_2 \leq w_3 < w_4 \dots$ of integer-valued random variables such that, for every $m = 0, 1, \dots$ and $a = 0, 1$ satisfying $w_{2m+1+a} - w_{2m+a} \geq 2$, the process $\{Y(t), w_{2m+a} < t < w_{2m+1+a}\}$ is a homogeneous Markov chain with transition probability matrix $(P_{ij,k}^{(a)})$ depending upon the condition $\{Y(w_{2m+a}) = k\}$;

(b) at the moments w_{2m+a} ($m = 0, 1, \dots; a = 0, 1$), for which $w_{2m+a} \neq w_{2m+1-a}$ holds, the change of the state of the process $\{Y(t), t \geq 0\}$ is the sum of two independent consecutive transitions: a Markovian one with transition matrix $(P_{ij,k}^{(1-a)})$ depending on the condition $\{Y(w_{2m-1+a}) = k\}$ and a transition with probability matrix $(p_{ij}^{(1-a)})$ ($a = 0, 1; i, j, k \in S$);

(c) at the moments w_{2m} ($m = 0, 1, \dots$) satisfying $w_{2m} = w_{2m+1}$, the change of the state of the process $\{Y(t), t \geq 0\}$ is the sum of three independent consecutive transitions: a Markovian one with transition probability matrix $(P_{ij,k}^{(1)})$ depending upon the condition $\{Y(w_{2m-1}) = k\}$, a transition with probability matrix $(p_{ij}^{(1)})$, and a transition with probability matrix $(p_{ij}^{(0)})$;

(d) for every $m = 0, 1, \dots, a = 0, 1$, and $k \in S$, the probability distributions

$$\Pr\{w_{2m+1+a} - w_{2m+a} = n \mid Y(w_{2m+a}) = k\} = f_k^{(a)}(n) \\ (n = a, a+1, \dots), \quad f_k^{(1)}(0) = 0, \quad f_k^{(0)}(0) < 1,$$

do not depend upon m .

The moments w_m ($m = 1, 2, \dots$) are called *process regeneration moments*, and the intervals $w_{m-1} \leq t < w_m$ — *Markov segments*. The change of the process state in the interval $w_{m-1} < t < w_m$ is called a *Markov transition*; at regeneration moments two transitions are distinguished: a Markovian one with transition matrix identical with that of the preceding segment, and a transition called *regenerative*.

The breakdown process $\{a(t), t = 0, 1, \dots\}$ related to a generalized piecewise Markov process is

$$a(t) = \begin{cases} 0 & \text{for } w_{2m} \leq t < w_{2m+1} \\ 1 & \text{for } w_{2m+1} \leq t < w_{2m+2} \end{cases} \quad (m = 0, 1, \dots),$$

where the moments w_m are defined for the generalized piecewise Markov process. The intervals $w_{2m+a} \leq t < w_{2m+1+a}$ in the breakdown process are called *breakdown segments* for $a = 0$ and *working segments* for $a = 1$.

3.2. Imbedded Markov chains. Assume, for simplicity, that for every $a = 0, 1$ the segment lengths $w_{2m+1+a} - w_{2m+a}$ ($m = 0, 1, \dots$) are independent, identically distributed and that the Markov transition matrices do not depend upon k :

$$f_k^{(a)}(n) = f^{(a)}(n), \quad (P_{ij,k}^{(a)}) = (P_{ij}^{(a)}) \quad (n = a, a+1, \dots; a = 0, 1).$$

Using the decomposition of the change of the process state at a regeneration moment into Markovian and regenerative state changes, denote by $Y^-(w_m)$ the state of the process $\{Y(t), t \geq 0\}$ at the moment w_m after the Markovian transition but before the regenerative transition.

Let $\{Y(t), t = 0, 1, \dots\}$ be a generalized piecewise Markov process characterized by $\{f^{(a)}(n)\}, (p_{ij}^{(a)}), (P_{ij}^{(a)}), a = 0, 1\}$. Consider four sequences of random variables,

$$R_{2m+a}^{(a)} = Y^-(w_{2m+a}) \quad (m = 1, 2, \dots), \quad S_{2m+a}^{(a)} = Y(w_{2m+a}) \quad (m = 0, 1, \dots),$$

where $a = 0, 1$. If $w_{2m} = w_{2m+1}$, then $Y(w_{2m}) = Y^-(w_{2m+1})$. They form homogeneous Markov chains, since $\{w_{2m+a}\}$ for every $a = 0, 1$ is the sequence of regeneration moments in the process $\{Y(t), t = 0, 1, \dots\}$. Denote by $(r_{ij}^{(a)})$ the transition matrices for the imbedded chains $\{R_{2m+a}^{(a)}\}$. To determine the matrix $(r_{ij}^{(0)})$ consider four transition probability matrices:

(i) the matrix $(p_{ij}^{(1)})$ of the regenerative transition at the end of a working segment;

(ii) the matrix $(p_{ij}^{(0,0)})$ of the transition of the Markov process in the breakdown segment,

$$(p_{ij}^{(0,0)}) = \sum_{n=0}^{\infty} f^{(0)}(n) (P_{ij}^{(0)})^n;$$

(iii) the matrix $(p_{ij}^{(0)})$ of the regenerative transition at the end of the breakdown segment;

(iv) the matrix $(p_{ij}^{(1,1)})$ of the transition of the Markov process in the working segment,

$$(p_{ij}^{(1,1)}) = \sum_{n=1}^{\infty} f^{(1)}(n) (P_{ij}^{(1)})^n.$$

The imbedded Markov chain $\{R_{2m}^{(0)}\}$ has, therefore, a transition probability matrix equal to the product

$$(r_{ij}^{(0)}) = (p_{ij}^{(1)})(p_{ij}^{(0,0)})(p_{ij}^{(0)})(p_{ij}^{(1,1)}).$$

The transition probability matrix for the imbedded Markov chain $\{R_{2m+1}^{(1)}\}$ can be formed analogously and is of the form

$$(r_{ij}^{(1)}) = (p_{ij}^{(0)})(p_{ij}^{(1,1)})(p_{ij}^{(1)})(p_{ij}^{(0,0)}).$$

In the same way the matrices $(s_{ij}^{(a)})$ for the Markov chains $\{S_{2m+a}^{(a)}\}$ ($a = 0, 1$) can be found.

Assume that the following limit probability distributions, being independent of the initial condition, exist:

$$\lim_{m \rightarrow \infty} \Pr\{R_{2m+a}^{(a)} = j | R_{2+a}^{(a)} = i\} = \varrho_j^{(a)}, \quad \lim_{m \rightarrow \infty} \Pr\{S_{2m+a}^{(a)} = j | S_a^{(a)} = i\} = \sigma_j^{(a)} \\ (a = 0, 1).$$

The limit probabilities satisfy the equalities

$$\varrho_j^{(a)} = \sum_i \varrho_i^{(a)} r_{ij}^{(a)}, \quad \sigma_j^{(a)} = \sum_i \sigma_i^{(a)} s_{ij}^{(a)},$$

and the entire probability conditions

$$\sum_j \varrho_j^{(a)} = \sum_j \sigma_j^{(a)} = 1 \quad (a = 0, 1).$$

The random variables $R_{2m}^{(0)}$ and $R_{2m+1}^{(1)}$ are related by the regenerative transition (i) and the Markovian transition (ii) in the breakdown segment, thus we have

$$(3.1) \quad \varrho_j^{(1)} = \sum_i \varrho_i^{(0)} r_{ij}^{(1,0)},$$

where $(r_{ij}^{(1,0)}) = (p_{ij}^{(1)})(p_{ij}^{(0,0)})$. The random variables $S_{2m+a}^{(a)}$ and $R_{2m+a}^{(a)}$ are related by the regenerative transition with matrix $(p_{ij}^{(1-a)})$, thus we have

$$(3.2) \quad \sigma_j^{(a)} = \sum_i \varrho_i^{(a)} p_{ij}^{(1-a)}.$$

3.3. Limit distributions. Consider the state probabilities

$$Q_j^{(a)}(t) = \Pr\{Y(t) = j, a(t) = a\} \quad (t = 0, 1, \dots; a = 0, 1; j \in S)$$

and the limit probabilities

$$q_j^{(a)} = \lim_{t \rightarrow \infty} Q_j^{(a)}(t) \quad (a = 0, 1; j \in S).$$

The following theorem gives the relation between the limit probability distributions of the state of the process $\{Y(t), a(t), t = 0, 1, \dots\}$ and the limit probability distributions of the state of the imbedded Markov chains.

THEOREM 3.1. *In the generalized piecewise Markov process $\{Y(t), t = 0, 1, \dots\}$, the limit probability distributions $\{q_j^{(a)}\}$, $\{\varrho_j^{(a)}\}$ and $\{\sigma_j^{(a)}\}$ ($a = 0, 1$), if they exist, satisfy the equation system*

$$(3.3) \quad q_j^{(a)} + \nu \varrho_j^{(1-a)} = \sum_i q_i^{(a)} P_{ij}^{(a)} + \nu \sigma_j^{(a)} \quad (a = 0, 1; j \in S),$$

where

$$\frac{1}{\nu} = \nu^{(0)} + \nu^{(1)}, \quad \nu^{(a)} = \sum_{n=1}^{\infty} n f^{(a)}(n).$$

Proof of Theorem 3.1. Let $X(t)$ denote the time to the nearest regeneration point after the moment t . The process $\{Y(t), a(t), X(t), t = 0, 1, \dots\}$ is a Markov chain determined on the states

$$(j, a, x) \in S' = S \times \{0, 1\} \times \{1, 2, \dots\}.$$

The investigation of the limit properties of the characteristics of this process can be restricted to stationary processes which are defined analogously, with an appropriate modification of the process in the first

and second segments. Write

$$(3.4) \quad P_j^{(a)}(x) = \Pr\{Y(t) = j, a(t) = a, X(t) = x\} \quad ((j, a, x) \in S').$$

Analyzing the state of the process at the moments t and $t+1$ we obtain, for $x = 1, 2, \dots$ and $j \in S$,

$$\begin{aligned} & \Pr\{Y(t+1) = j, a(t+1) = 0, X(t+1) = x\} \\ &= \sum_i \Pr\{Y(t) = i, a(t) = 0, X(t) = x+1\} P_{ij}^{(0)} + \\ &+ \sum_i \Pr\{Y(t) = i, a(t) = 1, X(t) = 1\} \sum_l P_{il}^{(1)} p_{lj}^{(1)} f^{(0)}(x), \\ & \Pr\{Y(t+1) = j, a(t+1) = 1, X(t+1) = x\} \\ &= \sum_i \Pr\{Y(t) = i, a(t) = 1, X(t) = x+1\} P_{ij}^{(1)} + \\ &+ \sum_i \Pr\{Y(t) = i, a(t) = 0, X(t) = 1\} \sum_l P_{il}^{(0)} p_{lj}^{(0)} f^{(1)}(x) + \\ &+ \sum_i \Pr\{Y(t) = i, a(t) = 1, X(t) = 1\} \sum_l P_{il}^{(1)} \sum_r p_{lr}^{(1)} f^{(0)}(0) p_{rj}^{(0)} f^{(1)}(x). \end{aligned}$$

Using (3.4) we have

$$\begin{aligned} (3.5) \quad & P_j^{(0)}(x) = \sum_i P_i^{(0)}(x+1) P_{ij}^{(0)} + \sum_i P_i^{(1)}(1) \sum_l P_{il}^{(1)} p_{lj}^{(1)} f^{(0)}(x), \\ & P_j^{(1)}(x) = \sum_i P_i^{(1)}(x+1) P_{ij}^{(1)} + \sum_i P_i^{(0)}(1) \sum_l P_{il}^{(0)} p_{lj}^{(0)} f^{(1)}(x) + \\ &+ \sum_i P_i^{(1)}(1) \sum_l P_{il}^{(1)} \sum_r p_{lr}^{(1)} f^{(0)}(0) p_{rj}^{(0)} f^{(1)}(x). \end{aligned}$$

Summation of (3.5) over x from 1 to ∞ gives

$$\begin{aligned} (3.6) \quad & q_j^{(0)} = \sum_i [q_i^{(0)} - P_i^{(0)}(1)] P_{ij}^{(0)} + \sum_i P_i^{(1)}(1) \sum_l P_{il}^{(1)} p_{lj}^{(1)} [1 - f^{(0)}(0)], \\ & q_j^{(1)} = \sum_i [q_i^{(1)} - P_i^{(1)}(1)] P_{ij}^{(1)} + \sum_i P_i^{(0)}(1) \sum_l P_{il}^{(0)} p_{lj}^{(0)} + \\ &+ \sum_i P_i^{(1)}(1) \sum_l P_{il}^{(1)} \sum_r p_{lr}^{(1)} f^{(0)}(0) p_{rj}^{(0)}. \end{aligned}$$

Consider now the expressions $P_i^{(a)}(1)$ for $a = 0, 1$, and $i \in S$. Let

$$P^{(a)}(x) = \sum_j P_j^{(a)}(x).$$

Summation of (3.5) over j leads to

$$\begin{aligned} P^{(0)}(x) &= P^{(0)}(x+1) + P^{(1)}(1) f^{(0)}(x), \\ P^{(1)}(x) &= P^{(1)}(x+1) + P^{(0)}(1) f^{(1)}(x) + P^{(1)}(1) f^{(0)}(0) f^{(1)}(x). \end{aligned}$$

Hence

$$(3.7) \quad \Pr\{a(t) = a, X(t) = x\} = P^{(a)}(x) = \nu F^{(a)}(x-1), \quad \frac{1}{\nu} = \nu^{(0)} + \nu^{(1)}.$$

Notice that

$$(3.8) \quad \begin{aligned} P_j^{(a)}(1) &= \Pr\{Y(t) = j, a(t) = a, X(t) = 1\} \\ &= \Pr\{a(t) = a, X(t) = 1\} \Pr\{Y(t) = j | a(t) = a, X(t) = 1\} \\ &= \nu F^{(a)}(0) P_j^{*(a)}, \end{aligned}$$

where $P_j^{*(a)} = \Pr\{Y(t) = j | a(t) = a, X(t) = 1\}$.

Analyzing the state of the process $\{Y(t), t = 0, 1, \dots\}$ at the moments w_{2m+a} and $w_{2m+a} - 1$ ($a = 0, 1$), by the definitions of the distributions $\{P_j^{*(a)}\}$ and $\{\varrho_j^{(a)}\}$ we obtain

$$\begin{aligned} \sum_i P_i^{*(1)} P_{ij}^{(1)} &= \varrho_j^{(0)}, \\ f^{(0)}(0) \sum_i P_i^{*(1)} \sum_k P_{ik}^{(1)} p_{kj}^{(1)} + (1 - f^{(0)}(0)) \sum_i P_i^{*(0)} P_{ij}^{(0)} &= \varrho_j^{(1)}. \end{aligned}$$

Thus, by (3.8) and (3.2), we have

$$\begin{aligned} \sum_i P_i^{(1)}(1) P_{ij}^{(1)} &= \nu \varrho_j^{(0)}, \\ \sum_i P_i^{(0)}(1) P_{ij}^{(0)} &= \nu \varrho_j^{(1)} - \nu f^{(0)}(0) \sigma_j^{(0)} \quad (j \in S). \end{aligned}$$

Using (3.6) and (3.2) we obtain Theorem 3.1.

3.4. Application. The generalized piecewise Markov processes in discrete time can be used to analyze queueing systems with priorities. In applications, they may, for example, be found in mathematical models of computer communication networks with time sharing (see [7] and [8]).

Consider a queueing system with two priority classes. Assume that the priority is pre-emptive and that the service of an item which has been interrupted is finished after servicing items with higher priority.

In the analysis of the service of lower priority items, the presence of higher priority items in the system can be interpreted as service line breakdown. The breakdown time distribution is the distribution of busy time of the system by higher priority items, and the working time of the service line is the time of waiting for a higher priority item.

Assume that the arrival streams of both priority classes are discrete batched Poisson streams, thus the numbers of arrivals $A(t)$ at the moments $t = 1, 2, \dots$ are independent random variables with the same distribution $\{p_j\}$. Assume that the system has one service line and that the service

time is geometrically distributed with characteristics given in Section 2.4. We do not deal with the number of items with higher priority, since the system $M^B/M/1$ is well known. We are, however, interested in the number of items with lower priority in the system or, equivalently, in the number of items in the system $M^B/M/1$ with service line breakdowns.

Let $\{Y(t), t = 0, 1, \dots\}$ be the number of items in the system and let $\{a(t), t = 0, 1, \dots\}$ denote the breakdown process of the service line. It is easily seen that $\{Y(t), t = 0, 1, \dots\}$ is a generalized piecewise Markov process, and $\{a(t), t = 0, 1, \dots\}$ is the breakdown process related to it.

THEOREM 3.2. *In the queueing system $M^B/M/1$ with service line breakdowns, the generating functions $Q^{(a)}(z)$, $R^{(a)}(z)$ and $S^{(a)}(z)$ ($a = 0, 1$) of the distributions $\{q_j^{(a)}\}$, $\{r_j^{(a)}\}$ and $\{s_j^{(a)}\}$ ($a = 0, 1$), respectively, satisfy the equalities*

$$(3.9) \quad Q^{(0)}(z) + \nu R^{(1)}(z) = Q^{(0)}(z)P(z) + \nu S^{(0)}(z),$$

$$(3.10) \quad Q^{(1)}(z) + \nu R^{(0)}(z) = \left[q_0 + (Q^{(1)}(z) - q_0) \left(\frac{b}{z} + 1 - b \right) \right] P(z) + \nu S^{(1)}(z),$$

$$(3.11) \quad R^{(0)}(z) = S^{(0)}(z), \quad R^{(1)}(z) = S^{(1)}(z),$$

$$(3.12) \quad R^{(1)}(z) = R^{(0)}(z)F^{(0)}(P(z)),$$

where $P(z)$ is the generating function of the distribution $\{p_j\}$, and $F^{(0)}(z)$ is the generating function of the distribution of service line breakdown duration.

Proof. Consider the process $\{Y(t), a(t), t = 0, 1, \dots\}$. It is not difficult to see that

$$Y(t+1) = (Y(t) - B(t+1)a(t+1))^+ + A(t+1) \quad (t = 0, 1, \dots).$$

The transition matrices on the Markov segments are thus of the form

$$(3.13) \quad P_{ij}^{(0)} = \begin{cases} 0 & \text{for } j < i, \\ p_{j-i} & \text{for } j \geq i, \end{cases} \quad P_{ij}^{(1)} = P_{ij},$$

where (P_{ij}) is given by (2.14), and the regenerative transition matrices are identity transition matrices

$$(3.14) \quad (P_{ij}^{(a)}) = (\delta_{ij}) \quad (a = 0, 1).$$

Using (3.3), (3.13) and (2.17), we get (3.9) and (3.10). Formula (3.11) follows from (3.14) and (3.2). Notice that the number of arrivals in the breakdown segment is equal to the sum of a random number of independent batches with the same distribution $\{p_j\}$, where the number of batch-

es is equal to the time length in the breakdown process $\{a(t), t = 0, 1, \dots\}$. Let $\{p_j^{(k)}\}$ be the k -th convolution of the distribution $\{p_j\}$ with itself. Then we have

$$p_{ij}^{(0,0)} = \begin{cases} 0 & \text{for } j < i, \\ \sum_{k=0}^{\infty} f^{(0)}(k) p_{j-i}^{(k)} & \text{for } j \geq i. \end{cases}$$

Hence and by (3.1) we have

$$R^{(1)}(z) = R^{(0)}(z) \sum_{k=0}^{\infty} f^{(0)}(k) P^k(z) = R^{(0)}(z) F^{(0)}(P(z)).$$

This completes the proof of Theorem 3.2.

Theorem 3.2 does not allow us to determine effectively the probability distributions found there. Now we give some remarks about the determination of one of these distributions, and all remaining distributions may be determined afterwards.

Let us define the modified time t' by contraction to points the periods of service time breakdowns at time t . After this transformation, the input stream of the system at time t' consists of the mixture of two arrival streams, the batched Poisson stream M^B of the items arriving at time t during the periods of work of the service line and the batched Poisson stream M^{B*} in which the batches are formed from those items at time t which arrive in the periods of service line breakdowns. At time t' we have thus the system $(M^B + M^{B*})/M/1$ which in the analysis of the number of items in the system is equivalent to the system $M^{B**}/M/1$ with suitably determined batched Poisson input stream. An analysis of that system is easily done and explicit formulas for the limit probabilities of the system state are known.

Let $\{Y'(t'), t' = 0, 1, \dots\}$ be the process defined as the number of items in the system at the moment t' . Assume that the moments w'_m ($m = 0, 1, \dots$) of arrivals in the stream M^{B*} are process regeneration moments. It is easy to see that this process is the piecewise Markov process defined in Section 2.1. Thus, using Theorem 2.1, we get the relations between the known state distribution of the process at time t' and the distributions of the imbedded Markov chains at regeneration moments. Notice that the imbedded Markov chain $R_m = Y'^-(w'_m)$ ($m = 0, 1, \dots$) has identical probability characteristics with the imbedded Markov chain $R_m^{(0)} = Y^-(w_{2m})$; the same holds for the Markov chains $S_m = Y'(w'_m)$ and $S_m^{(1)} = Y(w_{2m+1})$ ($m = 0, 1, \dots$). This allows us to determine effectively distributions occurring in Theorem 3.1.

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Received on 26. 10. 1976

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UOGÓLNIONE PROCESY PRZEDZIAŁAMI MARKOWSKIE

STRESZCZENIE

Definicja procesu przedziałami markowskiego w czasie dyskretnym jest analogiczna do definicji podanej przez Kuczurę [6], który rozważał przypadek czasu ciągłego. Przedmiotem analiz są procesy o segmentach markowskich jednorodnych i ich rozszerzenia na segmenty niejednorodne. W pracy podane są związki między stacjonarnymi rozkładami prawdopodobieństwa stanu procesu i pewnych włożonych łańcuchów Markowa.