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Received on 13. 2. 1976
and in revised form on 18. 6. 1976 (812)

ACTA ARITHMETICA
XXXIV (1978)

On the density of some sets of primes, II

by

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1. Denote by K an algebraic number field generated by an algebraic number ϑ. Denote further by n the degree of K and by Δ the discriminant of K.

The Dedekind zeta function $\zeta_K(s)$, $s = \sigma + it$ is defined for $\sigma > 1$ by the absolutely convergent series

$$\zeta_K(s) = \sum_{m=1}^{\infty} F(m) m^{-s},$$

where $F(m)$ denotes the number of ideals of the field K, having the norm equal to m.

The function $\zeta_K(s)$ can be continued over the whole complex plane as a regular function, except $s = 1$, where there is a simple pole.

In the region $\sigma > 1$ we have

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{m=1}^{\infty} G(m) m^{-s}, \tag{1.1}$$

where

$$G(m) = \sum_{(N\mathfrak{p})^l = m} \log N\mathfrak{p}$$

and series in (1.1) is absolutely convergent in this region (see [1], p. 89).

It was proved by E. Landau that $\zeta_K(s) \neq 0$ in the region

$$\sigma \geqslant 1 - \frac{1}{nC_1 \log t}, \quad t \geqslant C_2, \tag{1.2}$$

where C_1 is a positive numerical constant and the constant C_2 depends on the field K (see [1], p. 105). E. Landau did not express C_2 in terms of the degree n and the discriminant Δ of the field K. The purpose of this paper is to express C_2 of (1.2) in terms of the degree n and the discriminant Δ of the field K, to extend (1.2) to $-\infty < t < +\infty$ and also to determine the remainder-term in the prime ideal formula in terms of n and Δ in the case of K being a normal extension of the field of rational numbers and in the general case, under the hypothesis that there are no exceptional real zeros of $\zeta_K(s)$.

2. Write

$$\Psi(x, K) = \sum_{m\leqslant x} G(m), \quad \Psi_1(x, K) = \sum_{m\leqslant x}(x-m)G(m), \quad \Pi(x, K) = \sum_{N\mathfrak{p}\leqslant x} 1.$$

We shall prove the following lemmas, which were given without proofs and applied in the first paper of this series (see [6], Lemmas 1 and 2).

LEMMA A. *If K is a normal extension of the field Q of rational numbers and if C_4, C_5 are any positive numerical constants, then there exists a numerical constant $C_3 > 0$ such that*

$$\Pi(x, K) = \mathrm{li}\,x + O\left(x\exp\left(-C_3\frac{\log^{1/2}x}{n^{1/2}}\right)\right), \tag{2.1}$$

provided

$$1 \leqslant |\Delta| \leqslant \log^{C_4}x, \quad 1 \leqslant n \leqslant C_5\frac{\log_2 x}{\log_3 x}, \tag{2.2}$$

and the constant implied by the O-notation depends only on C_4 and C_5.

LEMMA B. *Write*

$$\omega(x, \Delta, n) = \frac{\log x}{\max(n^{1/2}\log^{1/2}x, \log|\Delta|)}. \tag{2.3}$$

If K is a normal extension of Q and if there exists a numerical constant $C_6 > 0$ such that $\zeta_K(s) \neq 0$ for $s > 1 - \dfrac{C_6}{\log(2|\Delta|)}$, then there exist numerical constants C_7 and C_8 such that

$$\Pi(x, K) = \mathrm{li}\,x + O\big(x\exp(-C_7\omega(x, \Delta, n))\big) \tag{2.4}$$

for $1 \leqslant |\Delta| \leqslant \exp\left(C_8\dfrac{\log x}{\log_2^2 x}\right)$, $x \geqslant e^e$ and the constant in the O-symbol is numerical.

3. The proofs of Lemmas A and B will rest on the following lemmas.

LEMMA 1. $\zeta_K(s) \neq 0$, *for* $\sigma > 1$, $-\infty < t < +\infty$ (see [1], p. 141).

LEMMA 2. *In the region $-\varepsilon/2 \leqslant \sigma \leqslant 3$ $(0 < \varepsilon < 2)$ we have the estimate*

$$|\zeta_K(s)(s-1)| \leqslant (C_9(\varepsilon))^n|\Delta|^{\frac{1+\varepsilon}{2}}(|t|+1)^{\frac{1+\varepsilon}{2}n+1}. \tag{3.1}$$

Proof (compare [5], Lemma 7). Consider

$$f(s) = \frac{\zeta_K(s)}{\zeta_K(1-s)}. \tag{3.2}$$

From the functional equation for $\zeta_K(s)$ it follows that

$$f(s) = \left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2}\left(\frac{\Gamma\big(\frac{1}{2}(1-s)\big)}{\Gamma(\frac{1}{2}s)}\right)^{r_1}\Delta^{1/2-s}\,2^{r_2(2s-1)}\pi^{\frac{n}{2}(2s-1)}, \tag{3.3}$$

where $n = 2r_2 + r_1$.

We estimate $f(s)$ on the line $s = -\varepsilon/2 + it$, using the following inequality (see [3], p. 395):

$$|\Gamma(s)| = (2\pi)^{1/2}|t|^{\sigma-1/2}e^{-\frac{\pi}{2}|t|}\big(1+O(|t|^{-1})\big) \quad (s = \sigma+it) \tag{3.4}$$

for $t\to\infty$. The constant in O depends only on σ.

From (3.3) and (3.4) we get

$$|f(-\varepsilon/2+it)| \leqslant (C_{10}(\varepsilon))^n|\Delta|^{\frac{1+\varepsilon}{2}}(|t|+1)^{\frac{1+\varepsilon}{2}n}$$

for $-\infty < t < +\infty$.

Since

$$|\zeta_K(1+\varepsilon/2-it)| \leqslant \zeta_K(\varepsilon/2+1) \leqslant \left(\frac{2+\varepsilon}{\varepsilon}\right)^n,$$

we have from (3.2)

$$\left|\zeta_K\left(-\frac{\varepsilon}{2}+it\right)\right| \leqslant (C_{11}(\varepsilon))^n|\Delta|^{\frac{1+\varepsilon}{2}}(|t|+1)^{\frac{1+\varepsilon}{2}n}. \tag{3.5}$$

Write

$$G(s) = \frac{\zeta_K(s)(s-1)}{(s+1)^{\frac{1+\varepsilon}{2}n+1}}. \tag{3.6}$$

From (3.5) we have

$$|G(-\varepsilon/2+it)| \leqslant (C_{12}(\varepsilon))^n|\Delta|^{(1+\varepsilon)/2}. \tag{3.7}$$

On the other hand,

$$|G(3+it)| \leqslant C_{13}^n. \tag{3.8}$$

Using the estimate

$$|(s-1)\zeta_K(s)| \leqslant a\exp\big((n+1)|t|\big)$$

(see [5], Lemma 5), which is valid in the strip $-1 < \sigma \leqslant 3$, where a depends on K, we get

$$|G(s)| \leqslant \alpha \exp\big((n+1)|t|\big) \tag{3.9}$$

for $-\varepsilon/2 \leqslant \sigma \leqslant 3$.

From (3.7)–(3.9) and the well-known theorem of Phragmen–Lindelöff we get

$$|G(s)| \leqslant \big(C_{14}(\varepsilon)\big)^n |\Delta|^{(1+\varepsilon)/2} \tag{3.10}$$

in the strip $-\varepsilon/2 \leqslant \sigma \leqslant 3$.

From (3.6) and (3.10) follows (3.1).

Remark. Similarly, we can prove that for $0 \leqslant \sigma \leqslant 3$

$$|\zeta_K(s)| \leqslant C^n\big(1+(|\Delta|\,|t|^n)^{1-\sigma}\big)\log^n(|\Delta|\,|t|), \qquad |t| \geqslant 2,$$

$$|(s-1)\zeta_K(s)| \leqslant C^n(1+|\Delta|^{1-\sigma})\log^n(2|\Delta|), \qquad |t| \leqslant 2.$$

Lemma 3. *In the region $1 < \sigma \leqslant 3$, $-\infty < t < +\infty$,*

$$\frac{1}{|\zeta_K(s)|} \leqslant \frac{(|\Delta|2^n)^M}{\sigma-1}, \tag{3.11}$$

where M is a numerical constant, $M > 1$.

Proof. For $\sigma > 1$ we have

$$\frac{1}{|\zeta_K(s)|} = \Big|\prod_{\mathfrak{p}}\big(1-(N\mathfrak{p})^{-s}\big)\Big| \leqslant \prod_{\mathfrak{p}}\big(1+(N\mathfrak{p})^{-\sigma}\big)$$

$$< \prod_{\mathfrak{p}}\big(1-(N\mathfrak{p})^{-\sigma}\big)^{-1} = \zeta_K(\sigma).$$

Therefore, owing to Lemma 2, we have (3.11).

Lemma 4. *For every $\varepsilon > 0$ there exists a positive constant $C_{15}(\varepsilon)$ such that for all fields K which are normal extensions of Q we have the inequality*

$$\operatorname{res}_1 \zeta_K(s) \geqslant \big(C_{15}(\varepsilon)\big)^{-n}|\Delta|^{-\varepsilon} \tag{3.12}$$

(see [2], p. 187).

Lemma 5. *Let $f(s)$ be a function regular in the circle $|s-s_0| \leqslant r$ and satisfying the inequality*

$$\left|\frac{f(s)}{f(s_0)}\right| \leqslant M.$$

If $f(s) \neq 0$ in the region $|s-s_0| \leqslant r/2$, $\operatorname{Re}(s-s_0) > 0$, then

$$\operatorname{Re}\frac{f'}{f}(s_0) \geqslant -\frac{4}{r}\log M,$$

$$\operatorname{Re}\frac{f'}{f}(s_0) \geqslant -\frac{4}{r}\log M + \operatorname{Re}\frac{h}{s-\varrho_1}, \tag{3.13}$$

where ϱ_1 denotes an arbitrary zero of order h of the function $f(s)$ in the region $|s-s_0| \leqslant r/2$, $\operatorname{Re}(s-s_0) \leqslant 0$.

Lemma 6. *Let $f(s)$ be a function regular in the circle $|s-s_0| \leqslant r$ and satisfying the conditions*

$$\left|\frac{f(s)}{f(s_0)}\right| \leqslant M, \qquad \left|\frac{f'}{f}(s_0)\right| \leqslant \frac{A}{r}\log M \qquad (A > 0).$$

If $f(s) \neq 0$ in the region $|s-s_0| \leqslant r$, $\operatorname{Re}(s-s_0) > -2r_1$, where $0 < r_1 < r/4$, then

$$\left|\frac{f'}{f}(s)\right| \leqslant \frac{B}{r}\log M, \qquad |s-s_0| \leqslant r_1,$$

where B depends on A, and is independent of f, r, r_1, M.

For the proofs of Lemmas 5 and 6 see [3], pp. 384–385.

Lemma 7. *There exists a numerical constant $C_{16} > 0$ such that $\zeta_K(s) \neq 0$ in the region*

$$\sigma > 1 - \frac{C_{16}}{\log\big(|\Delta|(|t|+2)^n\big)}, \qquad -\infty < t < +\infty, \tag{3.14}$$

except the hypothetical real simple zero β_1 of $\zeta_K(s)$.

Proof. We shall first prove that $\zeta_K(s) \neq 0$ for any $A > 0$ in the region

$$\sigma > 1 - \frac{C_{17}}{\log\big(|\Delta|\,(t+2)^n\big)}, \qquad t \geqslant \frac{A}{\log(|\Delta|2^n)}, \tag{3.15}$$

where C_{17} depends on A.

Let $\varrho = \beta+i\gamma$ be a zero of $\zeta_K(s)$ such that $\beta \geqslant \frac{1}{2}$, $\gamma \geqslant \dfrac{A}{\log(|\Delta|2^n)}$. Choose

$$\sigma_0 = 1 + \frac{C_{18}}{\log\big(|\Delta|\,(2\gamma+3)^n\big)},$$

where C_{18} is a sufficiently small constant depending on A and

$$s_0 = \sigma_0 + \gamma i, \qquad s_0' = \sigma_0 + 2\gamma i, \qquad |\Delta|(2\gamma+3)^n = T.$$

From Lemmas 2 and 3 it follows that there exists a constant $M_0 > 0$ such that

$$|(s-1)\zeta_K(s)| \leqslant \big(|\Delta|(|t|+2)^n\big)^{M_0}, \tag{3.16}$$

in the strip $0 \leqslant \sigma \leqslant 3$ and

$$\frac{1}{|\zeta_K(s)|} < \frac{(|\Delta|2^n)^{M_0}}{\sigma-1} \tag{3.17}$$

for $1 < \sigma \leqslant 3$.

Hence we get in the circles $|s-s_0|\leqslant 1$, $|s-s_0'|\leqslant 1$, $|s-\sigma_0|\leqslant 1$, the estimates

$$\left|\frac{\zeta_K(s)\,(s-1)}{\zeta_K(s_0)\,(s_0-1)}\right| \leqslant \frac{T^{M_0}(|\Delta|2^n)^{M_0}}{(\sigma_0-1)^2} \leqslant \frac{T^{3M_0}}{C_{18}^2},$$

$$\left|\frac{\zeta_K(s)\,(s-1)}{\zeta_K(s_0')\,(s_0'-1)}\right| \leqslant \frac{T^{3M_0}}{C_{18}^2}, \qquad \left|\frac{\zeta_K(s)\,(s-1)}{\zeta_K(\sigma_0)\,(\sigma_0-1)}\right| \leqslant \frac{T^{3M_0}}{C_{18}^2}.$$

We apply Lemma 5 to the function $(s-1)\zeta_K(s)$ in the circle $|s-s_0|\leqslant 1$.

If $\beta > \sigma_0 - 1/2$, then

$$-\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(s_0) < \operatorname{Re}\frac{1}{s_0-1} + 4\log\frac{T^{3M_0}}{C_{18}^2} - \frac{1}{\sigma_0-\beta}.$$

But

$$\operatorname{Re}\frac{1}{s_0-1} \leqslant \frac{1}{|s_0-1|} \leqslant \frac{1}{\gamma} \leqslant \frac{\log(|\Delta|2^n)}{A}.$$

Hence

$$-\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(s_0) < 12\left(1+\frac{1}{12A}\right)M_0\log T + 8\log C_{18}^{-1} - \frac{1}{\sigma_0-\beta}.$$

Similarly, from Lemma 5 for the circles $|s-s_0'|\leqslant 1$ and $|s-\sigma_0|\leqslant 1$ we get

$$-\operatorname{Re}\frac{\zeta_K'}{\zeta_K}(s_0') < 12\left(1+\frac{1}{24A}\right)M_0\log T + 8\log C_{18}^{-1},$$

$$-\frac{\zeta_K'}{\zeta_K}(\sigma_0) < \frac{1}{\sigma_0-1} + 12M_0\log T + 8\log C_{18}^{-1}.$$

Since

$$-\operatorname{Re}\left(3\,\frac{\zeta_K'}{\zeta_K}(\sigma_0) + 4\frac{\zeta_K'}{\zeta_K}(s_0) + \frac{\zeta_K'}{\zeta_K}(s_0')\right) \geqslant 0,$$

we have

$$\frac{3}{\sigma_0-1} + 36M_0\log T + 24\log C_{18}^{-1} + 48M_0\left(1+\frac{1}{12A}\right)\log T +$$
$$+ 32\log C_{18}^{-1} - \frac{4}{\sigma_0-\beta} + 12\left(1+\frac{1}{24A}\right)M_0\log T + 8\log C_{18}^{-1} \geqslant 0.$$

Therefore

$$\sigma_0-\beta \geqslant \frac{4(\sigma_0-1)}{3+96\left(1+\frac{1}{12A}\right)M_0C_{18} + 64C_{18}\log C_{18}^{-1}}$$

and

$$1-\beta \geqslant \left(\frac{4}{3+96\left(1+\frac{1}{12A}\right)M_0C_{18} + 64C_{18}\log C_{18}^{-1}} - 1\right)(\sigma_0-1).$$

Since $\lim_{C_{18}\to 0} C_{18}\log C_{18}^{-1} = 0$, it follows that for sufficiently small C_{18}

$$\beta < 1 - \frac{C_{18}}{7\log(|\Delta|(2\gamma+3)^n)}. \tag{3.18}$$

We therefore draw the conclusion that each zero

$$\varrho = \beta+\gamma i, \quad \beta > \sigma_0 - \tfrac{1}{2}, \quad \gamma \geqslant \frac{A}{\log(|\Delta|2^n)}$$

satisfies inequality (3.18).

Hence $\zeta_K(s) \neq 0$ in the region (3.15).

In the next step we shall prove that $\zeta_K(s) \neq 0$ for sufficiently small B and C_{19} in the region

$$\sigma > 1 - \frac{C_{19}}{\log(|\Delta|3^n)}, \quad 0 < t \leqslant \frac{B}{\log(|\Delta|3^n)}. \tag{3.19}$$

Put

$$\sigma_0 = 1 + \frac{C_{20}}{\log(|\Delta|3^n)} = 1 + \frac{C_{20}}{\log T_1}$$

where C_{20} is sufficiently small.

Suppose

$$B \leqslant \tfrac{1}{2}C_{20} \leqslant \tfrac{1}{4}\log 2. \tag{3.20}$$

Suppose further that $\varrho_1 = \beta + \gamma_1 i$ is a zero of $\zeta_K(s)$ such that

$$\beta > \sigma_0 - \tfrac{1}{4}, \quad 0 < \gamma_1 \leqslant \frac{B}{\log(|\Delta|3^n)} \leqslant \tfrac{1}{4}.$$

From (3.16), (3.17) we get in the circle $|s-\sigma_0|\leqslant 1$ the estimate

$$\left|\frac{\zeta_K(s)\,(s-1)}{\zeta_K(\sigma_0)\,(\sigma_0-1)}\right| < \frac{T^{3M_0}}{C_{20}^2}.$$

Since ϱ_1 is contained in the circle $|s-\sigma_0|\leqslant 1/2$ and $\operatorname{Im}\varrho_1 > 0$, it follows that there exists in this circle another zero of $\zeta_K(s)$, namely $\varrho_2 = \bar{\varrho}_1 = \beta - i\gamma_1$.

Applying Lemma 5 to the function $(s-1)\zeta_K(s-1)$ in $|s-\sigma_0|\leqslant 1$, we get

$$\frac{\zeta_K'}{\zeta_K}(\sigma_0) + \frac{1}{\sigma_0-1} \geqslant -12M_0\log(|\Delta|3^n) - 8\log C_{20}^{-1} + \frac{1}{\sigma_0-\varrho_1} + \frac{1}{\sigma_0-\bar{\varrho}_1}.$$

Hence

$$(3.21)\qquad \frac{2(\sigma_0-\beta)}{(\sigma_0-\beta)^2+\gamma_1^2} \leqslant \frac{1}{\sigma_0-1}+12M_0\log(|\Delta|3^n)+8\log C_{20}^{-1}.$$

On the other hand, from (3.20) we have

$$(3.22)\qquad \gamma_1 \leqslant \frac{B}{\log(|\Delta|3^n)} = \frac{B(\sigma_0-1)}{C_{20}} \leqslant \tfrac{1}{2}(\sigma_0-\beta).$$

From inequalities (3.21), (3.22) we at once infer that

$$\frac{8}{5(\sigma_0-\beta)} < \frac{1}{\sigma_0-1}+12M_0\log(|\Delta|3^n)+8\log C_{20}^{-1}$$

and for a sufficiently small C_{20}

$$\beta < 1-\tfrac{1}{3}(\sigma_0-1).$$

Therefore (3.19) follows.

In the last step of the proof we shall show that there exists a numerical constant C_{21} such that the function $\zeta_K(s)$ has not more than one simple zero on the segment

$$(3.23)\qquad 1-\frac{C_{21}}{\log(|\Delta|3^n)} < s < 1.$$

Put

$$\sigma_0 = 1+\frac{C_{22}}{\log(|\Delta|3^n)} = 1+\frac{C_{22}}{\log T_1},$$

where C_{22} is a sufficiently small numerical constant.

Suppose that $\varrho_i = \beta_i$ $(i=1,2)$ are such zeros of $\zeta_K(s)$ that

$$(3.24)\qquad \beta_i \geqslant \sigma_0-\tfrac{1}{2}, \quad \beta_2 \leqslant \beta_1.$$

From (3.16), (3.17) we get in the circle $|s-\sigma_0| \leqslant 1$ the estimate

$$\left|\frac{\zeta_K(s)(s-1)}{\zeta_K(\sigma_0)(\sigma_0-1)}\right| < \frac{T_1^{3M_0}}{C_{22}^2}.$$

Hence owing to Lemma 5

$$\frac{\zeta_K'}{\zeta_K}(\sigma_0)+\frac{1}{\sigma_0-1} \geqslant -12M_0\log(|\Delta|3^n)-8\log C_{22}^{-1}+\frac{1}{\sigma_0-\beta_1}+\frac{1}{\sigma_0-\beta_2},$$

and owing to (3.24)

$$\frac{2}{\sigma_0-\beta_2} \leqslant \frac{1}{\sigma_0-1}+12M_0\log(|\Delta|3^n)+8\log C_{22}^{-1}.$$

Therefore, if C_{22} is sufficiently small

$$\beta_2 < 1-\frac{1}{2}\frac{C_{22}}{\log(|\Delta|3^n)},$$

and this means that on the segment (3.23) the function $\zeta_K(s)$ has not more than one simple zero.

From (3.15), (3.19) and (3.23) Lemma 7 follows.

LEMMA 8. *Suppose that K is a normal extension of the field Q of rational numbers. Then, for any $\varepsilon > 0$,*

$$(3.25)\qquad \zeta_K(s) \neq 0 \quad \textit{for} \quad s > 1-\frac{(C(\varepsilon))^n}{|\Delta|^{\varepsilon}}.$$

Remark. If K is a quadratic field, then Lemma 8 implies the well-known theorem of Siegel on exceptional real zeros of $L(s,\chi)$ for real characters $\chi \bmod k$.

Proof. Suppose $0<\varepsilon<1$. Suppose further that there exists a real zero β_1', $0<\beta_1' \leqslant \frac{1}{4}$ of $\zeta_K(s)$. Then the function

$$\frac{\zeta_K(s)}{s^{r_1+r_2-1}(s-\beta_1')}$$

is regular in the circle $|s-\beta_1'| \leqslant \beta_1'+\varepsilon/2$. Hence from Lemma 2 and the maximum-modulus theorem we get

$$\left|\frac{\zeta_K(s)}{s^{r_1+r_2-1}(s-\beta_1')}\right| \leqslant (C_{23}(\varepsilon))^n|\Delta|^{(1+\varepsilon)/2}$$

for $|s-\beta_1'| \leqslant \beta_1'+\varepsilon/2$.

If $s=0$, we get

$$\frac{|\Delta|^{1/2}}{\beta_1' 2^{r_1+r_2}\pi^{r_2}}\operatorname{res}_1\zeta_K(s) \leqslant (C_{23}(\varepsilon))^n|\Delta|^{(1+\varepsilon)/2}.$$

Hence

$$(3.26)\qquad \beta_1' \geqslant (C_{24}(\varepsilon))^n|\Delta|^{-\varepsilon/2}\operatorname{res}_1\zeta_K(s).$$

Since the zeros of $\zeta_K(s)$ either lie on $\sigma=\frac{1}{2}$ or occur in pairs symmetrical about this line, from Lemma 4 and from (3.26) we get Lemma 8.

LEMMA 9. *Write*

$$E_1 = \begin{cases} 1 & \textit{if there is a real zero in the region (3.14)}, \\ 0 & \textit{otherwise}. \end{cases}$$

If K is any algebraic number field, then there exist numerical constants C_{25} and C_{26} such that

$$(3.27)\qquad \Psi(x,K) = x-E_1\frac{x^{\beta_1}}{\beta_1}+O\big(x\exp(-C_{25}\,\omega(x,\Delta,n))\big)$$

for

$$1 \leqslant |\Delta| \leqslant \exp\left(C_{26}\frac{\log x}{\log_2^2 x}\right), \qquad x \geqslant e^e,$$

and the constant implied by the O-notation is numerical.

Proof. Let us deduce the following formula:

$$-\frac{\zeta_K'}{\zeta_K}(s) = \frac{1}{s-1} - \frac{E_1}{s-\beta_1} + O\big(\log(|\Delta|(|t|+2)^n)\big) \tag{3.28}$$

valid in the region

$$\sigma \geqslant 1 - \frac{C_{27}}{\log(|\Delta|(|t|+2)^n)}, \tag{3.29}$$

where C_{27} is a numerical constant and the constant in O depends on C_{27}.

Put $t_0 \geqslant 0$, $s_0 = \sigma_0 + it_0$,

$$1 + \frac{C_{27}}{\log(|\Delta|(t_0+2)^n)} \leqslant \sigma_0 \leqslant 2.$$

From (3.16) and (3.17) we get in the circle $|s-\sigma_0| \leqslant 1$ the inequality

$$\left|\frac{\zeta_K(s)\,(s-1)}{\zeta_K(\sigma_0)\,(\sigma_0-1)}\right| \leqslant \frac{(|\Delta|(t_0+2)^n)^{4M_0}}{C_{27}^2}.$$

Hence from Lemma 5

$$-\frac{\zeta_K'}{\zeta_K}(\sigma_0) \leqslant \frac{1}{\sigma_0-1} + 16M_0\log(|\Delta|(t_0+2)^n) + 8\log C_{27}^{-1}$$
$$\leqslant \frac{M_1\log(|\Delta|(t_0+2)^n)}{C_{27}}.$$

Therefore we get

$$\left|\frac{\zeta_K'}{\zeta_K}(\sigma_0)\right| \leqslant \frac{M_1\log(|\Delta|(t_0+2)^n)}{C_{27}}. \tag{3.30}$$

Write

$$f(s) = \zeta_K(s)\,(s-1)\,(s-\beta_1)^{-E_1}.$$

From (3.30) we get

$$\left|\frac{f'}{f}(s_0)\right| = \left|\frac{\zeta_K'}{\zeta_K}(s_0) + \frac{1}{s_0-1} - \frac{E_1}{s_0-\beta_1}\right| \tag{3.31}$$
$$\leqslant \left|\frac{\zeta_K'}{\zeta_K}(\sigma_0)\right| + \frac{2}{\sigma_0-1} \leqslant \frac{M_2\log(|\Delta|(t_0+2)^n)}{C_{27}}.$$

Therefore (3.28) is valid for $\sigma \geqslant 1 + \dfrac{C_{27}}{\log(|\Delta|(|t|+2)^n)}$.

Suppose $t_0 \geqslant 0$, $s_0 = \sigma_0 + it_0$, $\sigma_0 = 1 + \dfrac{C_{27}}{\log(|\Delta|(t_0+2)^n)}$. In the circle $|s-s_0| \leqslant 1$ we have the inequality

$$\left|\frac{f(s)}{f(s_0)}\right| \leqslant \frac{(|\Delta|(t_0+2)^n)^{M_3}}{C_{27}^2}. \tag{3.32}$$

Applying Lemma 5 to the function $f(s)$ with $r = 1$,

$$r_1 = \frac{2C_{27}}{\log(|\Delta|(t_0+2)^n)} < \tfrac{1}{4}, \qquad C_{27} < \frac{C_{16}}{6}$$

and using (3.31), (3.32), we get

$$\left|\frac{f'}{f}(s)\right| \leqslant M_4\log(|\Delta|(t_0+2)^n)$$

in the circle $|s-s_0| \leqslant r_1$.

Hence

$$-\frac{\zeta_K'}{\zeta_K}(s) = \frac{1}{s-1} - \frac{E_1}{s-\beta_1} + O\big(\log(\Delta(|t|+2)^n)\big) \tag{3.33}$$

in the strip

$$1 - \frac{C_{27}}{\log(|\Delta|(|t|+2)^n)} \leqslant \sigma \leqslant 1 + \frac{3C_{27}}{\log(|\Delta|(|t|+2)^n)}.$$

From (3.31) and (3.33) follows (3.28).

In order to get (3.27) we shall estimate the function $\Psi_1(x, K)$. From the well-known formula

$$\frac{1}{2\pi i}\int_{C-\infty i}^{C+\infty i}\frac{y^{s-1}}{s(s+1)}\,ds = \begin{cases}0, & 0 < y \leqslant 1,\\ 1-1/y, & y \geqslant 1\end{cases} \tag{3.34}$$

for $C > 1$, $y > 0$, we get for $x \geqslant 1$

$$\frac{\Psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{C-\infty i}^{C+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta_K'}{\zeta_K}(s)\right)ds.$$

Hence

$$\frac{\Psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{C-\infty i}^{C+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta_K'}{\zeta_K}(s) - \frac{1}{s-1} + \frac{E_1}{s-\beta_1}\right)ds + \tag{3.35}$$

$$+\frac{1}{2\pi i}\int_{C-\infty i}^{C+\infty i}\left(\frac{1}{s-1}-\frac{E_1}{s-\beta_1}\right)\frac{x^{s-1}}{s(s+1)}\,ds$$

$$=\frac{1}{2}-E_1\frac{x^{\beta_1-1}}{\beta_1(\beta_1+1)}+$$

$$+\frac{1}{2\pi i}\int_{C-\infty i}^{C+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta_K'}{\zeta_K}(s)-\frac{1}{s-1}+\frac{E_1}{s-\beta_1}\right)ds+O(x^{-1}).$$

We shall estimate the integral

$$\frac{1}{2\pi i}\int_{C-\infty i}^{C+\infty i}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta_K'}{\zeta_K}(s)-\frac{1}{s-1}+\frac{E_1}{s-\beta_1}\right)ds$$

by the use of (3.28).

We define the contour L consisting of the following parts:

L_1: $s=C+it$, $-T\leqslant t\leqslant T$,

L_2: $s=\sigma+iT$, $1-\dfrac{C_{27}}{\log\big(|\Delta|(T+2)^n\big)}\leqslant\sigma\leqslant C$ (see (3.29)),

L_3: $s=1-\dfrac{C_{27}}{\log\big(|\Delta|(t+2)^n\big)}+it$, $0\leqslant t\leqslant T$

and of L_2', L_3' situated symmetrically to L_2, L_3.

Let

$$F(s)=-\frac{\zeta_K'}{\zeta_K}(s)-\frac{1}{s-1}+\frac{E_1}{s-\beta_1}.$$

From the Cauchy formula we get

$$\int_{L_1}\frac{x^{s-1}}{s(s+1)}F(s)\,ds=-\int_{L_2+L_3+L_2'+L_3'}\frac{x^{s-1}}{s(s+1)}F(s)\,ds. \tag{3.36}$$

From (3.28) we get

$$\int_{L_2}\frac{x^{s-1}}{s(s+1)}F(s)\,ds\xrightarrow[T\to\infty]{}0,$$

$$\left|\int_{L_3}\frac{x^{s-1}}{s(s+1)}F(s)\,ds\right|\leqslant C_{28}\int_0^T\frac{x^{-\frac{C_{27}}{\log(|\Delta|(t+2)^n)}}}{(2+t)^2}\log\big(|\Delta|(t+2)^n\big)dt \tag{3.37}$$

$$=C_{28}\int_1^{T+1}\frac{\exp\left(-\dfrac{C_{27}}{\log\big(|\Delta|(u+1)^n\big)}\log x-C_{27}\log(u+1)\right)}{(u+1)^{2-C_{27}}}\log\big(|\Delta|(u+1)^n\big)du.$$

Since

$$\min_{u\geqslant 1}\left(\frac{C_{27}}{\log\big(|\Delta|(u+1)^n\big)}\log x+C_{27}\log(u+1)\right)$$
$$\geqslant\frac{C_{29}\log x}{\max(n^{1/2}\log^{1/2}x,\log|\Delta|)}=C_{29}\,\omega(x,\Delta,n)$$

($n<C\log(2|\Delta|)$, see [2], p. 184), we have

$$\left|\int_{L_3}\frac{x^{s-1}}{s(s+1)}F(s)\,ds\right| \tag{3.38}$$
$$\leqslant C_{30}\exp\big(-C_{29}\,\omega(x,\Delta,n)\big)\int_1^\infty\frac{\log\big(|\Delta|(u+1)^n\big)}{(u+1)^{2-C_{27}}}du$$
$$\leqslant C_{31}\exp\big(-C_{29}\,\omega(x,\Delta,n)\big)\log(2|\Delta|).$$

Similarly

$$\left|\int_{L_3'}\frac{x^{s-1}}{s(s+1)}F(s)\,ds\right|\leqslant C_{31}\exp\big(-C_{29}\,\omega(x,\Delta,n)\big)\log(2|\Delta|) \tag{3.39}$$

and

$$\int_{L_2'}\frac{x^{s-1}}{s(s+1)}F(s)\,ds\xrightarrow[T\to\infty]{}0. \tag{3.40}$$

From (3.35)–(3.40) for $1\leqslant|\Delta|\leqslant\exp\left(C_{32}\dfrac{\log x}{\log_2^2 x}\right)$ (C_{32} is a sufficiently small numerical constant) we have

$$\frac{\Psi_1(x,K)}{x^2}=\frac{1}{2}-E_1\frac{x^{\beta_1-1}}{\beta_1(\beta_1+1)}+O\Big(\exp\big(-C_{33}\,\omega(x,\Delta,n)\big)\Big). \tag{3.41}$$

Suppose $x>C_{34}$, $h=h(x)$, $0<h<x/2$. Hence owing to (3.41) we have

$$\frac{\Psi_1(x\pm h,K)-\Psi_1(x,K)}{h} \tag{3.42}$$
$$=x\pm\frac{h}{2}-E_1\frac{x^{\beta_1}}{\beta_1}+O(hx^{\beta_1-1})+O\left(\frac{x^2}{h}\exp\big(-C_{33}\,\omega(x,\Delta,n)\big)\right).$$

From (3.42) and in view of the inequality

$$\frac{1}{h}\big(\Psi_1(x,K)-\Psi_1(x-h,K)\big)\leqslant\Psi(x,K)\leqslant\frac{1}{h}\big(\Psi_1(x+h,K)-\Psi_1(x,K)\big)$$

with $h=\dfrac{x}{2}\exp\left(-\dfrac{C_{33}}{2}\,\omega(x,\Delta,n)\right)$ Lemma 9 follows.

4. Proof of Lemmas A and B. We shall use the following formulae:

$$\sum_{m\leqslant x}\frac{G(m)}{\log m}=\int_{2}^{x}\frac{\Psi(t,K)}{t\log^2 t}\,dt+\frac{\Psi(x,K)}{\log x},\tag{4.1}$$

$$\sum_{m\leqslant x}\frac{G(m)}{\log x}=\Pi(x,K)+\tfrac{1}{2}\Pi(x^{1/2},K)+\tfrac{1}{3}\Pi(x^{1/3},K)+\ldots\tag{4.2}$$

$$=\Pi(x,K)+O\left(\frac{n}{\log x}x^{1/2}\right).$$

From Lemma 9 and Lemma 8 for $\varepsilon=1/4C_4$ it follows that

$$\Psi(x,K)=x+O\left(x\exp\left(-C_{25}\frac{\log^{1/2}x}{n^{1/2}}\right)\right)\tag{4.3}$$

for $1\leqslant|\Delta|\leqslant\log^{C_4}x$, $1\leqslant n\leqslant C_5\dfrac{\log_2 x}{\log_3 x}$.

Hence owing to (4.1)

$$\sum_{m\leqslant x}\frac{G(m)}{\log m}=\operatorname{li}x+O\left(x\exp\left(-C_{25}\frac{\log^{1/2}x}{n^{1/2}}\right)\right).\tag{4.4}$$

Combining (4.2) and (4.4), we get Lemma A. Similarly, the proof of Lemma B follows.

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Received on 13. 2. 1976
and in revised form on 18. 6. 1976 (813)

ACTA ARITHMETICA
XXXIV (1978)

On the zeros of the Riemann zeta-function and L-series

by

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1. Introduction. The object of this paper is to prove some theorems of which two typical ones are as follows.

THEOREM 1. *Let $\beta_0+i\gamma_0$ be any zero of $\zeta(s)$ with $\beta_0\geqslant\frac{1}{2}$ and $\gamma_0\geqslant 100$. With every positive constant λ not exceeding $\frac{1}{2}$ and every complex number $1+i\mu$ (μ real) let $D_\lambda(1+i\mu)$ denote the disc $|1+i\mu-s|\leqslant\lambda$. Then there exist effective positive absolute constants C_1, C_2, C_3, C_4 (depending only on λ) such that for all Y satisfying $C_1\log\log\gamma_0\leqslant Y\leqslant C_2(1-\beta_0)^{-1}$ there holds*

$$\sum_{\varrho}e^{-Y(1-\beta)}>C_3\big(Y(1-\beta_0)\big)^{-1}-C_4,$$

where $\varrho=\beta+i\gamma$ runs over all the zeros of $\zeta(s)$ which lie in $D_\lambda(1+i\gamma_0)$ and $D_\lambda(1+2i\gamma_0)$.

Remark. Since we let $C_1, C_2, \ldots, C_9$ depend on λ, in Theorems 1, 2 the upper bounds like $Y\leqslant C_2(1-\beta_0)^{-1}$ are plainly unnecessary. But we have retained them only for trivial reasons. However, we have good reasons to retain it in Theorem 3 which we state at the end of the paper.

THEOREM 2. *Let $\beta_1+i\gamma_1$ and $\beta_2+i\gamma_2$ be two zeros of $\zeta(s)$ with $\beta_1\geqslant\frac{1}{2}$, $\beta_2\geqslant\frac{1}{2}$, $100\leqslant\gamma_1<\gamma_2\leqslant 2\gamma_1$, $10^{-8}\leqslant\gamma_2-\gamma_1$ (10^{-8} is unimportant and can be replaced by smaller positive constant as well). Then there exist effective positive absolute constants C_5, C_6, C_7, C_8 and C_9 (depending only on λ) such that for all Y satisfying*

$$C_5\log\log\gamma_1\leqslant Y\leqslant C_6\min\big((1-\beta_1)^{-1},(1-\beta_2)^{-1}\big)$$

and all γ_1,γ_2 satisfying

$$\gamma_2-\gamma_1<\exp\big(C_7(\log\log\gamma_1)^{3/2}(\log\log\log\gamma_1)^{-2}\big)$$

there holds

$$\sum_{\varrho}e^{-Y(1-\beta)}>C_8Y^{-1}\min\big((1-\beta_1)^{-1},(1-\beta_2)^{-1}\big)-C_9,$$

where $\varrho=\beta+i\gamma$ runs over all the zeros of $\zeta(s)$ which lie in $D_\lambda(1+i\gamma_1)$ and $D_\lambda(1+i\gamma_2)$.