ω-models of second order arithmetic and admissible sets

by

W. Marek (Warszawa)

Abstract. It is proved that ω-model of second order arithmetic (without choice) is continuum of an admissible set. Problem of standard part of nonstandard models of various set theories is discussed.

This paper is devoted to ω-models of second order arithmetic and admissible sets and interconnections between them.

In § 0 we introduce the results and the notation, to be used in the sequel.

§ 1 is devoted to the interconnections and models of KP. We prove the following:

(a) Under interpretation by trees, the Δ_0-collection is provable in $\mathscr{A}_2^-$ (CA in Kreisel's notation). This fact extends former results by Srebrny and Marek [11] and Kreisel [9].

(b) If $\mathfrak{M}$ is an ω-model of $\mathscr{A}_2^-$, then there is an admissible set M such that $\mathfrak{M} = M \cap \wp(\omega)$.

This in turn leads us to another proof of the following result, extending that of Enderton [6]:

(c) If $\mathfrak{M}$ is a β-model of $\mathscr{A}_2^-$, then $\mathrm{Constr}^{\mathfrak{M}}$ is a β-model of $\mathscr{A}_2 + (X)$ Constr (X).

This in turn implies:

(d) The hard core of the transitive models of $Z^- + \Delta_0$-collection is L_{β_0} (where β_0 is the closure ordinal of the ramified analysis).

§ 2 is devoted to the discussion of various conservative extensions of $\mathscr{A}_2^-$. As pointed to us by G. Kreisel, the choice of such an extension is far from arbitrary. We propose a certain theory (in the language of set theory) which seems to correspond to the second order arithmetic $\mathscr{A}_2^-$. We then discuss various independence results.

§ 3 deals with the problem of the standard part of nonstandard models of various set theories. We are interested in models of the form L_α. Then very peculiar results are obtained;

(a) If L_α is a model of KP+V = HC, then L_α is not a standard part of a nonstandard admissible set.

Indeed, we find that: If $L_\alpha \models \mathrm{KP}$ and it is a standard part of a nonstandard admissible set, then L_α satisfies the Σ_2-collection and is power admissible.

These results naturally extend to relative constructibility, which leads to the following result:

(c) If $M \models \mathrm{KP}+\mathrm{V} = \mathrm{HC}$ and is a standard part of a nonstandard admissible set, then

$$M \models (x)(\mathrm{V} \neq L[x]) .$$

Unfortunately the most general problem, namely under what criterion an admissible set is a standard part of a nonstandard model of KP, remains unsolved.

We are grateful to Professor Mostowski, Krzysztof Apt, Marian Srebrny, Paweł Zbierski and other colleagues in Warsaw for discussions and advice.

§ 0. Preliminaries. There are two main groups of results to be used in the paper. The first group deals with models of second order arithmetic, the second with those of Kripke–Platek and other set theories.

I. Second order arithmetic. $\mathscr{A}_2^-$, second order arithmetic, is a two-sorted first-order theory based on a language with the function symbols $+, \cdot, S$, the relational symbol $<$ (for 0th sort objects) and the relational symbol $\in$. The axioms of $\mathscr{A}_2^-$ are: the Peano axioms for natural numbers (including the induction axiom as one sentence), the extensionality axiom and the full comprehension scheme.

$\mathscr{A}_2$ is $\mathscr{A}_2^-$ with the following scheme of choice:

$$(x)(\mathrm{E}Y)\Phi(x, Y)\rightarrow(\mathrm{E}Y)(x)\Phi(x, Y^{(x)}) \quad \text{(for all formulas } \Phi) .$$

If $\mathfrak{M} \subseteq \wp(\omega)$ then we say that $\mathfrak{M}$ is a *model* of $\mathscr{A}_2^-$ ($\mathscr{A}_2$) iff the structure $\langle\omega, \mathfrak{M}, \ldots\rangle$ is a model of $\mathscr{A}_2^-$ ($\mathscr{A}_2$).

$\mathfrak{M}$ is called a *β-model* iff $\mathfrak{M}$ preserves the notion of well-ordering. A β-model preserves all Σ_1^1 and Π_1^1 notions. If $\mathfrak{M}$ is a model, then an ordinal α is representable in $\mathfrak{M}$ iff there is an $X \in \mathfrak{M}$ such that $\overline{\{(x, y)\colon J(x, y) \in X\}} = \alpha$.

$$h(\mathfrak{M}) = \bigcup \{\alpha\colon \alpha \text{ is representable in } \mathfrak{M}\} .$$

ω_1^{CK} is the first non-recursive ordinal. Notice, that $h(\mathfrak{M}) \geqslant \omega_1^{\mathrm{CK}}$. Finally let us note that although the scheme of choice is unprovable in $\mathscr{A}_2^-$, we still have some partial inferences, namely the $\mathscr{A}_2^- \vdash \Pi_1^1$-scheme of choice (and thus the $\mathscr{A}_2^- \vdash \Sigma_2^1$-scheme of choice). This fact results from the following observation: the Kondo–Addison uniformization theorem is provable in $\mathscr{A}_2^-$.

II. Set theories. KP is the following theory in the language of set theory: extensionality, pairing, sum, infinity, Δ_0-comprehension, Δ_0-collection and full foundation scheme. KP^C is KP plus the following scheme of choice:

$$(*)_{\Delta_0} \qquad (x)_a(\mathrm{E}y)\varphi(x, y)\rightarrow(\mathrm{E}f)(\mathrm{func}(f)\ \&\ \mathrm{D}f = a\ \&\ (x)_a\varphi(x, fx))$$

(for all Δ_0-formulas φ).

Z^- is the following theory: Extensionality, pairing, sum, infinity, full comprehension scheme and foundation scheme. Z is Z^- plus the power set axiom. ZFC^- is KP plus the scheme $(*)$ for all formulas. Transitive models of KP are called *admissible sets*. Non-wellfounded models of KP are called *nonstandard admissible sets*.

If $\mathfrak{M} = \langle M, E\rangle$ is a structure, then x is called *standard* in $\mathfrak{M}$ iff

$$\neg(\mathrm{E}\underline{x})(n)(x_{n+1}\mathrm{E}x_n\ \&\ x = x_0) ,$$

otherwise x is called *nonstandard.*

If $\mathfrak{M}$ is a model of KP, then $\mathrm{Sp}\,\mathfrak{M}$ is a structure $\langle \mathrm{Sp}\,M, E \restriction \mathrm{Sp}\,M\rangle$ where $\mathrm{Sp}\,M$ is the collection of all standard elements of $\mathfrak{M}$. The system $\mathrm{Sp}\,\mathfrak{M}$ is transitive in $\mathfrak{M}$ (i.e., $\mathfrak{M}$ is an end extension of $\mathrm{Sp}\,\mathfrak{M}$). Unless $\mathfrak{M} = \mathrm{Sp}\,\mathfrak{M}$, $\mathrm{Sp}\,\mathfrak{M}$ is not definable in $\mathfrak{M}$. $\mathfrak{M}$ is also a rank extension of $\mathrm{Sp}\,\mathfrak{M}$, i.e., the rank of x, $\varrho(x)$, is standard iff x is standard. The following result is attributed to several persons. The proof may be found in Friedman [7]:

Proposition 1. *If $\mathfrak{M} \models \mathrm{KP}$ and $\mathfrak{M}$ is an ω-model, then $\mathrm{Sp}\,\mathfrak{M} \models \mathrm{KP}$.*

An admissible set is power-admissible iff it satisfies collection and comprehension for a wider class of formulas, namely those Δ_0 in the graph of power-set operation.

Let us note that the power-set of a standard element — if it exists — is also standard.

The following important result is due to Friedman [7]:

Proposition 2. *Every countable power-admissible set is of the form $\mathrm{Sp}\,\mathfrak{M}$ for some nonstandard power-admissible $\mathfrak{M}$. Conversely, $\mathrm{Sp}\,\mathfrak{M}$ is power-admissible whenever $\mathfrak{M}$ is.*

The formula $\mathrm{V} = \mathrm{HC}$ denotes the following sentence:

$$(x)(\mathrm{E}f)(f \text{ injects } x \text{ into } \omega) .$$

It is equivalent to the following model-theoretic form: $(\mathrm{HC})^{\mathfrak{M}} = |\mathfrak{M}|$. Analogically $\mathrm{V} = \mathrm{H}_K$ is the formula:

$$(x)(\mathrm{E}f)(f \text{ injects } x \text{ into } K) .$$

III. Interconnections. $\mathrm{Constr}\,(\cdot)$ is Addison's Σ_2^1 formula such that

$$\wp(\omega) \models \mathrm{Constr}\,[X] \leftrightarrow X \in L \cap \wp(\omega)$$

and $\mathscr{L}(\cdot)$ is a Σ_1 formula such that $\mathscr{L}(x) \leftrightarrow x \in L$ (where L is the constructible universe).

L_α is the αth constructible level (i.e., $L_0 = \emptyset$, $L_{\alpha+1} = \mathrm{Def}\,L_\alpha$, $L_\lambda = \bigcup_{\mu<\lambda} L_\mu$ for a limit λ) α is called a *gap* iff $(L_{\alpha+1}-L_\alpha) \cap \wp(\omega) = \emptyset$ α is the *beginning of a gap* iff α is a gap but $(\beta)_\alpha((L_\alpha-L_\beta) \cap \wp(\omega) \neq \emptyset)$. The following facts connect gaps, transitive models of ZFC^- and $\mathscr{A}_2$:

Proposition 3 (Marek and Srebrny [10]). *α is the beginning of a gap iff $L_\alpha \models \mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$ iff $L_\alpha \cap \wp(\omega)$ is a β-model of A_2 of the height α.*

Finally let us note the following Basis Principle of Gandy, which is useful in the construction of patological models:

If a $\Sigma_1^{1,T}$ collection of reals $\mathfrak{A}$ is nonempty, then $\mathfrak{A}$ contains an element of a hyperdegree less then those of O^T.

§ 1. Trees and admissible sets.

DEFINITION. A *tree* is a partial function X from ω to ω such that:

1^0 There is exactly one element $\mathrm{MAX}_X \in RX - DX$ such that

$$(a)_{DX}(\mathrm{E}n)(X^{(n)}(a) = \mathrm{MAX}_X).$$

2^0 X is wellfounded, i.e., there is no sequence $\underset{\sim}{x} \in {}^{\omega}DX$ such that

$$(n)(X(x_{n+1}) = x_n).$$

3^0 X has no automorphism, i.e., there is no nonidentical

$$\varphi\colon DX \cup RX \xrightarrow[\text{onto}]{\text{1-1}} DX \cup RX$$

such that

$$(x, y)_{DX \cup RX}(X(x) = y \leftrightarrow X(\varphi(x)) = \varphi(y)).$$

This definition is formalizable by a Π_1^1-formula $\mathrm{Tr}(\cdot)$.

DEFINITION. Let X, Y be trees, $a \in DX \cup RX$;

(a) $X_a = X \upharpoonright \{x \in DX \cup RX\colon (\mathrm{E}n)(X^{(n)} = a)\} \cup \{a\}$;

(b) $\mathrm{AMAX}_X = X^{-1} * \{\mathrm{MAX}_X\}$;

(c) $X \mathrm{Eq}\, Y \leftrightarrow$ There is an isomorphism of X and Y;

(d) $X \mathrm{Eps}\, Y \leftrightarrow (\mathrm{E}a)_{\mathrm{AMAX}_Y}(X \mathrm{Eq}\, Ya)$.

As defined, both Eq and Eps are Σ_1^1.

Using the method of Barwise–Gandy–Moschovakis [4], we have.

LEMMA 1.1. *Both* Eq *and* Eps *are* Δ_1^1.

Proof. It is enough to show Eq to be Π_1^1 on trees.

Let $\Gamma_{X,Y}(U)$ be the following operator:

$$\begin{aligned}\Gamma_{X,Y}(U) = \{\langle x, y, z\rangle\colon & [(a)_{X^{-1}*\{x\}}(\mathrm{E}b)_{Y^{-1}*\{y\}}(\langle a, b, 0\rangle \in U) \\ & \& (b)_{Y^{-1}*\{y\}}(\mathrm{E}a)_{X^{-1}*\{x\}}(\langle a, b, 0\rangle \in U) \& z = 0] \vee \\ & \vee [((\mathrm{E}a)_{X^{-1}*\{x\}}(b)_{Y^{-1}*\{y\}}(\langle a, b, 1\rangle \in U) \vee \\ & \vee (\mathrm{E}b)_{Y^{-1}*\{y\}}(a)_{X^{-1}*\{x\}}(\langle a, b, 1\rangle \in U)) \& (z = 1]\}.\end{aligned}$$

It is easy to show that $\langle x, y, 0\rangle \in \Gamma_{X,Y}^{\infty} \leftrightarrow X_x \mathrm{Eq}\, Y_y$.

Thus $X \mathrm{Eq}\, Y \leftrightarrow \langle \mathrm{MAX}_X, \mathrm{MAX}_Y, 0\rangle \in \Gamma_{X,Y}^{\infty}$, which is Π_1^1. ∎

Alternatively one can prove the lemma by using the strong reflexivity of $\mathscr{A}_2^-$.

Define $\|x\|_X = \{\|y\|_X\colon X(y) = x\}$ and $\|X\| = \|\mathrm{MAX}_X\|_X$. We call $\|X\|$ the *realization* of X.

We interpret the language of set theory within that of second order arithmetic as follows:

$$\begin{aligned}(x \in y)^T &= x \mathrm{Eps}\, y, \\ (x = y)^T &= x \mathrm{Eq}\, y, \\ (\Phi | \Psi)^T &= \Phi^T | \Psi^T, \\ (\mathrm{E}x\Phi)^T &= (\mathrm{E}x)(\mathrm{Tr}(x) \& \Phi^T).\end{aligned}$$

LEMMA 1.2. *If* Φ *is a* Δ_0*-formula, then* Φ^T *is equivalent to both a* Π_1^1*-formula and a* Σ_1^1*-formula.*

Proof. By induction on the length of Φ, using the fact that both Eps and Eq are Δ_1^1. ∎

It is well known that while working in $\mathscr{A}_2$ we have a uniform procedure allowing us to find the form of Φ^T in the analytic hierarchy when Φ is in Levy's hierarchy. In the case of $\mathscr{A}_2^-$ (i.e., without the axiom of choice) there is no such general method.

One can ask what statements of the language of set theory are true under the interpretation $(\cdot)^T$.

The following results have been obtained by several investigators:

1) (Kreisel, Zbierski) $(\mathrm{ZFC}^-)^T$ is provable in $\mathscr{A}_2$;

2) (Zbierski) $(\mathrm{V} = \mathrm{HC})^T$ is provable in $\mathscr{A}_2^-$;

3) (Kreisel, Marek & Srebrny) $(\mathrm{Z}^-)^T$ is provable in $\mathscr{A}_2^-$.

Our purpose is the following addition to 2) and 3).

THEOREM 1.1. $(\Delta_0$*-collection*$)^T$ *is provable in* $\mathscr{A}_2^-$.

Proof. Assume $((x)_a(\mathrm{E}y)\Phi(x, y))^T$, i.e.,

$$(X)(X \mathrm{Eps} A \,\&\, \mathrm{Tr} X \to (\mathrm{E}Y)(\mathrm{Tr}(Y) \& \Phi^T) \text{ and } \mathrm{Tr} A.$$

Since $X \mathrm{Eps} A \,\&\, \mathrm{Tr} A \to \mathrm{Tr} X$, this reduces to

$$(X)(X \mathrm{Eps} A \to (\mathrm{E}Y)(\mathrm{Tr}(Y) \& \Phi^T)) \& \mathrm{Tr}(A).$$

But

$$X \mathrm{Eps} A \leftrightarrow (\mathrm{E}a)_{\mathrm{AMAX}_A}(X \mathrm{Eq} A_a).$$

Thus, by the substitutivity of the relation Eq we get

$$(a)_{\mathrm{AMAX}_A}(\mathrm{E}Y)(\mathrm{Tr}(Y) \& \Phi^T).$$

Using the Π_1^1-choice, we get

$$(\mathrm{E}Y)(a)_{\mathrm{AMAX}_A}(\mathrm{Tr}(Y^{(a)}) \& \Phi^T(A_a, Y^{(a)})).$$

Clearly we may assume that $DY = \mathrm{AMAX}_A$.

We now proceed as follows: We divide $\omega - \{0\}$ effectively into ω-parts. In the nth part we copy as an initial segment $Y^{(n)}$ (this needs no choice since natural numbers are wellordered). Then we glue the copies together as they are being produced and put 0 on the top of the constructed relation. There may be, however, one small difficulty. Namely, the function constructed may have an automorphism. It is clear, however, that subfunctions starting in almost maximal elements have no automorphisms. We now erase the superfluous parts in the relation, getting a tree. Thus if Z is the tree obtained as above we have

$$(a)_{\mathrm{AMAX}_A}(Y^{(a)}\mathrm{Eps}\,Z)\,.$$

Thus since Z is a tree we get

$$(a)_{\mathrm{AMAX}_A}(\mathrm{E}\,Y)(Y\mathrm{Eps}\,Z\,\&\,\Phi^T(A_a,\,Y)\,.$$

And so

$$(X)(X\mathrm{Eps}\,A\to(\mathrm{E}\,Y)(Y\mathrm{Eps}\,Z\,\&\,\Phi^T(X,\,Y))\,.$$

Finally we have

$$(\mathrm{E}Z)(\mathrm{Tr}(Z)\,\&\,(X)(X\mathrm{Eps}\,A\to\ldots))\,,$$

which gives the Δ_0-collection. ■

COROLLARY 1.1 (Model-theoretic version I). *If* $\mathfrak{M}\models\mathscr{A}_2^-$ *then*

$$\langle\mathrm{Tr}\,\mathfrak{M},\,\mathrm{Eps}^{\mathfrak{M}},\,\mathrm{Eq}^{\mathfrak{M}}\rangle\models Z^-+\Delta_0\text{-}coll.+\mathrm{V}=\mathrm{HC}\,.$$

(Note that we consider here models for the equality relation).

COROLLARY 1.2 (Model-theoretic version II). *If* $\mathfrak{M}\models\mathscr{A}_2^-$ *then*

$$\langle\mathrm{Tr}^{\mathfrak{M}},\,\mathrm{Eps}^{\mathfrak{M}}\rangle/_{\mathrm{Eq}^{\mathfrak{M}}}\models Z^-+\Delta_0\text{-}coll.+\mathrm{V}=\mathrm{HC}\,.$$

In the case where $\mathfrak{M}$ is an ω-model, so is the model from Corollary 1.2. One easily proves that the continuum of the latter model is isomorphic to $\mathfrak{M}$. Using Proposition 1 from § 0. Preliminaries, we get

THEOREM 1.2. *If $\mathfrak{M}$ is an ω-model of $\mathscr{A}_2^-$, then there is an admissible set A such that*

$$(*)\qquad \mathfrak{M}=A\cap\wp(\omega)\,.$$

Moreover, this admissible set can be chosen so as to satisfy V = HC.

Under the latter assumption this admisible set is unique and consists exactly of the family of the realizations of trees in $\mathfrak{M}$. Finally, under this assumption,

$$A\cap On=h(\mathfrak{M})\,.$$

Warning. In this theorem we mean trees in $\mathfrak{M}$ and not sets satisfying in $\mathfrak{M}$ formula $\mathrm{Tr}(\cdot)$. One easily shows that, if X is a tree and $X\in\mathfrak{M}$, then $\mathfrak{M}\models\mathrm{Tr}[X]$.

However, if $(X)(\mathfrak{M}\models\mathrm{Tr}[X]\to\mathrm{Tr}(X))$, then $\mathfrak{M}$ is necessarily a β-model.

Proof of Theorem 1.2. Consider $\mathcal{N}=\langle\mathrm{Tr}^{\mathfrak{M}},\,\mathrm{Eps}^{\mathfrak{M}}\rangle/_{\mathrm{Eq}^{\mathfrak{M}}}$. By Corollary 2 it is an admissible set (possibly nonstandard). As we noticed, $\wp(\omega)^{\mathcal{N}}$ is isomorphic to $\mathfrak{M}$.

Using Proposition 1 of the preliminaries, we get $A=\mathrm{Sp}\,\mathcal{N}$, a transitive admissible set with a continuum isomorphic to $\mathfrak{M}$.

The uniqueness of A follows from the fact that if the tree X belongs to the admissible set A, then its realization $\|X\|$ also belongs to A. (Once more we point out that we mean that X is a tree and not only that it satisfies formula $\mathrm{Tr}(\cdot)$ in $\mathfrak{M}$).

However, if A is an admissible set satisfying V = HC, then every element of A is a realization of a tree in A. (An appropriate tree is obtained from any enumeration of $\mathrm{TC}(x)$.)

Finally, the last claim follows from a similar reasoning concerning wellorderings. ■

The unique model A from Theorem 1.2 is called $A^{\mathfrak{M}}$.

THEOREM 1.3. *There is a $\mathfrak{M}$ such that $A^{\mathfrak{M}}$ does not satisfy Σ_1 comprehension.*

We first prove a lemma interesting *per se*.

LEMMA 1.3. *If A is an admissible set satisfying Σ_1-comprehension, then L^A also satisfies Σ_1-comprehension.*

Proof. If $\varphi\in\Sigma_1$ then φ^L is also Σ_1.

Let $a\in L^A$. By Σ_1-comprehension there is γ such that witnesses for φ^L may be found in L_γ ($\gamma<A\cap On$). But then we may use Δ_0-comprehension (since the existential quantifier in φ^L may be bound by L_γ), which holds in L^A.

Proof of Theorem 1.3. Since $A\cap On=L^A\cap On$, it is enough to have $A^{\mathfrak{M}}$ of projectible height. By Gandy's basis theorem, since $(\cdot)$ is a code for an ω-model of $\mathscr{A}_2^-$ is Δ_1^1, we have a set X which is a code for an ω-model of $\mathscr{A}_2^-$ such that $\omega_1^X=\omega_1^{\mathrm{CK}}$. Taking $\mathfrak{M}=\{X^{(a)}\colon a\in\omega\}$, we get an $\mathfrak{M}$ such that $h(\mathfrak{M})=\omega_1^{\mathrm{CK}}$. Thus $A^{\mathfrak{M}}\cap On=\omega_1^{\mathrm{CK}}$. But $L_{\omega_1^{\mathrm{CK}}}$ does not satisfy Σ_1-comprehension. ■

Let us remark here that since $\omega_1^X=\omega_{\omega_1^{\mathrm{CK}}}$ we have $L_{\omega_1^X}[X]=L_{\omega_1^{\mathrm{CK}}}[X]$. On the other hand, $X\in L_{\omega_1^X}[X]$ and so $\mathfrak{M}\in L_{\omega_1^X}[X]$.

The fact that an admissible set may contain a structure (say, an admissible set) with the same height is well known. Here, however, let us point out one interesting fact. If N is a code for an ω-model of KP, then there is an injection of $L_{\omega_1^{\mathrm{CK}}}$ into N. We can find an injection which is recursively enumerable (partial recursive) but cannot be extended to a total recursive one. This follows from the fact that the existence of such a map implies the existence of the power set of ω in $L_{\omega_1^{\mathrm{CK}}}[M]$, which cannot happen. In particular, $L_{\omega_1^{\mathrm{CK}}}$ does not contain a code for an ω-model of KP (this also follows from the fact that $L_{\omega_1^{\mathrm{CK}}}\cap\wp(\omega)=$ H. A.). Using the construction from the proof of Theorem 3 combined with the result of Sacks–Jensen–Friedman on the form of countable admissible ordinals, we get

THEOREM 1.4. *For every countable admissible ordinal α there is an admissible set A such that $A\cap On=\alpha$ and $A\cap\wp(\omega)$ is a model of $\mathscr{A}_2^-$ of the height α.*

Part of this result was previously proved by Ms. M. Dubiel.

We now pass to the case where $\mathfrak{M}$ is a β-model.

In this case $A^{\mathfrak{M}}$ is isomorphic to $\langle\mathrm{Tr}^{\mathfrak{M}},\,\mathrm{Eps}^{\mathfrak{M}}\rangle/_{\mathrm{Eq}^{\mathfrak{M}}}$ and therefore satisfies the whole of $Z^- + \Delta_0$-collection+V = HC. This follows from the fact that $\mathrm{Tr}(\cdot)$ is Π_1^1 and so is absolute w.r.t. β-models of $\mathscr{A}_2^-$.

The following lemma clarifies the connection between the formulas $\mathrm{Constr}(\cdot)$ and $\mathscr{L}(\cdot)$.

LEMMA 1.4. *If $\mathfrak{M}$ is a β-model of $\mathscr{A}_2^-$, then*

$$\mathfrak{M} \models \mathrm{Constr}[X] \leftrightarrow A^{\mathfrak{M}} \models \mathscr{L}[X].$$

Proof. To make the situation clear let us choose an appropriate form of the formulas $\mathrm{Constr}(\cdot)$ and $\mathscr{L}(\cdot)$.

$$\begin{aligned}\mathrm{Constr}(X) \leftrightarrow (\mathrm{E}Y)(\mathrm{E}Z)(&Z \text{ is a well- ordering}\\ &\& \ Y \text{ is a sequence with domain } \mathrm{Fld}\,Z\\ &\& \ Y \text{ is a sequence constructor}\\ &\& \text{ The last term of } Y \text{ codes } X),\end{aligned}$$

where Y is a sequence constructor means that Y is a sequence such that Y_x is a tree obtained from Y_z and Y_t by appropriate operations (as prescribed by Gödel's functions K and L).

$$\begin{aligned}\mathscr{L}(x) \leftrightarrow (\mathrm{E}f)(\mathrm{E}\alpha)(&\mathrm{Ord}(\alpha) \,\&\, f \text{ is a sequence on } \alpha\\ &\& f \text{ is a sequence constructor}\\ &\& \text{ the last term of } f \text{ is } x).\end{aligned}$$

Now we prove $\to$ as follows. Since $\mathfrak{M}$ is a β-model, the formulas X is a tree and X is a wellordering are absolute. But if T is a tree (wellordering) and $T \in A^{\mathfrak{M}}$, then $\|T\|$ $(\overline{T})$ belongs to $A^{\mathfrak{M}}$ and also the realization function (similarity function) is in $A^{\mathfrak{M}}$.

Thus if Y and Z are an appropriate sequence and a wellordering (as required by the formula $\mathrm{Constr}(\cdot)$), then we define a sequence f and an ordinal α as follows: $\alpha = \overline{T}$;

$$\langle\beta, x\rangle \in f \leftrightarrow (\text{for a unique } s \text{ such that } \overline{Z \upharpoonright s} = \beta,\ \|y_s\| = x)$$

f and α make $\mathscr{L}(x)$ true.

To prove $\leftarrow$ it is enough to prove that an appropriate sequence constructor exists. So let α be an ordinal. Since $A^{\mathfrak{M}} \models \mathrm{V} = \mathrm{HC}$, there is a T such that $\overline{T} = \alpha$. It is enough to prove that there is a sequence constructor on $\mathrm{Fld}\,T$. Here is the point where we use the Σ_2^1-scheme of choice.

Clearly it is enough to consider the case where α is limit. Then, for all $x \in \mathrm{Fld}\,T$ there is a sequence constructor on $T \upharpoonright O_T(x)$. By the Σ_2^1-scheme of choice there is a function which to every x gives an appropriate sequence. We now produce an appropriate sequence on $\mathrm{Fld}\,T$. ■

Remark. The assumption that $\mathfrak{M}$ is a β-model is necessary, as is shown by the following example: Let $\mathfrak{M}$ be a β-model of $\mathscr{A}_2$. Let $\mathfrak{M}_1$ be an elementary extension of $\mathfrak{M}$ such that

1) $\mathfrak{M}_1$ is not a β-model,
2) $h(\mathfrak{M}) = h(\mathfrak{M}_1)$.

We may assume that $\mathfrak{M} \models (X)$ Constr. Then also $\mathfrak{M}_1 \models (X)$ Constr. Consider now $A^{\mathfrak{M}}$ and $A^{\mathfrak{M}_1}$. $A^{\mathfrak{M}} \models (x)\mathscr{L}(x)$. Since $h(A^{\mathfrak{M}}) = h(A^{\mathfrak{M}_1})$, we have

$$L^{A^{\mathfrak{M}}} = L^{A^{\mathfrak{M}_1}}.$$

In fact $L^{A^{\mathfrak{M}}} = A^{\mathfrak{M}}$. Now

$$x \in A^{\mathfrak{M}_1} \to (A^{\mathfrak{M}_1} \models \mathscr{L}(x) \to x \in A^{\mathfrak{M}}).$$

Thus

$$X \in A^{\mathfrak{M}_1} \to (A^{\mathfrak{M}_1} \models \mathscr{L}(X) \to X \in \mathfrak{M}).$$

Thus, for $X \in \mathfrak{M}_1 - \mathfrak{M}$

$$\mathfrak{M}_1 \models \mathrm{Constr}[X] \,\&\, A^{\mathfrak{M}_1} \models \neg\mathscr{L}[X].$$

The existence of an extension of $\mathfrak{M}$ satisfying 1) and 2) was shown by Ms. M. Dubiel.

However, let us point that if $A^{\mathfrak{M}} \models \mathscr{L}[X]$, then necessarily $\mathfrak{M} \models \mathrm{Constr}[X]$.

LEMMA 1.5. *If $\mathfrak{M}$ is a β-model, then $h(\mathfrak{M})$ is a gap ordinal.*

Proof. Let $\alpha = h(\mathfrak{M})$. $A^{\mathfrak{M}}$ is admissible and $h(A^{\mathfrak{M}}) = \alpha$, and so we have

$$L^{A^{\mathfrak{M}}} = L_\alpha.$$

Thus if $X \in (L_{\alpha+1} - L_\alpha) \cap \wp(\omega)$, then X is definable over $A^{\mathfrak{M}}$. But $A^{\mathfrak{M}}$ models the whole of Z^- and so $X \in A^{\mathfrak{M}}$.

Thus $(L_{\alpha+1} - L_\alpha) \cap \wp(\omega) \subseteq A^{\mathfrak{M}}$. But, among the sets of natural numbers constructed in $L_{\alpha+1}$ there is an arithmetical copy of L_α, E_α (here we use the result of Boolos and Putnam [3]). Thus $E_\alpha \in A^{\mathfrak{M}}$ and since $A^{\mathfrak{M}}$ is admissible, the result of the contraction of E_α is also in $A^{\mathfrak{M}}$. Thus $L_\alpha \in A^{\mathfrak{M}}$ and so $h(A^{\mathfrak{M}}) > \alpha$, which contradicts the assumption. ■

Using Proposition 3 of § 0. Preliminaries, we get:

THEOREM 1.5 (Enderton). *If $\mathfrak{M}$ is a β-model of $\mathscr{A}_2^-$, then* $\mathrm{Constr}^{\mathfrak{M}}$ *is a β-model of $\mathscr{A}_2 + (X)\mathrm{Constr}\,X$.*

Proof. Let $\alpha = h(\mathfrak{M})$. By the above lemma

$$\mathrm{Constr}^{\mathfrak{M}} = L^{A^{\mathfrak{M}}} \cap \wp(\omega) = L_\alpha \cap \wp(\omega).$$

But, since α is a gap, there is a beginning of a gap β, $\beta \leqslant \alpha$, such that

$$L_\alpha \cap \wp(\omega) = L_\beta \cap \wp(\omega).$$

But $L_\beta \cap \wp(\omega)$ is a β-model of $\mathscr{A}_2 + (X)\mathrm{Constr}(X)$. ■

COROLLARY 1.3. *If $\mathfrak{M}$ is a β-model of $\mathscr{A}_2^-$, then there is a β-model of $\mathscr{A}_2$, $\mathfrak{N}$, such that $\mathfrak{N} \subseteq \mathfrak{M}$.*

COROLLARY 1.4. *If A is a transitive model of the $Z^- + \Delta_0$-collection, then there is a transitive model B of* ZFC$^-$, $B \subseteq A$.

Proof. If A is a transitive model of $Z^- + \Delta_0$-coll., then $A \cap \wp(\omega)$ is a β-model of $\mathscr{A}_2^-$. (Here the Σ_1-comprehension is used to show that $A \cap \wp(\omega)$ is a β-model!). Consider now $\mathrm{Constr}^{A\cap\wp(\omega)}$. It is a β-model of $\mathscr{A}_2$. $A^{\mathrm{Constr}^{A\cap\wp(\omega)}} \subseteq A$. But $A^{\mathrm{Constr}^{A\cap\wp(\omega)}}$ models ZFC$^-$. ■

Corollary 1.5. *Under the assumptions of Corollary 2 if* $h(L^A \cap \wp(\omega)) = h(A)$, *then B may be chosen of the same height as A. If* $h(L^A \cap \wp(\omega)) < h(A)$, *then B may be found in A.*

Proof $A^{\mathrm{Constr}^{A\cap\wp(\omega)}} = L_\alpha$ for some α.

If $\alpha = h(A)$ then the first part is true.

If $\alpha < h(A)$ then $L_\alpha \in A$ and the second part holds. ■

Let us point out that the analysis of the proof shows that Constr is an inner interpretation of $\mathscr{A}_2$ in $\mathscr{A}_2^-$.

Let us also remark that, as pointed out by W. Powell, the fact that $\mathrm{Constr}^{\mathfrak{M}}$ is a β-model does not imply that $\mathfrak{M}$ is a β-model. The proof may be found in Apt and Marek [2].

It follows from the results of Kreisel [9] and Zbierski [12] that, if $\mathfrak{M}$ is a β-model of $\mathscr{A}_2$, then $h(\mathfrak{M})$ is Σ_n-admissible for all n.

The situation drastically changes in the case of $\mathscr{A}_2^-$. For instance, consider as $\mathfrak{M}$ a continuum of a Levy model N in which $\omega_1 = \omega_\omega^L$ (as presented in Cohen [5]). It is easy to calculate $h(\mathfrak{M}) = \omega_1^N = \omega_\omega^{LN}$. The latter ordinal is not Σ_2 admissible. In this case $\mathrm{Constr}^{\mathfrak{M}}$ is codable within $\mathfrak{M}$ and $h(\mathrm{Constr}^{\mathfrak{M}}) = \omega_1^{LN}$.

We may add to our list of corollaries the following:

Corollary 1.6. *The hard core of transitive models of $Z^- + \Delta_0$-collection is exactly L_{β_0}, the least transitive model of* ZFC$^-$ *(where β_0 is the height (and at the same time the closure ordinal) of ramified analysis).*

Let us finally note that there is an extension of Theorem 2 for models of some fragments of analysis. Since Δ_2^1-comprehension is sufficient to prove the Σ_2^1-scheme of choice (one has to check what is exactly necessary for Kondo–Addison), an appropriate version of Theorem 2 holds for models of the Δ_2^1-comprehension axiom.

The results of our Section 1 lead us to the following two directions:

(a) What does a hard core for subsystems of $Z^- + \Delta_0$-coll. look like?

(b) What do standard parts of admissible sets look like?

§ 2. Independence results.

Hard core problem.

Definition. The rank of the tree X is the *least ordinal* α such that there is a norm for X on α, i.e., there is an $f\colon DX \cup RX \xrightarrow{\text{onto}} \alpha$ such that $X(x) = y \to f(x) \in f(y)$.

This definition is formalizable within $\mathscr{A}_2^-$ with a wellordering instead of an ordinal.

Lemma 2.1. *If $\mathfrak{M}$ is an ω-model of $\mathscr{A}_2^-$, and $X \in \mathfrak{M}$ is a tree, then X has rank in $\mathfrak{M}$.*

Proof. Instead of directly using the Π_1^1-choice, work in $A^{\mathfrak{M}}$. If X is a tree, then $\|X\| \in A^{\mathfrak{M}}$ and $\mathrm{TC}(\|X\|)$ belongs to $A^{\mathfrak{M}}$. Then $\varphi(a) = \varrho(\|X_a\|)$ is the desired function. ■

Lemma 2.2. *If T is a wellordering, $T \in \mathfrak{M}$, $X \in \mathfrak{M}$, and X is a tree then*:

$$\mathfrak{M} \models T \textit{ is a rank of } X \leftrightarrow T \textit{ is a rank of } X.$$

Proof. By induction on the height of X. ■

Definition. The following formulas are called a *bounded collection scheme*:

$$(z)[(x)_a(\mathrm{E}y)(\Phi(x,y) \,\&\, y \subseteq z) \to (\mathrm{E}t)(x)_a(\mathrm{E}y)_t(y \subseteq z \,\&\, \Phi(x,y))].$$

The following is immediate:

Lemma 2.3. *$Z \vdash$ bounded collection.*

Proof. $\wp(z)$ serves as t. ■

Definition. (a) $\mathrm{Tr}^{<\alpha,\mathfrak{M}} = \{X\colon X \text{ is a tree of rank } \langle \alpha \,\&\, X \in \mathfrak{M}\}$.

(b) If $U \subseteq \mathrm{Tr}$ then $\overline{U} = \{\|X\|\colon X \in U\}$.

Theorem 2.1. *If $\mathfrak{M}$ is an ω-model of $\mathscr{A}_2^-$ and $\omega < \alpha < h(\mathfrak{M})$, for a limit α, then $\overline{\mathrm{Tr}^{<\alpha,\mathfrak{M}}}$ is a transitive model of Z^-+bounded Δ_0-coll.+V = HC+$\ulcorner$HF exists$\urcorner$.*

Proof. $\mathrm{Tr}^{<\alpha,\mathfrak{M}}$ is a definable subclass of $\mathfrak{M}$. Indeed, by Lemmas 1 and 2 we find that if $\overline{T} = \alpha$ then:

$$X \in \mathrm{Tr}^{<\alpha,\mathfrak{M}} \leftrightarrow \mathfrak{M} \models \mathrm{Tr}[X] \,\&\, \text{Rank of } X \text{ is less than } T.$$

Since T is a wellordering, if the rank of X is less than T, then T is really a tree. The proof of full comprehension follows as in the general case. Bounded the Δ_0-collection is proved in the same way as the Δ_0-collection was proved in § 1. V = HC is proved as before. HF is a realization of a recursive tree of rank ω. The transitivity of $\overline{\mathrm{Tr}^{<\alpha,\mathfrak{M}}}$ follows from the fact that $X \mathrm{Eps}\, Y \,\&\, \ulcorner$Rank of Y is $T\urcorner$ implies that, for some $s \in \mathrm{Fld}\, T$, Rank of X is $T \restriction S$. ■

Corollary 2.1. *Z^-+bounded Δ_0-coll.+V = HC+HF exist not $\vdash \Delta_0$-coll.*

Proof. $\mathrm{Tr}^{<\alpha,\mathfrak{M}}$ has height α. If the Δ_0-collection were provable in the above theory, then α would necessarily be admissible, which does not always happen. ■

In particular, Z^- not $\vdash \Delta_0$-coll.

We show that Z^- not $\vdash$ Bounded — Δ_0-coll.

Theorem 2.2. *There is a transitive model M of* Z^- +V = HC *in which bounded — Δ_0-collection is false. Moreover, M can be chosen so that* HF $\notin M$ *(Thus $M \models \neg$HF exists).*

Produce the model M as follows:

Let $A = \{a_i\}_{i\in\omega}$, $B = \{b_i\}_{i\in\omega}$ be two disjoint sets of individuals. We form $\mathrm{HF}(A \cup B)$, the family of hereditarity finite sets over $A \cup B$. Thus the elements of

$\mathrm{HF}(A \cup B)$ are built from a_i's and b_j's by pairings and unions. Now let $\mathfrak{M}$ be a denumerable model of $\mathscr{A}_2^-$, $\mathfrak{M} = \{X_n\}_{n\in\omega}$. For $t \in \mathrm{HF}(A \cup B)$ define

$$\|t\| = \begin{cases} n & \text{if} \quad t = a_n\,, \\ X_n & \text{if} \quad t = b_n\,, \\ \{\|S\|\colon S \in \mathrm{HF}(A \cup B)^t\} & \text{otherwise}\,. \end{cases}$$

Let $M = \{\|t\|\colon t \in \mathrm{HF}(A \cup B)\}$. M is transitive. M models Z^-. (This is shown by an appropriate coding of M in $\mathfrak{M}$ and by comprehension in $\mathfrak{M}$.) $M \models \mathrm{V} = \mathrm{HC}$, $\omega \in M$. But $\mathrm{HF} \notin M$ and bounded Δ_0-collection is false in M.

The first fact is obtained from the following result:

If $x \in M$, $\varrho(x) = \omega$, then there is a $y \in \mathrm{HF}$ and an $X \subseteq \omega$ such that $x = y \cup X$ (but HF is not of this form).

Similarly, for all n, $\omega - n \in M$ but $\{\omega - n\colon n \in \omega\} \notin M$. This implies that bounded Δ_0-collection is false in M.

THEOREM 2.3. Z^- *+bounded* Δ_0*-coll.* $\vdash$ *If every tree of rank* $\leqslant \alpha$ *has a realization, then every tree of rank* $< \alpha + \omega$ *has a realization.*

Proof. We proceed by induction; assume that every tree of rank $\leqslant \alpha + n$ has a realization. We show that trees of rank $\alpha + n + 1$ have a realization.

Let X be a tree of rank $\alpha + n + 1$. Form X' as follows:

$$X' = X - [(\{\mathrm{MAX}_X\} \times \mathrm{AMAX}_X) \cup \bigcup_{x \in \mathrm{AMAX}_X} (\{x\} \times \mathrm{AMAX}_{X_x})] \cup$$
$$\cup [\{\mathrm{MAX}_X\} \times \bigcup_{x \in \mathrm{AMAX}_X} \mathrm{AMAX}_{X_x}]\,.$$

We depict X' as follows:

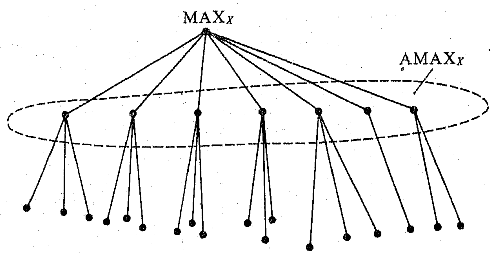

Fig. 1

Fig. 1: X' is formed from X by erasing AMAX_X and linking the elements immediately below AMAX_X directly with MAX_X. The rank of X' is $\alpha + n$ and $(x)\mathrm{AMAX}_X(\|X_x\| \subseteq \|X'\|)$. Now using bounded Δ_0-coll., we get the desired result.

COROLLARY 2.2. *If* M *is a transitive model of* Z^- *+bounded* Δ_0*-coll.+HF exists, then the realizations of all hyperarithmetic trees of rank less than* $\omega + \omega$ *are in* M.

Proof. Since HF belongs to M, trees of rank ω which are in M have a realization in M (since they are included in HF). But, in particular, $M \cap \wp(\omega) \models \mathscr{A}_2^-$ and so

$\mathrm{H.A.} \subseteq M$ (by the results of Grzegorczyk–Mostowski–Ryll-Nardzewski [8]). By Theorem 2.3 all trees of rank $< \omega + \omega$ in M have realizations in M. Thus all H. A. trees of rank less than $\omega + \omega$ have a realization in M. ■

Let H be the hard core of all transitive models of Z^-+bounded Δ_0-coll.+HF exists.

THEOREM 2.4. $L_{\omega_1^{\mathrm{CK}}} \cap R_{\omega+\omega} \subseteq H$.

Proof. $L_{\omega_1^{\mathrm{CK}}}$ consists exactly of realizations of H. A. trees. Indeed, $L_{\omega_1^{\mathrm{CK}}} \cap \wp(\omega) = \mathrm{H.A.}$ Since $L_{\omega_1^{\mathrm{CK}}}$ is admissible, therefore, if X is a tree, $X \in L_{\omega_1^{\mathrm{CK}}}$, then $\|X\| \in L_{\omega_1^{\mathrm{CK}}}$. On the other hand, if $x \in L_{\omega_1^{\mathrm{CK}}}$, then $\mathrm{TC}(x) \in L_{\omega_1^{\mathrm{CK}}}$ and, since the latter satisfies $\mathrm{V} = \mathrm{HC}$, $\mathrm{TC}(x)$ may be enumerated in $L_{\omega_1^{\mathrm{CK}}}$. From the enumeration it is easy to construct a tree X such that $\|X\| = x$. Clearly X belongs to $L_{\omega_1^{\mathrm{CK}}}$. ■

Similarly we show that:

If M is an admissible set, $M \models \mathrm{V} = \mathrm{HC}$, then $M = \overline{\mathrm{Tr} \cap M}$. Notice that Tr^M may be different from $\mathrm{Tr} \cap M$. In particular,

$$\mathrm{Tr} \cap L_{\omega_1^{\mathrm{CK}}} \subsetneq \mathrm{Tr}^{L_{\omega_1^{\mathrm{CK}}}}\,.$$

We finish this section with the following

Conjecture: $H = L_{\omega_1^{\mathrm{CK}}} \cap R_{\omega+\omega}$.

Let us note that the conjecture easily reduces to a certain omitting type problem for Borel sets.

§ 3. Standard parts of nonstandard admissible sets. We consider here the following problem:

Let A be an admissible set. Under what assumption is there a nonstandard admissible set $\mathfrak{M}$ such that $A = \mathrm{Sp}\,\mathfrak{M}$?

The only known positive result was obtained by Friedman [7], who proved that a power-admissible set is a standard part of a nonstandard admissible set. In the same paper he announced that $L_{\omega_1^{\mathrm{CK}}}$ is not a standard part of any nonstandard admissible set.

DEFINITION. (a) Adm is the *class of all admissible sets.*

(b) If $T \supseteq \mathrm{KP}$, we define U_T to be a *class of those admissible sets which are standard parts of nonstandard models of* T.

(c) $V_T = \mathrm{Adm} - U_T$.

DEFINITION. KP^C is KP plus the scheme of choice for Δ_0-formulas.

Let $\mathfrak{M} = \langle M, E\rangle$ be a model of KP. $\mathrm{HC}^{\mathfrak{M}}$ is a submodel of $\mathfrak{M}$ consisting of all hereditarily countable sets in $\mathfrak{M}$. Clearly $\mathrm{HC}^{\mathfrak{M}}$ is transitive in $\mathfrak{M}$.

THEOREM 3.1. *If* $\mathfrak{M} = \langle M, E\rangle \models \mathrm{KP}^C$, *then*

$$\langle \mathrm{HC}^{\mathfrak{M}}, E \restriction \mathrm{HC}^{\mathfrak{M}}\rangle \models \mathrm{KP}^C\,.$$

Proof. If $\mathfrak{M} \models \mathrm{V} = \mathrm{HC}$, then $\mathrm{HC}^{\mathfrak{M}} = M$ and there is nothing to prove. Since $\mathrm{HC}^{\mathfrak{M}}$ is transitive in M, Δ_0-formulas are absolute and so Δ_0-comprehension holds in $\mathrm{HC}^{\mathfrak{M}}$.

Similarly, since $HC^{\mathfrak{M}}$ is definable in $\mathfrak{M}$, the foundation scheme holds in $HC^{\mathfrak{M}}$. It is enough to prove the Δ_0-choice scheme in $HC^{\mathfrak{M}}$.

Let us note that Δ_0-choice implies Σ_1 choice. Assume now that

$$HC^{\mathfrak{M}} \models (x)_a(Ey)\varphi(x,y)\,.$$

Then also

$$\mathfrak{M} \models (x)_a(Ey)\varphi(x,y)$$

and finally

$$\mathfrak{M} \models (x)_a(Ey)\ (y \text{ is hereditarily countable } \&\ \varphi(x,y))\,.$$

But "y is hereditarily countable" is Σ_1 and so by Σ_1 choice we have a function f on a such that $\mathfrak{M} \models (x)_a\varphi(x,fx)$.

Since Df is denumerable and $f \subseteq HC^{\mathfrak{M}}$, we have $f \in HC^{\mathfrak{M}}$, which shows Δ_0-choice. ■

LEMMA 3.1. *If* $\mathfrak{M}$ *is a nonstandard admissible set, then* $\mathrm{Sp}\,\mathfrak{M}$ *is not definable in* $\mathfrak{M}$.

Proof. If φ defines $\mathrm{Sp}\,\mathfrak{M}$ in $\mathfrak{M}$, then by foundation there is a $y \subseteq \mathrm{Sp}\,\mathfrak{M}$ such that $\neg\varphi(y)$. But then $y \in \mathrm{Sp}\,\mathfrak{M}$ and $\neg\varphi(y)$. ■

COROLLARY. *If* M *is an admissible set,* $M \models V = HC$ $\mathfrak{N} = \langle N, E\rangle$ *is a nonstandard admissible set,*

$$\textit{if} \quad M = \mathrm{Sp}\,\mathfrak{N} \quad \textit{then} \quad M \neq HC^{\mathfrak{N}}\,.$$

THEOREM 3.2. *If* M *is an admissible set,* $\omega \in M$, $M \models V = HC$ *and* M *has* β*-property, then* $M \in V_{KPC}$.

Proof. Assume $M \in U_{KPC}$. Then there is a $\mathfrak{N}$ such that $M = \mathrm{Sp}^{\mathfrak{N}}$, $\mathfrak{N}$ nonstandard.

By Theorem 3.1 we may assume that $\mathfrak{N} \models V = HC$ since $M = \mathrm{Sp}\,HC^{\mathfrak{N}}$. Let ζ be a nonstandard ordinal in $\mathfrak{N}$. ζ is denumerable and let T be an ordering on ω such that $\mathfrak{N} \models T \sim \zeta$. Then $\mathfrak{N} \models \ulcorner T$ is a wellordering$\urcorner$ and, since M is smaller than $\mathfrak{N}$, $M \models \ulcorner T$ is a wellordering$\urcorner$, and so T is a wellordering, contradicting the fact that ζ is nonstandard.

LEMMA 3.2. *If* $L_\alpha \in U_{KP}$ *then* $L_\alpha \in U_{KPC}$.

Proof. Let $\mathfrak{N}$ be nonstandard admissible and such that $L_\alpha = \mathrm{Sp}\,\mathfrak{N}$. Consider $L^{\mathfrak{N}}$.

It is nonstandard since it has the same ordinals as $\mathfrak{N}$. The constructibility formula $\mathscr{L}(\cdot)$ is Σ_1, and so

$$\mathrm{Sp}\,\mathfrak{N} \models \mathscr{L}[x] \rightarrow \mathfrak{N} \models \mathscr{L}[x]\,.$$

But $L_\alpha \models (x)\mathscr{L}(x)$, and so $L_\alpha \subseteq L^{\mathfrak{N}}$. Thus L_α is $\mathrm{Sp}\,L^{\mathfrak{N}}$. (Because a smaller model cannot have more standard sets and $L^{\mathfrak{N}}$ is transitive in $\mathfrak{N}$.)

But $L^{\mathfrak{N}} \models KP + V = L$, and so $L^{\mathfrak{N}} \models KP^C$. ■

LEMMA 3.3. *If* M *is an admissible set,* $M \models V = HC$, *then* $M \models \ulcorner\wp(\omega)$ *does not exist*$\urcorner$.

Proof. Assume $M \models \ulcorner y$ is $\wp(\omega)\urcorner$. Then y is enumerated in M and we produce a diagonal set. ■

THEOREM 3.3. *If* L_α *is admissible,* $\omega \in \alpha$, $L_\alpha \models V = HC$, *then* $L_\alpha \in V_{KP}$.

Proof. Let $\mathfrak{N}$ be a nonstandard admissible set such that $L_\alpha = \mathrm{Sp}\,\mathfrak{N}$. Let ζ be a nonstandard ordinal. If $X \subseteq \omega$ and $X \in \mathfrak{N}$, then $X \in \mathrm{Sp}^{\mathfrak{N}}$ (i.e. $X \in L_\alpha$). Then

$$\wp(\omega)^{L_\alpha} = \wp(\omega)^{L_\zeta^{\mathfrak{N}}}\,.$$

But $\wp(\omega)^{L_\zeta^{\mathfrak{N}}} \in \mathfrak{N}$. Since $\wp(\omega)^{\mathfrak{N}} = \wp(\omega)^{L_\alpha}$ and the power-set of a standard set (if it exists) is standard, therefore $\wp(\omega)^{L_\alpha} \in L_\alpha$, which contradicts $V = HC$ (by Lemma 3.3). ■

COROLLARY 3.1 (Friedman). $L_{\omega_1^{CK}} \in V_{KP}$.

Proof. $L_{\omega_1^{CK}} \models V = HC$. ■

COROLLARY 3.2. *If* $\mathfrak{M} = \langle M, E\rangle$ *is a nonstandard admissible set, being an* ω*-model,* $\mathfrak{M} \models V = HC$, *then* $\mathrm{Sp}\,\mathfrak{M} \models V \neq L$.

Proof. Since $\omega \in \mathrm{Sp}\,\mathfrak{M}$, therefore $\mathrm{Sp}\,\mathfrak{M} \models V = HC$. If in addition $\mathrm{Sp}\,\mathfrak{M} \models V = L$, then there is an $\alpha > \omega$ such that $\mathrm{Sp}\,\mathfrak{M} = L_\alpha$, which contradicts Theorem 3.3. ■

Note that $\mathfrak{M}$ may satisfy $V = L$ and still $\mathrm{Sp}\,\mathfrak{M} \models V \neq L$; indeed, some standard elements will be constructed through nonstandard ordinals. Thus we have:

THEOREM 3.4. *If* $\mathfrak{M} = \langle M, E\rangle$ *is a nonstandard admissible set, which is an* ω*-model,* $\mathfrak{M} \models V = HC = L$, *then there is a subset* X *of* ω *such that*

$$\mathrm{Sp}(\mathfrak{M}) \models \neg\mathscr{L}[X] \quad \textit{and} \quad \mathfrak{M} \models \mathscr{L}[X]\,.$$

DEFINITION. Let $X \subseteq \omega$. X is called *pseudo constructible* in T ($PC^T(X)$) iff there is an ω-model $\mathfrak{M}$ of T such that X is constructible in $\mathfrak{M}$.

THEOREM 3.5 (Putnam). $PC^{KP} = \wp(\omega)$.

Proof. The statement $PC^{KP} = \wp(\omega)$ is a Π_2^1 statement true in L. ■

In the case of PC^{ZFC} an analogous theorem is true under the assumption that there is a standard model of ZFC with uncountable height. The latter assumption is provable in the Kelley–Morse theory of classes.

An ω-model of KP in which X is constructible may be found in $L_{\omega_1^X+1}[X]$.

After the above digression let us come back to the reasoning of Theorem 3.3 and corollary. The construction can be generalized in two directions:

(a) To relative constructibility.

(b) To models of $V = H_K$.

THEOREM 3.3*. *If* $\mathfrak{M} = \langle M, E\rangle$ *is a nonstandard admissible set which is an* ω*-model* $\mathfrak{M} \models V = HC$, *then* $\mathrm{Sp}\,\mathfrak{M} \models (x)(V \neq L[x])$.

THEOREM 3.3**. *If* $\mathfrak{M} = \langle M, E\rangle$ *is a nonstandard admissible set,* K *a standard ordinal in* $\mathfrak{M}$, $\mathfrak{M} \models V = H_K$, *then* $\mathrm{Sp}\,\mathfrak{M} \models V \neq L$.

COROLLARY 3.3 (Final version). *If* $\mathfrak{M} = \langle M, E\rangle$ *is a nonstandard admissible set,* K *a standard ordinal in* $\mathfrak{M}$, $\mathfrak{M} \models V = H_K$, *then*

$$\mathrm{Sp}\,\mathfrak{M} \models (x)(V \neq L[x])\,.$$

There are admissible sets A satisfying both $(x)\,(\mathrm{V} \neq \mathrm{L}[x])$ and $\mathrm{V} = \mathrm{HC}$. Does then $A \in U_{\mathrm{KP}}$ necessarily hold?

LEMMA 3.5. *If L_α is admissible, $L_\alpha \models (n)\aleph_n$ exists and $L_\alpha \models \aleph_\omega$ does not exist, then $L_\alpha \in \mathrm{V}_{\mathrm{KP}}$.*

Proof. Assume $L_\alpha \in U_{\mathrm{KP}}$. Let $\mathfrak{M}$ be a (nonstandard) extension of L_α such that $L_\alpha = \mathrm{Sp}\,\mathfrak{M}$. Then $\mathfrak{M}$ is a rank extension of L_α, i.e.:

If $x \in M$, $\varrho(x) \in \mathrm{Sp}\,\mathfrak{M}$, then $x \in \mathrm{Sp}\,\mathfrak{M}$. Thus the cardinals of L_α are cardinals in $\mathfrak{M}$. Thus $\mathfrak{M} \models (n)(\aleph_n$ exists). There are nonstandard ordinals in $\mathfrak{M}$. Pick one, say ζ. Then $\aleph_n^{L_\alpha} = \aleph_n^{L_\zeta^{\mathfrak{W}}}$. Thus $\{\aleph_n^{L_\zeta^{\mathfrak{W}}}\colon n \in \omega\} \in \mathfrak{M}$ and finally $\bigcup_{n\in\omega} \aleph_n^{L_\zeta^{\mathfrak{W}}} \in \mathfrak{M}$. But the latter ordinal is α. ■

Similarly we get:

THEOREM 3.6. *If L_α is an admissible set, $L_\alpha \models (\zeta)_\beta$ ($\aleph_\zeta$ exists) and $L_\alpha \models \aleph_\beta$ does not exist, then $L_\alpha \in \mathrm{V}_{\mathrm{KP}}$.*

COROLLARY 3.4. *If $L_\alpha \in U_{\mathrm{KP}}$, then L_α satisfies the power set axiom.*

COROLLARY 3.5. *If $L_\alpha \in U_{\mathrm{KP}}$, then L_α satisfies (ζ) ($\aleph_\zeta$ exists).*

COROLLARY 3.6. *If $L_\alpha \in U_{\mathrm{KP}}$, then L_α satisfies (ζ) (H_ζ exists).*

Proof. It is enough to prove that, for ζ being a cardinal of L_α, $L_\zeta = H_\zeta^{L_\alpha}$. This, however, is a standard reasoning. ■

COROLLARY 3.7. *If $L_\alpha \in U_{\mathrm{KP}}$, then L_α is recursively inaccessible and thus L_α has β-property.*

Proof. If ζ is a cardinal of L_α, then $L_\alpha \models L_\zeta$ is admissible and thus L_ζ is admissible: By Corollary 3.5 α is a limit of cardinals of L_α and so is a limit of admissibles. ■

LEMMA 3.6. *Assume $L_\alpha = \mathrm{Sp}\,\mathfrak{M}$, $\mathfrak{M}$ nonstandard, $\omega \in L_\alpha$, $\mathfrak{M} \models \mathrm{KP}^C$. Then $L_\alpha <_1 \mathfrak{M}$.*

Proof. Let $\mathfrak{M} \models (\mathrm{E}x)\,\varphi(x, x_1, \ldots, x_k)$, $\varphi \in \Delta_0$ when $x_1, \ldots, x_k \in L_\alpha$. Applying the Skolem–Löwenheim theorem within $\mathfrak{M}$ to the set $\mathrm{TC}(\{x, x_1, \ldots, x_k\})$, we get a subsystem of the latter set containing $H_\varkappa^{L_\alpha}$ (where $x_1, \ldots, x_n \in H_\varkappa^{L_\alpha}$) and of power $\varkappa$. Since this system is of power $\varkappa$ and wellfounded in L_α, its copy in $(\varkappa^+)^{L_\alpha}$ is wellfounded in L_α. Thus, by Corollary 3.7 it is wellfounded. Thus it has a realization in L_α and so we get a transitive system $\mathfrak{N} \in L_\alpha$,

$$\mathfrak{N} \models (\mathrm{E}x)\,\varphi(x, x_1, \ldots, x_k)\,.$$

Thus

$$L_\alpha \models (\mathrm{E}x)\,\varphi(x, x_1, \ldots, x_k).\ ■$$

THEOREM 3.7. *If $L_\alpha \in U_{\mathrm{KP}}$, then L_α is Σ_2-admissible and power admissible.*

Proof. Assume $L_\alpha = \mathrm{Sp}\,\mathfrak{M}$, $\mathfrak{M}$ nonstandard; we may — as before — assume $\mathfrak{M} \models \mathrm{V} = \mathrm{L}$. Let ζ be a nonstandard ordinal in $\mathfrak{M}$.

Then $L_\alpha <_1 L_\zeta^{\mathfrak{M}}$ (by Lemma 3.6). Assume now $L_\alpha \models (x)_a(\mathrm{E}y)\,\varphi$ where $\varphi \in \Pi_1$; then

$$\mathfrak{M} \models (x)_a(\mathrm{E}y)_{L_\zeta^{\mathfrak{M}}}\,\varphi\,.$$

We now modify φ and write instead $\tilde{\varphi}$: $\varphi \,\&\, y$ is of least possible rank. Then

$$\mathfrak{M} \models (x)_a(\mathrm{E}y)_{L_\zeta}(\tilde{\varphi})_{L_\zeta}\,.$$

But any y which makes $(\tilde{\varphi})^{L_\zeta}$ true is necessarily standard. Now, using the fact that $\mathfrak{M}$ is choice-admissible, we find the appropriate function f. But f has a standard domain and takes only standard values, and thus f is itself standard. Thus Rf is a standard set making Σ_2 collection true in L_α. ■

Power admissibility is shown by the same reasoning with the use, instead of the lemma, of the following fact: If L_α is $\mathrm{Sp}\,\mathfrak{M}$ then L_α preserves $\Delta_0(\wp)$ formulas.

But Friedman [7] shows that if L_α is power admissible, then $L_\alpha \in U_{\mathrm{KP}}$, and thus:

COROLLARY 3.8. *$L_\alpha \in U_{\mathrm{KP}}$ iff L_α is power admissible (for $\alpha \in \omega_1$).*

COROLLARY 3.9. *If L_α is power admissible, then L_α is Σ_2-admissible.*

The converse implication is obviously false — take β_0, as α.

Also for admissible sets of form other than L_α it is easy to show a set which is power admissible but not Σ_2-admissible. Indeed, Friedman [7] shows that for any countable admissible ordinal α there is a power admissible A such that $A \cap On = \alpha$. But for $\alpha = \omega_1^{\mathrm{CK}}$ no such A is Σ_2-admissible.

Applying the Gandy–Basis theorem together with the Sacks–Friedman–Jensen theorem on the form of countable admissible ordinals, we get:

THEOREM 3.8. *If $\alpha \in \omega_1$, α admissible, then there is an admissible set $A \in U_{\mathrm{KP+V=HC}}$ such that $A \cap On = \alpha$.*

This theorem does not follow from the results of Friedman [7] since he finds an $A \in U_{\mathrm{KP}}$ of the height α but one which is power admissible and so necessarily satisfying $\mathrm{V} \neq \mathrm{HC}$.

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Accepté par la Rédaction le 18. 8. 1975

A decidable $\aleph_0$-categorical theory with a non-recursive Ryll-Nardzewski function

by

James H. Schmerl * (Storrs, Conn.)

Abstract. According to the theorem of Ryll-Nardzewski, a complete first-order theory T is $\aleph_0$-categorical iff for each $n<\omega$, the set $S_n(T)$ of its n-types is finite. For such theories T, we call the function $n\to|S_n(T)|$ the Ryll-Nardzewski function of T. Waszkiewicz asked if the Ryll-Nardzewski function of a decidable theory is recursive. It is shown that this question has a negative answer. More specifically, for any Turing degree $\boldsymbol{b}$ there is a function $G\colon \omega\to\omega$ of degree $\boldsymbol{b}$ with the following property: whenever $\boldsymbol{a}$ is a degree such that $\boldsymbol{b}$ is recursively enumerable in $\boldsymbol{a}$, then there is a complete, $\aleph_0$-categorical theory of degree $\boldsymbol{a}$ whose Ryll-Nardzewski function is G.

According to the classic theorem of Ryll-Nardzewski [1], a complete first-order theory T is $\aleph_0$-categorical iff, for each $n<\omega$, the set $S_n(T)$ of its n-types is finite. For such theories T let us denote the function $n\to|S_n(T)|$ by R_T, which we call the Ryll-Nardzewski function of the theory T. The following question was posed by Waszkiewicz in [2]: Is the Ryll-Nardzewski function of a decidable $\aleph_0$-categorical theory always recursive? It is the purpose of this note to give a negative answer to this question (1). More generally, we consider a relativized version of Waszkiewicz's question: If the Turing degree of T is $\boldsymbol{a}$, then what are the possible Turing degrees $\boldsymbol{b}$ of R_T? It is a straightforward matter to show that $\boldsymbol{b}$ must be recursively enumerable in $\boldsymbol{a}$. Our theorem shows that this is the only restriction. What is perhaps more surprising is that the Ryll-Nardzewski function of degree $\boldsymbol{b}$ can be chosen independently of $\boldsymbol{a}$.

THEOREM. *For any Turing degree* $\boldsymbol{b}$ *there is a function* $G\colon \omega\to\omega$ *of degree* $\boldsymbol{b}$ *with the following property: whenever* $\boldsymbol{a}$ *is a degree such that* $\boldsymbol{b}$ *is recursively enumerable in* $\boldsymbol{a}$, *then there is a complete* $\aleph_0$-*categorical theory of degree* $\boldsymbol{a}$ *in a language consisting of one binary relation symbol whose Ryll-Nardzewski function is* G.

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* Research partially supported by NSF Grant GP-32463.

(1) A negative answer was also given by E. Herrmann of Humboldt University in Berlin. His results, obtained independently of ours, are contained in a manuscript entitled "About Lindenbaum functions of $\aleph_0$-categorical theories of finite similarity type". The examples he gets are not, however, as extensive as ours.