On the Cauchy problem for linear partial differential-functional equations of first order

by Z. KAMONT (Gdańsk)

Abstract. The Cauchy problem for linear partial differential-functional equations of first order

$$(a) \quad \frac{\partial z(x, y)}{\partial x} = f(x, y, z(x, y), z(a_1(x, y), \beta_1(x, y)), \dots, z(a_m(x, y), \beta_m(x, y))) \frac{\partial z(x, y)}{\partial y} + \\ + g(x, y, z(x, y), z(a_1(x, y), \beta_1(x, y)), \dots, z(a_m(x, y), \beta_m(x, y))), \\ z(x, y) = \varphi(x, y) \quad \text{for } (x, y) \in I$$

is considered, where $f, g, a_i, \beta_i, \varphi$ are given functions and I is the initial set of the form.

$$I = \{(x, y): x \in (p_0, 0], y \in (-\infty, +\infty)\}, \quad p_0 < 0.$$

In this paper it is shown that under certain conditions concerning the functions $f, g, a_i, \beta_i, \varphi$, there exists a solution $\bar{z}(x, y)$ of problem (a). The estimation of the existence domain of solutions of (a) is given.

I. Assumptions and definitions. Assume that functions $f(x, y, z_0, z_1, \dots, z_m)$, $g(x, y, z_0, z_1, \dots, z_m)$, $\varphi(x, y)$, $a_i(x, y)$, $\beta_i(x, y)$, $i = 1, 2, \dots, m$, are given and that $p_0 < a_i(x, y) \leq x$ for $(x, y) \in E_0$, $i = 1, \dots, m$, where $p_0 < 0$ (in particular, we admit the case $p_0 = -\infty$) and

$$E_0 = \{(x, y): x \in [0, a), y \in (-\infty, +\infty)\}.$$

We consider the initial problem of the form

$$(1) \quad z_x(x, y) = f(x, y, z(x, y), z(a_1(x, y), \beta_1(x, y)), \dots, z(a_m(x, y), \beta_m(x, y))) \frac{\partial z(x, y)}{\partial y} + \\ + g(x, y, z(x, y), z(a_1(x, y), \beta_1(x, y)), \dots, z(a_m(x, y), \beta_m(x, y))), \\ z(x, y) = \varphi(x, y) \quad \text{for } (x, y) \in I,$$

where I is the initial set of the form

$$I = \{(x, y): x \in (p_0, 0], y \in (-\infty, +\infty)\}.$$

Applying theorems on the estimation of the existence domain of solutions of linear partial differential equations of first order and theorems on the differential inequalities we shall prove that there exists a unique solution of the Cauchy problem (1). We also give the estimation of the existence domain of solutions of (1).

We introduce

ASSUMPTION H_1 . We assume that

1° The functions $f(x, y, z_0, z_1, \dots, z_m)$ and $g(x, y, z_0, z_1, \dots, z_m)$ are of class C^2 in the set

$$\Omega = \{(x, y, z_0, \dots, z_m): x \in [0, \alpha), y, z_0, z_1, \dots, z_m \in (-\infty, +\infty)\}.$$

2° The functions $\alpha_i(x, y)$, $\beta_i(x, y)$, $i = 1, 2, \dots, m$, are of class C^2 in the set E_0 and

$$(2) \quad p_0 < \alpha_i(x, y) \leq x \quad \text{for } (x, y) \in E_0, i = 1, \dots, m.$$

3° The initial function $\varphi(x, y)$ is of class C^2 in the initial set I .

4° There exist constants $A, L \geq 0$ such that

$$\left. \begin{array}{l} |f_x|, |f_y|, |f_{z_i}| \\ |g_x|, |g_y|, |g_{z_i}| \end{array} \right\} \leq A, \quad i = 0, 1, 2, \dots, m,$$

$$|f| \leq L.$$

5° There exist constants $B, C \geq 0$ such that

$$|\varphi_x|, |\varphi_y| \leq B,$$

$$\left| \frac{\partial \alpha_i}{\partial x} \right|, \left| \frac{\partial \alpha_i}{\partial y} \right|, \left| \frac{\partial \beta_i}{\partial x} \right|, \left| \frac{\partial \beta_i}{\partial y} \right| \leq C, \quad i = 1, \dots, m.$$

6° The following consistency condition

$$\begin{aligned} \varphi_x(0, y) = & f(0, y, \varphi(0, y), \varphi(\alpha_1(0, y), \beta_1(0, y)), \dots \\ & \dots, \varphi(\alpha_m(0, y), \beta_m(0, y)))\varphi_y(0, y) + \\ & + g(0, y, \varphi(0, y), \varphi(\alpha_1(0, y), \beta_1(0, y)), \dots, \varphi(\alpha_m(0, y), \beta_m(0, y))) \end{aligned}$$

is satisfied for $y \in (-\infty, +\infty)$.

ASSUMPTION H_2 . We assume that there exists a constant $D \geq 0$ such that

$$1^\circ \quad \left. \begin{array}{l} |f_{xx}|, |f_{xy}|, |f_{xz_i}|, |f_{yy}|, |f_{yz_i}|, |f_{z_i z_j}| \\ |g_{xx}|, |g_{xy}|, |g_{xz_i}|, |g_{yy}|, |g_{yz_i}|, |g_{z_i z_j}| \end{array} \right\} \leq D, \quad i, j = 0, 1, \dots, m.$$

$$2^\circ \quad \left\{ \begin{array}{l} \left| \frac{\partial^2 \alpha_i}{\partial x^2} \right|, \left| \frac{\partial^2 \alpha_i}{\partial x \partial y} \right|, \left| \frac{\partial^2 \alpha_i}{\partial y^2} \right| \\ \left| \frac{\partial^2 \beta_i}{\partial x^2} \right|, \left| \frac{\partial^2 \beta_i}{\partial x \partial y} \right|, \left| \frac{\partial^2 \beta_i}{\partial y^2} \right| \end{array} \right\} \leq D, \quad i = 1, \dots, m,$$

$$3^\circ \quad |\varphi_{xx}|, |\varphi_{xy}|, |\varphi_{yy}| \leq D.$$

Let $E = \{(x, y): x \in [0, b), y \in (-\infty, +\infty)\}$, where

$$b = \min(a, p, q)$$

and

$$p = [2A(1+2mC)(1+B+2mBC)]^{-1},$$

$$q = [A(1+B)(1+2mrC)]^{-1},$$

$$r = 2B + \frac{1}{1+2mC}.$$

II. The existence of the sequence of successive approximations. We define the sequence $\{z_n\}$ in the following way: $z_0(x, y)$ is an arbitrary function such that the function

$$(3) \quad u_0(x, y) = \begin{cases} z_0(x, y) & \text{for } (x, y) \in E, \\ \varphi(x, y) & \text{for } (x, y) \in I, \end{cases}$$

is of class C^2 in $E \cup I$ and

$$(4) \quad \left| \frac{\partial z_0(x, y)}{\partial x} \right|, \left| \frac{\partial z_0(x, y)}{\partial y} \right| \leq B,$$

$$\left| \frac{\partial^2 z_0(x, y)}{\partial x^2} \right|, \left| \frac{\partial^2 z_0(x, y)}{\partial x \partial y} \right|, \left| \frac{\partial^2 z_0(x, y)}{\partial y^2} \right| \leq D$$

for $(x, y) \in E$.

The function $z_{n+1}(x, y)$ is the solution of the Cauchy problem

$$(5) \quad \begin{aligned} z_x(x, y) &= f^{(n)}(x, y, z(x, y))z_y(x, y) + g^{(n)}(x, y, z(x, y)), \\ z(0, y) &= \omega(y), \quad y \in (-\infty, +\infty), \end{aligned}$$

where

$$(6) \quad \begin{aligned} &f^{(n)}(x, y, z) \\ &= f(x, y, z, u_n(\alpha_1(x, y), \beta_1(x, y)), \dots, u_n(\alpha_m(x, y), \beta_m(x, y))), \\ &g^{(n)}(x, y, z) \\ &= g(x, y, z, u_n(\alpha_1(x, y), \beta_1(x, y)), \dots, u_n(\alpha_m(x, y), \beta_m(x, y))) \end{aligned}$$

and

$$(7) \quad u_n(x, y) = \begin{cases} z_n(x, y) & \text{for } (x, y) \in E, \\ \varphi(x, y) & \text{for } (x, y) \in I, \end{cases}$$

$$(8) \quad \omega(y) = \varphi(0, y), \quad y \in (-\infty, +\infty).$$

LEMMA 1. *If Assumption H_1 is satisfied, then:*

(I_n) *The functions z_n are well defined and are of class C^2 in E .*

(II_n) *For $(x, y) \in E$ we have*

$$(9) \quad \left| \frac{\partial z_n(x, y)}{\partial x} \right| \leq \lambda(x), \quad \left| \frac{\partial z_n(x, y)}{\partial y} \right| \leq \lambda(x),$$

where

$$(10) \quad \lambda(x) = \frac{B + A(1 + B + 2mBC)x}{1 - A(1 + 2mC)(1 + B + 2mBC)x}$$

and

$$(11) \quad \left| \frac{\partial z_n(x, y)}{\partial x} \right| \leq r, \quad \left| \frac{\partial z_n(x, y)}{\partial y} \right| \leq r, \quad (x, y) \in E.$$

(III_n) *The functions $f^{(n)}(x, y, z)$ and $g^{(n)}(x, y, z)$ are of class C^2 in the set*

$$\Omega_0 = \{(x, y, z): x \in [0, b), y, z \in (-\infty, +\infty)\}$$

and

$$|f_x^{(n)}|, |f_y^{(n)}|, |g_x^{(n)}|, |g_y^{(n)}| \leq N_0,$$

where

$$N_0 = A(1 + 2mrC).$$

Proof. It follows from conditions 1°-4° of Assumption H_1 and by (3), (4), (6), (7) that conditions (I₀)-(III₀) are satisfied. Let us suppose that for some $n \geq 0$ conditions (I_n)-(III_n) hold. Now we shall prove that (I_{n+1})-(III_{n+1}) also hold.

Let us consider the initial problem (5), where the functions $f^{(n)}$, $g^{(n)}$ and ω are defined by (6)-(8). It follows from Assumption H_1 and by (I_n)-(III_n) that there exists a solution z_{n+1} of the Cauchy problem (5). This solution is defined and is of class C^2 in the set

$$\tilde{E} = \{(x, y): x \in [0, \tilde{b}), y \in (-\infty, +\infty)\},$$

where

$$\tilde{b} = \min \left[b, \frac{1}{A(1+B)(1+2mrC)} \right]$$

(see [1], [2]). Since

$$b \leq [A(1+B)(1+2mrC)]^{-1},$$

we have $\tilde{b} = b$ and $\tilde{E} = E$. Condition (I_{n+1}) is proved.

To prove (II_{n+1}) , first observe that

$$(12) \quad |f_x^{(n)}|, |f_y^{(n)}|, |g_x^{(n)}|, |g_y^{(n)}| \leq A[1+2mC\lambda(x)]$$

and

$$(13) \quad |f_z^{(n)}| \leq A, \quad |g_z^{(n)}| \leq A, \quad |f^{(n)}| \leq L.$$

Let $t > 0$ be a constant such that $b < t/L$. Put

$$T(b, t, L) = \{(x, y) : x \in [0, b], |y| \leq t - Lx\}.$$

Now, we will show that the differential inequality

$$(14) \quad \left| \frac{\partial}{\partial x} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right) \right| \leq A \left[1 + 2mC\lambda(x) + \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right|^2 \right] + \\ + L \left| \frac{\partial}{\partial y} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right) \right|, \quad (x, y) \in T(b, t, L)$$

and the initial inequality

$$(15) \quad \left| \frac{\partial z_{n+1}(0, y)}{\partial y} \right| \leq B, \quad y \in [-t, t]$$

hold.

Indeed, it follows from (6), (7) (9) and (12), (13) that

$$\left| \frac{\partial}{\partial x} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right) \right| \leq |f_y^{(n)}(x, y, z_{n+1}(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| + \\ + |f_z^{(n)}(x, y, z_{n+1}(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right|^2 + \\ + |f^{(n)}(x, y, z_{n+1}(x, y))| \left| \frac{\partial}{\partial y} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right) \right| + \\ + |g_y^{(n)}(x, y, z_{n+1}(x, y))| + |g_z^{(n)}(x, y, z_{n+1}(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| \\ \leq A \left[1 + 2mC\lambda(x) + \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right|^2 \right] + \\ + L \left| \frac{\partial}{\partial y} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right) \right|, \quad (x, y) \in T(b, t, L),$$

which completes the proof of (14).

Hence, by condition 4° of Assumption Π_1 and by (8) we obtain (15). From (14), (15) it follows that (see [3], Theorem 37.1)

$$\left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| \leq \lambda_0(x), \quad (x, y) \in T(b, t, L),$$

where $\lambda_0(x)$ is the solution of the initial problem

$$(16) \quad \frac{du}{dx} = A[1 + 2mC\lambda(x) + u]^2, \quad u(0) = B.$$

Since $\lambda(x)$ is the solution of (16) for $x \in [0, 2p)$, we have

$$(17) \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| \leq \lambda(x), \quad (x, y) \in T(b, t, L).$$

Because for each point $(x, y) \in E$ there exists a $t > 0$ such that $(x, y) \in T(b, t, L)$, then we have by (17)

$$(18) \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| \leq \lambda(x) \quad \text{for } (x, y) \in E.$$

In a similar way we can prove that

$$(19) \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial x} \right| \leq \lambda(x) \quad \text{for } (x, y) \in E.$$

Since $\lambda(p) = r$ and $b \leq p$, it follows from (18) and (19) that

$$\left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| \leq r, \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial x} \right| \leq r, \quad (x, y) \in E,$$

which completes the proof of (Π_{n+1}) .

As an immediate corollary of (I_n) and (12) we obtain (III_{n+1}) . Now, Lemma 1 follows by induction.

III. The convergence of the sequence $\{z_n\}$. We adopt the following notations:

$$\bar{T}(b_0, t, L_0) = \{(x, y): x \in [0, b_0], |y| \leq t - L_0 x\},$$

where $0 < b_0 < b$, $t > 0$, $L_0 = \max(L, C)$, $b_0 < t/L_0$,

$$M = \max_{(x, y) \in \bar{T}(b_0, t, L_0)} \left| \frac{\partial z_0(x, y)}{\partial x} - f(x, y, z_0(x, y), u_0(\alpha_1(x, y), \beta_1(x, y)), \dots, u_0(\alpha_m(x, y), \beta_m(x, y))) \frac{\partial z_0(x, y)}{\partial y} - g(x, y, z_0(x, y), u_0(\alpha_1(x, y), \beta_1(x, y)), \dots, u_0(\alpha_m(x, y), \beta_m(x, y))) \right|,$$

$$K = \frac{M}{mA(1+r)}, \quad N = mA(1+r)e^{A(1+r)b}.$$

LEMMA 2. If Assumption H_1 is satisfied, then

$$(21) \quad |z_n(x, y) - z_{n-1}(x, y)| \leq K \frac{(Nx)^n}{n!}, \quad n = 1, 2, \dots, (x, y) \in \bar{T}(b_0, t, L_0).$$

Proof. According to Lemma 1 we have for $n = 1$

$$\begin{aligned} \left| \frac{\partial z_1(x, y)}{\partial x} - \frac{\partial z_0(x, y)}{\partial x} \right| &\leq |f^{(0)}(x, y, z_0(x, y))| \left| \frac{\partial z_1(x, y)}{\partial y} - \frac{\partial z_0(x, y)}{\partial y} \right| + \\ &+ |f^{(0)}(x, y, z_0(x, y)) - f^{(0)}(x, y, z_1(x, y))| \left| \frac{\partial z_0(x, y)}{\partial y} \right| + \\ &+ |g^{(0)}(x, y, z_1(x, y)) - g^{(0)}(x, y, z_0(x, y))| + \\ &+ \left| f^{(0)}(x, y, z_0(x, y)) \frac{\partial z_0(x, y)}{\partial y} + g^{(0)}(x, y, z_0(x, y)) - \frac{\partial z_0(x, y)}{\partial x} \right| \\ &\leq rA |z_1(x, y) - z_0(x, y)| + L_0 \left| \frac{\partial z_1(x, y)}{\partial y} - \frac{\partial z_0(x, y)}{\partial y} \right| + \\ &+ A |z_1(x, y) - z_0(x, y)| + M. \end{aligned}$$

From these inequalities and from (3), (5), (8) we see that the function $z_1(x, y) - z_0(x, y)$ satisfies the following conditions:

$$\begin{aligned} \left| \frac{\partial}{\partial x} (z_1(x, y) - z_0(x, y)) \right| \\ \leq A(1+r) |z_1(x, y) - z_0(x, y)| + M + L_0 \left| \frac{\partial}{\partial y} (z_1(x, y) - z_0(x, y)) \right|, \\ (x, y) \in \bar{T}(b_0, t, L_0), \end{aligned}$$

$$|z_1(0, y) - z_0(0, y)| = 0, \quad y \in [-t, t].$$

In virtue of Theorem 37.1 in [3] we have

$$(22) \quad |z_1(x, y) - z_0(x, y)| \leq v_1(x) \quad \text{for } (x, y) \in \bar{T}(b_0, t, L_0),$$

where $v_1(x)$ is the solution of the initial problem

$$\frac{du}{dx} = A(1+r)u + M, \quad u(0) = 0.$$

Since

$$v_1(x) = \frac{M}{A(1+r)} [e^{A(1+r)x} - 1]$$

and

$$e^\tau - 1 \leq \tau e^\tau \quad \text{for } \tau \geq 0,$$

it follows from (22) that

$$|z_1(x, y) - z_0(x, y)| \leq \frac{M}{A(1+r)} e^{A(1+r)b} A(1+r)x = K \frac{Nx}{1!},$$

$$(x, y) \in \bar{T}(b_0, t, L_0),$$

which completes the proof of (21) for $n = 1$.

Now suppose that inequality (21) is true for some $n \geq 1$. In virtue of Lemma 1 and by Assumption H_1 we have

$$\begin{aligned} & \left| \frac{\partial z_{n+1}(x, y)}{\partial x} - \frac{\partial z_n(x, y)}{\partial x} \right| \\ & \leq |f^{(n)}(x, y, z_{n+1}(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + \\ & \quad + |f^{(n)}(x, y, z_{n+1}(x, y)) - f^{(n)}(x, y, z_n(x, y))| \left| \frac{\partial z_n(x, y)}{\partial y} \right| + \\ & \quad + |f^{(n)}(x, y, z_n(x, y)) - f^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial z_n(x, y)}{\partial y} \right| + \\ & \quad + |g^{(n)}(x, y, z_{n+1}(x, y)) - g^{(n)}(x, y, z_n(x, y))| + \\ & \quad + |g^{(n)}(x, y, z_n(x, y)) - g^{(n-1)}(x, y, z_n(x, y))| \\ & \leq L_0 \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + A(r+1)|z_{n+1}(x, y) - z_n(x, y)| + \\ & \quad + A(r+1) \sum_{i=1}^m |u_n(\alpha_i(x, y), \beta_i(x, y)) - u_{n-1}(\alpha_i(x, y), \beta_i(x, y))|. \end{aligned}$$

Since $(\alpha_i(x, y), \beta_i(x, y)) \in \bar{T}(b_0, t, L_0)$ for $(x, y) \in \bar{T}(b_0, t, L_0)$ and for $\alpha_i(x, y) \geq 0$, it follows that the function $z_{n+1}(x, y) - z_n(x, y)$ satisfies the differential inequality

$$\begin{aligned} & \left| \frac{\partial}{\partial x} (z_{n+1}(x, y) - z_n(x, y)) \right| \leq A(r+1)|z_{n+1}(x, y) - z_n(x, y)| + \\ & \quad + A(1+r)mK \frac{(Nx)^n}{n!} + L_0 \left| \frac{\partial}{\partial y} (z_{n+1}(x, y) - z_n(x, y)) \right|, \\ & \quad (x, y) \in \bar{T}(b_0, t, L_0). \end{aligned}$$

Since

$$z_{n+1}(0, y) - z_n(0, y) = 0, \quad y \in [-t, t],$$

we finally obtain (see [3], Theorem 37.1)

$$(23) \quad |z_{n+1}(x, y) - z_n(x, y)| \leq v_{n+1}(x), \quad (x, y) \in \bar{T}(b_0, t, L_0),$$

where $v_{n+1}(x)$ is the solution of the Cauchy problem

$$\frac{du}{dx} = A(r+1)u + A(r+1)Km \frac{(Nx)^n}{n!}, \quad u(0) = 0.$$

Because

$$v_{n+1}(x) = \frac{mKN^n}{A^n(1+r)^n} \left[e^{A(1+r)x} - 1 - \frac{A(1+r)x}{1!} - \frac{(A(1+r)x)^2}{2!} - \dots \right. \\ \left. \dots - \frac{(A(1+r)x)^n}{n!} \right]$$

and

$$e^\tau - 1 - \frac{\tau}{1!} - \frac{\tau^2}{2!} - \dots - \frac{\tau^n}{n!} \leq \frac{\tau^{n+1}}{(n+1)!} e^\tau, \quad \tau \geq 0,$$

we conclude by (23) that

$$|z_{n+1}(x, y) - z_n(x, y)| \leq \frac{mKN^n}{A^n(1+r)^n} \frac{[A(1+r)x]^{n+1}}{(n+1)!} e^{A(1+r)x} \\ \leq mA(1+r)e^{A(1+r)b} N^n K \frac{x^{n+1}}{(n+1)!} = K \frac{(Nx)^{n+1}}{(n+1)!}.$$

Thus we obtain (21) by induction.

IV. The convergence of the sequence $\{\partial z_n / \partial y\}$. At first we prove that the second order derivatives of the functions z_n , $n = 0, 1, 2, \dots$, are uniformly bounded in E .

LEMMA 3. *If Assumptions H_1, H_2 are satisfied, then*

$$(24) \quad \left| \frac{\partial^2 z_n}{\partial y^2} \right|, \left| \frac{\partial^2 z_n}{\partial x \partial y} \right|, \left| \frac{\partial^2 z_n}{\partial x^2} \right| \leq \mu(x), \quad n = 0, 1, \dots, (x, y) \in E,$$

where

$$\mu(x) = \left[D + \frac{A_0}{B_0 + C_0} \right] e^{(B_0 + C_0)x} - \frac{A_0}{B_0 + C_0}, \\ A_0 = D(1+r) \{ [1+r+2mrC]^2 + 2mrA \}, \\ B_0 = 4mA(1+r), \\ C_0 = N_0 + A(2+3r+2mrC).$$

Proof. From (4) and by Assumption H_2 we conclude that estimations (24) are true for $n = 0$.

Let us suppose that evaluation (24) holds for some $n \geq 0$. We will prove that

$$(25) \quad \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right|, \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial x \partial y} \right|, \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial x^2} \right| \leq \mu(x), \\ (x, y) \in \bar{T}(b_0, t, L_0),$$

where $\bar{T}(b_0, t, L_0)$ is the set defined in III.

Setting

$$(26) \quad V_{n+1}(x, y, h) = \frac{1}{h} \left[\frac{\partial z_{n+1}(x, y+h)}{\partial y} - \frac{\partial z_{n+1}(x, y)}{\partial y} \right],$$

$$(x, y), (x, y+h) \in \bar{T}(b_0, t, L_0)$$

we will prove that there exists a non-negative function $\theta(h)$ such that

$$(27) \quad \lim_{h \rightarrow 0} \theta(h) = 0$$

and

$$(28) \quad \left| \frac{\partial}{\partial x} (V_{n+1}(x, y, h)) \right| \leq A_0 + B_0 \mu(x) + C_0 |V_{n+1}(x, y, h)| + \theta(h) +$$

$$+ L_0 \left| \frac{\partial}{\partial y} (V_{n+1}(x, y, h)) \right|, \quad (x, y), (x, y+h) \in \bar{T}(b_0, t, L_0).$$

Substituting $z_{n+1}(x, y)$ and $z_{n+1}(x, y+h)$ into (5) and differentiating the identities thus obtained with respect to y we get

$$(29) \quad \left| \frac{\partial V_{n+1}(x, y, h)}{\partial x} \right| = \left| \frac{1}{h} \left[\frac{\partial^2 z_{n+1}(x, y+h)}{\partial x \partial y} - \frac{\partial^2 z_{n+1}(x, y)}{\partial x \partial y} \right] \right|$$

$$\leq \left| \frac{1}{h} [f_y^{(n)}(x, y+h, z_{n+1}(x, y+h)) - f_y^{(n)}(x, y, z_{n+1}(x, y))] \right| \left| \frac{\partial z_{n+1}(x, y+h)}{\partial y} \right| +$$

$$+ |f_y^{(n)}(x, y, z_{n+1}(x, y))| |V_{n+1}(x, y, h)| +$$

$$+ \left| \frac{1}{h} [f_z^{(n)}(x, y+h, z_{n+1}(x, y+h)) - f_z^{(n)}(x, y, z_{n+1}(x, y))] \right| \left| \frac{\partial z_{n+1}(x, y+h)}{\partial y} \right|^2 +$$

$$+ \left| \frac{1}{h} \left[\left(\frac{\partial z_{n+1}(x, y+h)}{\partial y} \right)^2 - \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right)^2 \right] \right| \cdot |f_z^{(n)}(x, y, z_{n+1}(x, y))| +$$

$$+ \left| \frac{1}{h} [g_y^{(n)}(x, y+h, z_{n+1}(x, y+h)) - g_y^{(n)}(x, y, z_{n+1}(x, y))] \right| +$$

$$+ \left| \frac{1}{h} [g_z^{(n)}(x, y+h, z_{n+1}(x, y+h)) - g_z^{(n)}(x, y, z_{n+1}(x, y))] \right| \cdot \left| \frac{\partial z_{n+1}(x, y+h)}{\partial y} \right| +$$

$$+ |g_z^{(n)}(x, y, z_{n+1}(x, y))| |V_{n+1}(x, y, h)| +$$

$$+ \left| \frac{1}{h} [f^{(n)}(x, y+h, z_{n+1}(x, y+h)) - f^{(n)}(x, y, z_{n+1}(x, y))] \right| \cdot \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right| +$$

$$+ |f^{(n)}(x, y, z_{n+1}(x, y))| \left| \frac{1}{h} \left[\frac{\partial^2 z_{n+1}(x, y+h)}{\partial y^2} - \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right] \right|.$$

To prove that $V_{n+1}(x, y, h)$ satisfies inequality (28) we first show that

$$(30) \quad \left. \begin{aligned} & \left| \frac{1}{h} [f_v(P_n(x, y+h)) - f_v(P_n(x, y))] \right| \\ & \left| \frac{1}{h} [f_{z_i}(P_n(x, y+h)) - f_{z_i}(P_n(x, y))] \right| \\ & (i = 0, 1, 2, \dots, m) \\ & \left| \frac{1}{h} [g_v(P_n(x, y+h)) - g_v(P_n(x, y))] \right| \\ & \left| \frac{1}{h} [g_{z_i}(P_n(x, y+h)) - g_{z_i}(P_n(x, y))] \right| \\ & (i = 0, 1, 2, \dots, m) \end{aligned} \right\} \leq D[1 + r(1 + 2mC)]$$

and

$$(31) \quad \left| \frac{1}{h} [f(P_n(x, y+h)) - f(P_n(x, y))] \right| \leq A[1 + r(1 + 2mC)],$$

where

$$(32) \quad P_n(x, y) = (x, y, z_{n+1}(x, y), u_n(\alpha_1(x, y), \beta_1(x, y)), \dots, u_n(\alpha_m(x, y), \beta_m(x, y))).$$

By Assumption H_2

$$(33) \quad \left| \frac{1}{h} [f_v(P_n(x, y+h)) - f_v(P_n(x, y))] \right| \leq D \left[1 + \left| \frac{z_{n+1}(x, y+h) - z_n(x, y)}{h} \right| + \sum_{i=1}^m \left| \frac{1}{h} (u_n(\alpha_i(x, y), \beta_i(x, y)) - u_n(\alpha_i(x, y), \beta_i(x, y))) \right| \right].$$

Since

$$\left| \frac{\partial u_n(x, y)}{\partial y} \right|, \left| \frac{\partial u_n(x, y)}{\partial x} \right| \leq r \quad \text{for } (x, y) \in E \cup I, \quad n = 0, 1, 2, \dots$$

and

$$\left| \frac{\alpha_j(x, y+h) - \alpha_j(x, y)}{h} \right| \leq C, \quad \left| \frac{\beta_j(x, y+h) - \beta_j(x, y)}{h} \right| \leq C, \\ j = 1, \dots, m,$$

we get

$$\begin{aligned} & \left| \frac{1}{h} [u_n(\alpha_j(x, y+h), \beta_j(x, y+h)) - u_n(\alpha_j(x, y), \beta_j(x, y))] \right| \\ & \leq r \left| \frac{\alpha_j(x, y+h) - \beta_j(x, y)}{h} \right| + r \left| \frac{\beta_j(x, y+h) - \beta_j(x, y)}{h} \right| \leq 2rC, \\ & j = 1, 2, \dots, m. \end{aligned}$$

Now, by the last inequalities we obtain from (33)

$$\left| \frac{1}{h} [f_v(P_n(x, y+h)) - f_v(P_n(x, y))] \right| \leq D[1+r+2mrC].$$

By a similar argument we get the remaining inequalities in (30) and (31). Now, we prove that

$$\begin{aligned} (34) \quad & \left| \frac{1}{h} [f_v^{(n)}(x, y+h, z_{n+1}(x, y+h)) - f_v^{(n)}(x, y, z_{n+1}(x, y))] \right| \\ & \leq D[1+r(1+2mC)][1+2mrC] + 2mA[2C\mu(x) + rD] \end{aligned}$$

and

$$\begin{aligned} (35) \quad & \left| \frac{1}{h} [g_v^{(n)}(x, y+h, z_{n+1}(x, y+h)) - g_v^{(n)}(x, y, z_{n+1}(x, y))] \right| \\ & \leq D[1+r(1+2mC)][1+2mrC] + 2mA[2C\mu(x) + rD]. \end{aligned}$$

To prove (34) observe that by (6) and (30)

$$\begin{aligned} (36) \quad & \left| \frac{1}{h} [f_v^{(n)}(x, y+h, z_{n+1}(x, y+h)) - f_v^{(n)}(x, y, z_{n+1}(x, y))] \right| \\ & \leq \left| \frac{1}{h} [f_v(P_n(x, y+h)) - f_v(P_n(x, y))] \right| + \\ & + \sum_{j=1}^m \left\{ \left| \frac{1}{h} [f_{z_j}(P_n(x, y+h)) - f_{z_j}(P_n(x, y))] \right| \times \right. \\ & \times \left| \frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial x} \frac{\partial \alpha_j(x, y+h)}{\partial y} + \right. \\ & + \left. \frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial y} \frac{\partial \beta_j(x, y+h)}{\partial y} \right| + \\ & + |f_{z_j}(P_n(x, y))| \left| \frac{1}{h} \left[\frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial x} \frac{\partial \alpha_j(x, y+h)}{\partial y} + \right. \right. \\ & + \left. \left. \frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial y} \frac{\partial \beta_j(x, y+h)}{\partial y} \right] \right| \end{aligned}$$

$$\left| \frac{\partial u_n(\alpha_j(x, y), \beta_j(x, y))}{\partial x} \frac{\partial \alpha_j(x, y)}{\partial y} - \frac{\partial u_n(\alpha_j(x, y), \beta_j(x, y))}{\partial y} \frac{\partial \beta_j(x, y)}{\partial y} \right| \Bigg\}.$$

It follows from Lemma 1 that

$$\begin{aligned} & \left| \frac{1}{h} \left[\frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial x} \frac{\partial \alpha_j(x, y+h)}{\partial y} - \frac{\partial u_n(\alpha_j(x, y), \beta_j(x, y))}{\partial x} \frac{\partial \alpha_j(x, y)}{\partial y} \right] \right| \\ & \leq \left| \frac{1}{h} \left[\frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial x} - \frac{\partial u_n(\alpha_j(x, y), \beta_j(x, y))}{\partial x} \right] \right| \cdot \left| \frac{\partial \alpha_j(x, y)}{\partial y} \right| + \\ & \quad + \left| \frac{\partial u_n(\alpha_j(x, y), \beta_j(x, y))}{\partial x} \right| \cdot \left| \frac{1}{h} \left[\frac{\partial \alpha_j(x, y+h)}{\partial y} - \frac{\partial \alpha_j(x, y)}{\partial y} \right] \right| \\ & \leq 2C\mu(x) + rD. \end{aligned}$$

By a similar argument we get

$$\left| \frac{1}{h} \left[\frac{\partial u_n(\alpha_j(x, y+h), \beta_j(x, y+h))}{\partial y} \frac{\partial \beta_j(x, y+h)}{\partial y} - \frac{\partial u_n(\alpha_j(x, y), \beta_j(x, y))}{\partial y} \frac{\partial \beta_j(x, y)}{\partial y} \right] \right| \leq 2C\mu(x) + rD.$$

Hence it follows from (30)–(32), (36) that

$$\begin{aligned} & \left| \frac{1}{h} [f_y^{(n)}(x, y+h, z_{n+1}(x, y+h)) - f_y^{(n)}(x, y, z_{n+1}(x, y))] \right| \\ & \leq D[1 + r(1 + 2mC)] + m\{2rCD[1 + r(1 + 2mC)] + 2A[2C\mu(x) + rD]\} \\ & = D[1 + r(1 + 2mC)](1 + 2mrC) + 2mA[2C\mu(x) + rD]; \end{aligned}$$

this completes the proof of (34).

In a similar way we obtain inequality (35).

It follows from (26) that there exists a non-negative function $\theta_0(h)$ such that

$$(37) \quad \lim_{h \rightarrow 0} \theta_0(h) = 0$$

and

$$(38) \quad \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right| \leq |V_{n+1}(x, y, h)| + \theta_0(h), \quad (x, y) \in \bar{T}(b_0, t, L_0),$$

$$|V_{n+1}(x, y, h)| \leq \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right| + \theta_0(h), \quad (x, y) \in \bar{T}(b_0, t, L_0)$$

for $h \in [0, b_0]$.

Finally, by (29)–(32), (34), (35), (38) we have

$$\begin{aligned}
 \left| \frac{\partial V_{n+1}(x, y, h)}{\partial x} \right| &\leq rD[1+r(1+2mC)](1+2mrC) + 2mrA[2C\mu(x)+rD] + \\
 &+ N_0|V_{n+1}(x, y, h)| + r^2D[1+r(1+2mC)] + 2Ar|V_{n+1}(x, y, h)| + \\
 &+ D[1+r(1+2mC)](1+2mrC) + 2mA[2C\mu(x)+rD] + rD[1+r(1+2mC)] + \\
 &+ A|V_{n+1}(x, y, h)| + A[1+r(1+2mC)](|V_{n+1}(x, y, h)| + \theta_0(h)) + \\
 &+ L \left| \frac{\partial}{\partial y} (V_{n+1}(x, y, h)) \right| \\
 &\leq A_0 + B_0\mu(x) + C_0|V_{n+1}(x, y, h)| + \\
 &+ \theta(h) + L_0 \left| \frac{\partial}{\partial y} (V_{n+1}(x, y, h)) \right|,
 \end{aligned}$$

where

$$\theta(h) = \theta_0(h)A[1+r+2mrC];$$

and this means that the differential inequality (28) is satisfied.

By (26), (38) and from Assumption H_2 we see that the following initial inequality

$$|V_{n+1}(0, y, h)| \leq D + \theta_0(h), \quad y \in [-t, t]$$

is satisfied.

Applying Theorem 37.1 of [3] to the function $V_{n+1}(x, y, h)$ we obtain

$$(39) \quad |V_{n+1}(x, y, h)| \leq \mu_h(x) \quad \text{for } (x, y), (x, y+h) \in \bar{T}(b_0, t, L_0),$$

where $\mu_h(x)$ is the solution of the initial problem

$$\frac{du}{dx} = A_0 + B_0\mu(x) + C_0u + \theta(h), \quad u(0) = D + \theta_0(h).$$

Let us consider the Cauchy problem

$$\frac{du}{dx} = A_0 + B_0\mu(x) + C_0u, \quad u(0) = D.$$

The function $u = \mu(x)$ is the solution of this problem. Since

$$\lim_{h \rightarrow 0} \theta(h) = \lim_{h \rightarrow 0} \theta_0(h) = 0,$$

it follows (from theorems on the continuous dependence of the solution of Cauchy problem on the initial values and on the right-hand sides) that

$$\lim_{h \rightarrow 0} \mu_h(x) = \mu(x) \quad \text{uniformly with respect to } x \in [0, b_0].$$

The last condition and (26), (39) imply that

$$(40) \quad \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right| \leq \mu(x) \quad \text{for } (x, y) \in \bar{T}(b_0, t, L_0).$$

In a similar way we obtain

$$(41) \quad \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial x^2} \right| \leq \mu(x), \quad \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial x \partial y_j} \right| \leq \mu(x) \quad \text{for } (x, y) \in \bar{T}(b_0, t, L_0).$$

For each point $(x, y) \in E$ we can choose b_0, t so large that $(x, y) \in \bar{T}(b_0, t, L_0)$. In virtue of (40), (41) we get

$$(42) \quad \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right|, \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial x^2} \right|, \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial x \partial y} \right| \leq \mu(x)$$

for $(x, y) \in E$.

Since estimations (24) for some $n \geq 0$ imply inequalities (42) (for $n+1$) and evaluations (24) are satisfied for $n = 0$, we obtain Lemma 3 by induction. Now we prove the following

LEMMA 4. If Assumptions H_1 and H_2 are satisfied, then

$$(43) \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| \leq \tilde{B} \frac{(Px)^n}{n!},$$

$$n = 0, 1, 2, \dots, (x, y) \in \bar{T}(b_0, t, L_0),$$

where

$$(44) \quad \begin{aligned} \tilde{A} &= H(1+r) + bNK(Nb+m)(As+Dr+Dr^2), \\ \tilde{B} &= \max[r+B, N_0+2Ar+A], \\ s &= \mu(b), \\ H &= M(Nb+m)[D(1+2rmC)be^{A(1+r)b} + AC] + ACm\tilde{B}(1+L), \\ P &= \max[N, \tilde{A}\tilde{B}^{-1}e^{\tilde{B}b}]. \end{aligned}$$

Proof. Since

$$\left| \frac{\partial z_1(x, y)}{\partial y} - \frac{\partial z_0(x, y)}{\partial y} \right| \leq r+B \leq \tilde{B},$$

it follows that (43) is satisfied for $n = 0$.

Let us suppose that for some $n \geq 1$ the inequality

$$(45) \quad \left| \frac{\partial z_n(x, y)}{\partial y} - \frac{\partial z_{n-1}(x, y)}{\partial y} \right| \leq \tilde{B} \frac{(Px)^{n-1}}{(n-1)!}, \quad [(x, y) \in \bar{T}(b_0, t, L_0)]$$

is true. We shall prove that

$$(46) \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| \leq \tilde{B} \frac{(Px)^n}{n!}, \quad (x, y) \in \bar{T}(b_0, t, L_0).$$

At first we prove that the differential inequality

$$(47) \quad \left| \frac{\partial}{\partial x} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right| \\ \leq \tilde{A} \frac{(Px)^{n-1}}{(n-1)!} + \tilde{B} \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + \\ + L_0 \left| \frac{\partial}{\partial y} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right|, \quad (x, y) \in \bar{T}(b_0, t, L_0),$$

holds.

Since

$$\frac{\partial z_{n+1}(x, y)}{\partial x} = f^{(n)}(x, y, z_{n+1}(x, y)) \frac{\partial z_{n+1}(x, y)}{\partial y} + g^{(n)}(x, y, z_{n+1}(x, y))$$

and

$$\frac{\partial z_n(x, y)}{\partial x} = f^{(n-1)}(x, y, z_n(x, y)) \frac{\partial z_n(x, y)}{\partial y} + g^{(n-1)}(x, y, z_n(x, y)),$$

then

$$(48) \quad \left| \frac{\partial}{\partial x} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right| \\ \leq |f_y^{(n)}(x, y, z_{n+1}(x, y)) - f_y^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| + \\ + |f_y^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + \\ + |f_z^{(n)}(x, y, z_{n+1}(x, y)) - f_z^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right|^2 + \\ + |f_z^{(n-1)}(x, y, z_n(x, y))| \left| \left(\frac{\partial z_{n+1}(x, y)}{\partial y} \right)^2 - \left(\frac{\partial z_n(x, y)}{\partial y} \right)^2 \right| + \\ + |f^{(n)}(x, y, z_{n+1}(x, y)) - f^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} \right| + \\ + |f^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial^2 z_{n+1}(x, y)}{\partial y^2} - \frac{\partial^2 z_n(x, y)}{\partial y^2} \right| + \\ + |g_y^{(n)}(x, y, z_{n+1}(x, y)) - g_y^{(n-1)}(x, y, z_n(x, y))| + \\ + |g_z^{(n)}(x, y, z_{n+1}(x, y)) - g_z^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} \right| + \\ + |g_z^{(n-1)}(x, y, z_n(x, y))| \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right|.$$

Now, setting for $(x, y) \in \bar{T}(b_0, t, L_0)$

$$\tilde{P}_n(x, y) = (x, y, z_n(x, y), u_{n-1}(\alpha_1(x, y), \beta_1(x, y)), \dots, u_{n-1}(\alpha_m(x, y), \beta_m(x, y)))$$

we prove that

$$(49) \quad \left\{ \begin{array}{l} |f_v(\tilde{P}_{n+1}(x, y)) - f_v(\tilde{P}_n(x, y))| \\ |f_{z_i}(\tilde{P}_{n+1}(x, y)) - f_{z_i}(\tilde{P}_n(x, y))| \\ (i = 0, 1, \dots, m) \\ |g_v(\tilde{P}_{n+1}(x, y)) - g_v(\tilde{P}_n(x, y))| \\ |g_{z_i}(\tilde{P}_{n+1}(x, y)) - g_{z_i}(\tilde{P}_n(x, y))| \\ (i = 0, 1, \dots, m) \end{array} \right\} \leq DK(Nb + m) \frac{(Nx)^n}{n!},$$

$$n = 0, 1, 2, \dots, (x, y) \in \bar{T}(b_0, t, L_0),$$

and

$$(50) \quad |f(\tilde{P}_{n+1}(x, y)) - f(\tilde{P}_n(x, y))| \leq AK(Nb + m) \frac{(Nx)^n}{n!},$$

$$n = 0, 1, 2, \dots, (x, y) \in \bar{T}(b_0, t, L_0).$$

By Assumption H_2 and Lemma 2 we have

$$\begin{aligned} & |f_v(\tilde{P}_{n+1}(x, y)) - f_v(\tilde{P}_n(x, y))| \\ & \leq D |z_{n+1}(x, y) - z_n(x, y)| + \\ & \quad + D \sum_{i=1}^m |u_n(\alpha_i(x, y), \beta_i(x, y)) - u_{n-1}(\alpha_i(x, y), \beta_i(x, y))| \\ & \leq DK \frac{(Nx)^{n+1}}{(n+1)!} + DmK \frac{(Nx)^n}{n!} \leq DK(Nb + m) \frac{(Nx)^n}{n!}. \end{aligned}$$

By a similar argument we get the remaining inequalities in (49) and (50).

Now, we prove that

$$(51) \quad |f_v^{(n)}(x, y, z_{n+1}(x, y)) - f_v^{(n-1)}(x, y, z_n(x, y))| \leq H \frac{(Px)^{n-1}}{(n-1)!},$$

$$n = 1, 2, \dots, (x, y) \in \bar{T}(b_0, t, L_0)$$

and

$$(52) \quad |g_v^{(n)}(x, y, z_{n+1}(x, y)) - g_v^{(n-1)}(x, y, z_n(x, y))| \leq H \frac{(Px)^{n-1}}{(n-1)!},$$

$$n = 1, 2, \dots, (x, y) \in \bar{T}(b_0, t, L_0).$$

Because

$$\begin{aligned}
& \left| \frac{\partial z_n(x, y)}{\partial x} - \frac{\partial z_{n-1}(x, y)}{\partial x} \right| \\
& \leq |f(\tilde{P}_n(x, y)) - f(\tilde{P}_{n-1}(x, y))| \left| \frac{\partial z_n(x, y)}{\partial y} \right| + \\
& \quad + |f(\tilde{P}_{n-1}(x, y))| \left| \frac{\partial z_n(x, y)}{\partial y} - \frac{\partial z_{n-1}(x, y)}{\partial y} \right| + \\
& \quad + |g(\tilde{P}_n(x, y)) - g(\tilde{P}_{n-1}(x, y))| \\
& \leq (rA + A) |z_n(x, y) - z_{n-1}(x, y)| + \\
& \quad + (rA + A) \sum_{i=1}^m |u_{n-1}(\alpha_i(x, y), \beta_i(x, y)) - u_{n-2}(\alpha_i(x, y), \beta_i(x, y))| + \\
& \quad + L \left| \frac{\partial z_n(x, y)}{\partial y} - \frac{\partial z_{n-1}(x, y)}{\partial y} \right| \\
& \leq AK(r+1) \frac{(Nx)^n}{n!} + AK(r+1)m \frac{(Nx)^{n-1}}{(n-1)!} + L \left| \frac{\partial z_n(x, y)}{\partial y} - \frac{\partial z_{n-1}(x, y)}{\partial y} \right|,
\end{aligned}$$

it follows from Lemma 2 and from (45), (49), (50) that

$$\begin{aligned}
& |f_y^{(n)}(x, y, z_{n+1}(x, y)) - f_y^{(n-1)}(x, y, z_n(x, y))| \\
& \leq |f_y(\tilde{P}_{n+1}(x, y)) - f_y(\tilde{P}_n(x, y))| + \\
& \quad + \sum_{i=1}^m |f_{z_i}(\tilde{P}_{n+1}(x, y)) - f_{z_i}(\tilde{P}_n(x, y))| \left| \frac{\partial u_n(\alpha_i(x, y), \beta_i(x, y))}{\partial x} \frac{\partial \alpha_i(x, y)}{\partial y} + \right. \\
& \quad \quad \quad \left. + \frac{\partial u_n(\alpha_i(x, y), \beta_i(x, y))}{\partial y} \frac{\partial \beta_i(x, y)}{\partial y} \right| + \\
& \quad + \sum_{i=1}^m |f_{z_i}(\tilde{P}_n(x, y))| \left[\left| \frac{\partial u_n(\alpha_i(x, y), \beta_i(x, y))}{\partial x} - \right. \right. \\
& \quad \quad \quad \left. \left. - \frac{\partial u_{n-1}(\alpha_i(x, y), \beta_i(x, y))}{\partial x} \right| \left| \frac{\partial \alpha_i(x, y)}{\partial y} \right| + \right. \\
& \quad \quad \left. + \left| \frac{\partial u_n(\alpha_i(x, y), \beta_i(x, y))}{\partial y} - \frac{\partial u_{n-1}(\alpha_i(x, y), \beta_i(x, y))}{\partial y} \right| \left| \frac{\partial \beta_i(x, y)}{\partial y} \right| \right] \\
& \leq DK(Nb + m) \frac{(Nx)^n}{n!} + 2rCmDK(Nb + m) \frac{(Nx)^n}{n!} +
\end{aligned}$$

$$\begin{aligned}
& + ACm \left[AK(r+1) \frac{(Nx)^n}{n!} + AK(r+1)m \frac{(Nx)^{n-1}}{(n-1)!} + L\tilde{B} \frac{(Px)^{n-1}}{(n-1)!} \right] + \\
& + ACm\tilde{B} \frac{(Px)^{n-1}}{(n-1)!} \\
& \leq M(Nb+m) [D(1+2rmC)be^{A(1+r)b} + AC] \frac{(Nx)^{n-1}}{(n-1)!} + ACm\tilde{B}(L+1) \frac{(Px)^{n-1}}{(n-1)!} \\
& \leq H \frac{(Px)^{n-1}}{(n-1)!},
\end{aligned}$$

which completes the proof of (51).

Similarly we prove (52).

From (48)–(52) and by Lemma 3 we have

$$\begin{aligned}
& \left| \frac{\partial}{\partial x} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right| \\
& \leq rH \frac{(Px)^{n-1}}{(n-1)!} + N_0 \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + \\
& + 2Ar \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + r^2 DK(Nb+m) \frac{(Nx)^n}{n!} + \\
& + AK(Nb+m)s \frac{(Nx)^n}{n!} + L_0 \left| \frac{\partial}{\partial y} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right| + \\
& + H \frac{(Px)^{n-1}}{(n-1)!} + rDK(Nb+m) \frac{(Nx)^n}{n!} + A \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right|.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \left| \frac{\partial}{\partial x} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right| \\
& \leq H(1+r) \frac{(Px)^{n-1}}{(n-1)!} + \frac{(Nx)^{n-1}}{(n-1)!} NbK(Nb+m)[As + Dr + Dr^2] + \\
& + \tilde{B} \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| + L_0 \left| \frac{\partial}{\partial y} \left(\frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right) \right|.
\end{aligned}$$

The last inequality and (44) imply the differential inequality (47).

Since

$$\frac{\partial z_{n+1}(0, y)}{\partial y} - \frac{\partial z_n(0, y)}{\partial y} = 0 \quad \text{for } y \in [-t, t],$$

it follows that (see [3], Theorem 37.1)

$$(53) \quad \left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| \leq \tilde{u}(x), \quad (x, y) \in \bar{T}(b_0, t, L_0),$$

where $\tilde{u}(x)$ is the solution of the initial problem

$$\frac{du}{dx} = \tilde{A} \frac{(Px)^{n-1}}{(n-1)!} + \tilde{B}u, \quad u(0) = 0.$$

Since

$$\tilde{u}(x) = \frac{\tilde{A}P^{n-1}}{\tilde{B}^n} \left[e^{\tilde{B}x} - 1 - \frac{\tilde{B}x}{1!} - \frac{(\tilde{B}x)^2}{2!} - \dots - \frac{(\tilde{B}x)^{n-1}}{(n-1)!} \right]$$

and

$$e^\tau - 1 - \frac{\tau}{1!} - \frac{\tau^2}{2!} - \dots - \frac{\tau^{n-1}}{(n-1)!} \leq \frac{\tau^n}{n!} e^\tau \quad \text{for } \tau \geq 0,$$

we have

$$\tilde{u}(x) \leq \frac{\tilde{A}P^{n-1}}{\tilde{B}^n} \frac{(\tilde{B}x)^n}{n!} e^{\tilde{B}x} \leq \tilde{B} \frac{e^{\tilde{B}b} \tilde{A}}{\tilde{B}} \frac{P^{n-1}x^n}{n!} \leq \tilde{B} \frac{(Px)^n}{n!}.$$

The last inequality and (53) imply

$$\left| \frac{\partial z_{n+1}(x, y)}{\partial y} - \frac{\partial z_n(x, y)}{\partial y} \right| \leq \tilde{B} \frac{(Px)^n}{n!}.$$

Inequality (46) is thus proved.

Now, Lemma 4 follows by induction.

V. Theorems on the existence and uniqueness of solutions.

THEOREM 1. *If Assumptions H_1 and H_2 are satisfied, then there exists in E a unique solution $\bar{z}(x, y)$ of the Cauchy problem (1). In an arbitrary closed and bounded domain contained in E the sequence $\{z_n(x, y)\}$ defined by (3)–(8) and the sequences of partial derivatives*

$$\left\{ \frac{\partial z_n(x, y)}{\partial y} \right\}, \left\{ \frac{\partial z_n(x, y)}{\partial x} \right\}$$

are uniformly convergent to the function $\bar{z}(x, y)$ and its derivatives $\bar{z}_y(x, y)$ and $\bar{z}_x(x, y)$, respectively.

Proof. From Lemma 2 it follows that the sequence $\{z_n(x, y)\}$ is uniformly convergent in $\bar{T}(b_0, t, L_0)$. By Lemma 4 the sequence

$$\left\{ \frac{\partial z_n(x, y)}{\partial y} \right\}$$

is uniformly convergent in $\bar{T}(b_0, t, L_0)$.

Because

$$\begin{aligned} (54) \quad \frac{\partial z_{n+1}(x, y)}{\partial x} &= f(x, y, z_{n+1}(x, y), u_n(\alpha_1(x, y), \beta_1(x, y)), \dots \\ &\quad \dots, u_n(\alpha_m(x, y), \beta_m(x, y))) \frac{\partial z_{n+1}(x, y)}{\partial y} + \\ &\quad + g(x, y, z_{n+1}(x, y), u_n(\alpha_1(x, y), \beta_1(x, y)), \dots, u_n(\alpha_m(x, y), \beta_m(x, y))) \end{aligned}$$

for $n = 0, 1, 2, \dots$ and $(x, y) \in E$, it follows from Assumptions H_1 and H_2 that the sequence

$$\left\{ \frac{\partial z_n(x, y)}{\partial x} \right\}$$

is uniformly convergent in $\bar{T}(b_0, t, L_0)$.

Put

$$\bar{z}(x, y) = \lim_{n \rightarrow \infty} z_n(x, y)$$

for $(x, y) \in \bar{T}(b_0, t, L_0)$. Then, from (7) and (54) it follows that $\bar{z}(x, y)$ is the solution of (1) in $\bar{T}(b_0, t, L_0)$.

For an arbitrary closed and bounded domain $\bar{E}^*$ such that $\bar{E}^* \subset E$ we can choose $b_0, t, 0 < b_0 < b, t > 0$, so large that $\bar{E}^* \subset \bar{T}(b_0, t, L_0)$. Hence the function $\bar{z}(x, y)$ is the solution of (1) in E .

The uniqueness of the solution $\bar{z}(x, y)$ follows immediately from the theorems on the differential inequalities given in [4] (see also [5]). Thus the proof of Theorem 1 is completed.

The Cauchy problem considered in Theorem 1 is such that the initial set I is the zone

$$I = \{(x, y): x \in (p_0, 0], y \in (-\infty, +\infty), p_0 < 0\}$$

and the initial function φ is a function of two variables. In the following Theorem 2 we give sufficient conditions for the existence and uniqueness of solutions of such a Cauchy problem for linear partial differential-functional equations of first order, in which the initial set I_0 is the line

$$I_0 = \{(x, y): x = 0, y \in (-\infty, +\infty)\}$$

and the initial function φ_0 is a function of one variable.

THEOREM 2. Assume that:

1° The functions $f(x, y, z_0, z_1, \dots, z_m)$ and $g(x, y, z_0, z_1, \dots, z_m)$ satisfy conditions 1°, 4° of Assumption H_1 and condition 1° of Assumption H_2 .

2° The functions $\alpha_i(x, y), \beta_i(x, y), i = 1, 2, \dots, m$, are of class C^2 in the set E_0 and

$$0 \leq \alpha_i(x, y) \leq x \quad \text{for } (x, y) \in E_0, i = 1, 2, \dots, m.$$

3° The initial function $\varphi_0(y)$ is of class C^2 in the set I_0 .

4° There exist constants $B, C \geq 0$ such that

$$|\varphi'_0(y)| \leq B,$$

$$\left| \frac{\partial \alpha_i(x, y)}{\partial x} \right|, \left| \frac{\partial \alpha_i(x, y)}{\partial y} \right|, \left| \frac{\partial \beta_i(x, y)}{\partial x} \right|, \left| \frac{\partial \beta_i(x, y)}{\partial y} \right| \leq C,$$

$$i = 1, 2, \dots, m.$$

5° The functions $\alpha_i(x, y)$, $\beta_i(x, y)$, $i = 1, 2, \dots, m$, satisfy condition 2° of Assumption H_2 and

$$|\varphi_0''(y)| \leq D, \quad y \in (-\infty, +\infty).$$

Under these assumptions there exists in the set E a unique solution $\bar{z}(x, y)$ of the Cauchy problem

$$(55) \quad \frac{\partial z(x, y)}{\partial x} = f(x, y, z(x, y), z(\alpha_1(x, y), \beta_1(x, y)), \dots, z(\alpha_m(x, y), \beta_m(x, y))) \frac{\partial z(x, y)}{\partial y} + \\ + g(x, y, z(x, y), z(\alpha_1(x, y), \beta_1(x, y)), \dots, z(\alpha_m(x, y), \beta_m(x, y))), \\ z(0, y) = \varphi_0(y), \quad y \in (-\infty, +\infty).$$

In an arbitrary closed and bounded domain contained in E the sequence $\{z_n(x, y)\}$ defined by:

$$z_0(x, y) = \varphi_0(y), \quad (x, y) \in E,$$

$z_{n+1}(x, y)$ is the solution of the initial problem

$$z_x = f^{(n)}(x, y, z)z_y + g^{(n)}(x, y, z), \quad z(0, y) = \varphi_0(y),$$

where

$$f^{(n)}(x, y, z) = f(x, y, z, z_n(\alpha_1(x, y), \beta_1(x, y)), \dots, z_n(\alpha_m(x, y), \beta_m(x, y))),$$

$$g^{(n)}(x, y, z) = g(x, y, z, z_n(\alpha_1(x, y), \beta_1(x, y)), \dots, z_n(\alpha_m(x, y), \beta_m(x, y))),$$

is uniformly convergent to the solution $\bar{z}(x, y)$ of (55) and the sequences of partial derivatives

$$\left\{ \frac{\partial z_n(x, y)}{\partial x} \right\}, \left\{ \frac{\partial z_n(x, y)}{\partial y} \right\}$$

are uniformly convergent to the derivatives $\bar{z}_x(x, y)$ and $\bar{z}_y(x, y)$, respectively.

The proof of this theorem is similar to the proof of Theorem 1.

References

- [1] E. Kamke, *Differentialgleichungen*, Leipzig 1969.
- [2] J. Peraśówna, *Sur le domaine d'existence des intégrales de l'équation $p = f(x, y, z)$, $q = g(x, y, z)$* , Ann. Soc. Polon. Math. 12 (1933), p. 1-5.
- [3] J. Szarski, *Differential inequalities*, Warszawa 1965.
- [4] K. Zima, *Sur les équations aux dérivées partielles du premier ordre à argument fonctionnel*, Ann. Polon. Math. 22 (1969), p. 49-59.
- [5] — *On a differential inequality with a lagging argument*, ibidem 18 (1966), p. 227-233.

Reçu par la Rédaction 27. 5. 1976