On topological factors of 3-dimensional locally connected continuum embeddable in E^3

by

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Abstract. In this paper we shall show that if a locally connected continuum embeddable in E^3 is a Cartesian product of non-trivial topological factors, then one of the factors is an arc or a simple closed curve. Then with the help of Claytor's results [2] we shall prove that if one of the factors is a simple closed curve then the other is flat (embeddable in E^2). If one of the factors is an arc, then with the triviality of the first and the second Čech cohomology groups, the other factor has to be flat. This theorem leads to the following corollary: every proper topological factor of an AR embeddable in E^3 is flat.

A space Y is said to be a *topological factor of a non-empty space* X if there exists a space Z such that $Y \times Z$ is homeomorphic to X. The space Y is a *proper topological factor of* X if Y is a topological factor of X and it is not homeomorphic to X and contains more than one point. By E^n we will denote an n-dimensional Euclidean space. When α is an arc, its interior will be denoted by $\mathring{\alpha}$.

1. On 1-dimensional factors. In this section we shall consider 1-dimensional topological factors of a 3-dimensional locally connected continuum embeddable in E^3. A set T is said to be a *triod* if it is the union of three arcs $\widehat{pa_1}, \widehat{pa_2}, \widehat{pa_3}$, pairwise disjoint outside point p. Let us notice that the following simple lemma holds:

LEMMA 1.1. *A non-empty non-degenerate locally connected continuum X which does not contain a triod is an arc or a simple closed curve.*

Next we shall prove

LEMMA 1.2. *A Cartesian product of two triods is not embeddable in E^3.*

Proof. By F^n we denote the n-dimensional skeleton of a $(2n+2)$-dimensional simplex. By G^p we denote a juncture $F^{k_1-1} * F^{k_2-1} * \ldots * F^{k_s-1}$ such that $\sum_{i=1}^{s} k_i = p+1$. Then, as was proved by Claude Weber [4], the cone over G^{n-1} is not quasi-embeddable in R^{2n-1}. Hence, if we denote by K_1 the juncture $F^0 * F^1$ of two 0-dimensional skeletons of a 2-dimensional simplex, the cone over K_1 is not quasi-embeddable in E^3. Hence it is not embeddabled in E^3. It is easy to see that the product of two triods is homeomorphic to the cone over K_1.

Now we can observe that the following theorem holds:

THEOREM 1.1. *If a locally connected continuum embeddable in E^3 is a Cartesian product if non-trivial factors, then one of the factors is an arc or a simple closed curve.*

Proof. Let $X\times Y \underset{\text{top}}{=} Z\subset E^3$. By Lemma 1.2 the product of two triods is not embeddabled in E^3; hence X or Y does not contain a triod. By Lemma 1.1 a locally connected continuum that does not contain a triod is an arc or a simple closed curve.

2. The case of a 1-dimensional factor being a simple closed curve. At the beginning of this section we shall give some definitions and quote a theorem by Claytor [2].

DEFINITION 2.1. A graph is said to be *Kuratowski's graph* K_1 if it is a union of six points a, b, c, d, p, q, and of disjoint open arcs $\overset{\circ}{\widehat{ab}}, \overset{\circ}{\widehat{bd}}, \overset{\circ}{\widehat{dq}}, \overset{\circ}{\widehat{qa}}, \overset{\circ}{\widehat{ac}}, \overset{\circ}{\widehat{dc}}, \overset{\circ}{\widehat{cp}}, \overset{\circ}{\widehat{pb}}, \overset{\circ}{\widehat{qp}}$, that do not contain any of the points a, b, c, d, p, q.

This definition is equivalent to the definition of K_1 on page 141.

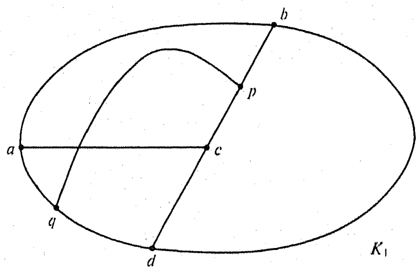

Fig. 1

DEFINITION 2.2. A graph is said to be *Kuratowski's graph* K_2 if it is homeomorphic to a 1-dimensional skeleton of 4-dimensional simplex. We shall denote the ramification points as in Figure 2 by a, b, c, p, q.

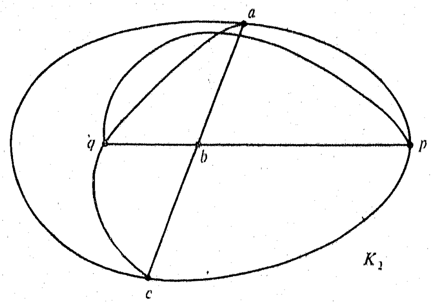

Fig. 2

DEFINITION 2.3. Let the graph Z_i be as in Figure 3. Consider the family of graphs $\{Z_i\}_{i\in N}$, where N is the set of natural numbers, such that, for every $i = 1, 2, \ldots, Z_i\cap Z_{i+1} = \{b_i\}, Z_i\cap Z_j = \emptyset$ if $|j-i|>1$ and the diameters of sets Z_i converge to zero. Let $b_\infty = \lim\limits_{i\to\infty} b_i$ and let the intersection of the arc $\widehat{b_\infty p}$ with $\bigcup\limits_{i=1}^{\infty} Z_i$ be empty. Then we obtain *Kuratowski's curve* $K_3 \overset{\text{df}}{=} \bigcup\limits_{i=1}^{\infty} Z_i \cup \widehat{b_\infty p}$.

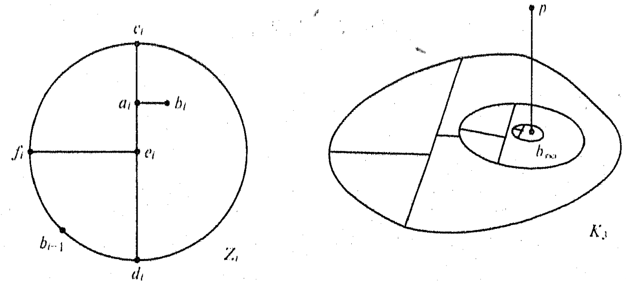

Fig. 3

DEFINITION 2.4. Let the graph R_i be as in Figure 4. Consider the family of graphs $\{R_i\}_{i\in N}$, N-natural numbers such that for every $i = 1, 2, \ldots, R_i\cap R_{i+1} = \{a_{i+1}\}$, $R_i\cap R_j = \emptyset$ if $j = i+1, i-1$ and the diameters of the sets R_i are convergent to zero as i diverges to infinity. Let $\alpha_\infty = \lim\limits_{i\to\infty}\alpha_i$ and $\widehat{a_\infty p}\cap \bigcup\limits_{i=1}^{\infty} R_i = \emptyset$. Then we have *Kuratowski's curve* $K_4 \overset{\text{df}}{=} \bigcup\limits_{i=1}^{\infty} R_i \cup \widehat{a_\infty p}$.

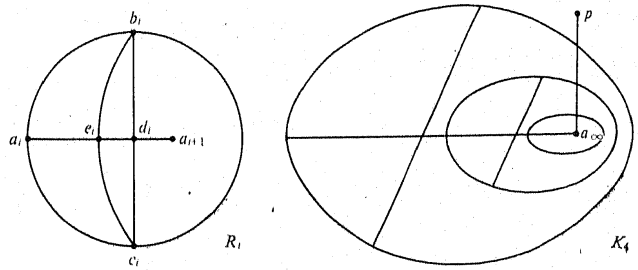

Fig. 4

CLAYTOR'S THEOREM. *A locally connected continuum X is embeddable in S^2 if and only if X does not contain any of the Kuratowski curves K_1, K_2, K_3, K_4.*

Now we shall formulate the main theorem of this section:

THEOREM 2.1. *If a locally connected continuum embeddable in E^3 is homeomorphic to the Cartesian product of a space Y and a simple closed curve, then the space Y is flat (Y is embeddable in E^2).*

Proof. The space $S^2 \times S^1$ is a compact 3-dimensional manifold without boundary, and thus it is not embeddable in E^3. By Claytor's Theorem it suffices to prove that the spaces $K_i \times S^1$ are not embeddable in E^3 for $i = 1, 2, 3, 4$. To show this we shall prove several lemmas.

LEMMA 2.1. *Let $S_i \underset{\text{top}}{=} S^1$ for $i = 1, 2, \ldots, n$; $\bigcup_{i=1}^{k} S_i \cap S_{k+1} = \langle 0, 1\rangle$ for $1 \leqslant k \leqslant n-1$ and let h be a homeomorphic embedding of $\bigcup_{i=1}^{n} S_i \times S^1$ in E^3. Then the set $Z = h\big(\bigcup_{i=1}^{n} S_i \times S^1\big)$ dissects E^3 into precisely $n+1$ components. Moreover, if $Z' = h\big(\bigcup_{i=1}^{n-1} S_i \times S^1\big)$ dissects E^3 into the components $A_1, \ldots, A_n$ and $h(S_n \times S^1)$ dissects E^3 into the components B_1, B_2 and $Z \setminus Z' \subset A_n$, then the set Z dissects E^3 into the components $A_1, \ldots, A_{n-1}$, $A_n \cap B_1$, $A_n \cap B_2$.*

Proof. We proceed by induction on n.

If $n = 1$, then for an arbitrary embedding h the set $h(S_1 \times S^1)$, being homeomorphic to the torus, dissects the space E^3 into precisely two components.

We shall show that the set $Z = h\big(\bigcup_{i=1}^{n} S_i \times S^1\big)$ dissects E^3 into $n+1$ components. From the inductive assumption it follows that the set $Z' = h\big(\bigcup_{i=i}^{n-1} S_i \times S^1\big)$ dissects E^3 into n components $A_1, \ldots, A_n$, such that $\bar{A}_i \setminus A_i \underset{\text{top}}{=} S^1 \times S^1$. The set $Z \setminus Z'$ is equal to $h(J \times S^1)$ where J is an open arc, and so it is connected and it is contained in $E^3 \setminus Z'$; hence it is contained in one of the components $A_1, \ldots, A_n$, say in A_n. We have

$$E^3 \setminus Z = E^3 \setminus (Z' \cup h(S_n \times S^1)) = (E^3 \setminus Z') \cap (E^3 \setminus h(S_n \times S^1)).$$

The set $h(S_n \times S^1)$ dissects E^3 into two components, B_1 and B_2. Notice that for every $i = 1, \ldots, n-1$ the component A_i is contained in $E^3 \setminus h(S_n \times S^1)$, hence, being a connected set, it has to be contained either in B_1 or in B_2. Hence

$$E^3 \setminus Z = A_1 \cup \ldots \cup A_{n-1} \cup (A_n \cap B_1) \cup (A_n \cap B_2).$$

The set $T = \bar{A}_n \setminus A_n$ is a torus and

$$T \cup h(J \times S') \underset{\text{top}}{=} \bigcup_{i=1}^{2} S_i \times S^1.$$

Hence the set $T \cup h(J \times S')$ dissects E^3 into three components one of which is $E^3 \setminus \bar{A}_n$. It is to observe that the sets $B_1 \cap A_n$ and $B_2 \cap A_n$ are non-empty, and so they have to be connected. Thus $A_1, \ldots, A_{n-1}, A_n \cap B_1, A_n \cap B_2$ are the components of the set $E^3 \setminus Z$, which proves the lemma.

LEMMA 2.2. *Let I', J', K' be arcs with end-points a and b and with pairwise disjoint interiors. Let h be the embedding of the set $(I' \cup J' \cup K') \times S^1$ in E^3. We denote:*

$$T_1 = h((I' \cup J') \times S^1), \qquad I = h(\mathring{I}' \times S^1),$$
$$T_2 = h((I' \cup K') \times S^1), \qquad J = h(\mathring{J}' \times S^1),$$
$$T_3 = h((J' \cup K') \times S^1), \qquad K = h(\mathring{K}' \times S^1).$$

For every $i = 1, 2, 3$ the set T_i dissects E^3 into the bounded component A_i and the unbounded component B_i. Then $I \subset B_3$ and $K \subset B_1$ iff $J \subset A_2$.

Proof. First we shall show that $I \subset B_3$ and $K \subset B_1$ implies $J \subset A_2$.

The set $T_1 \cup T_2$ is equal to $T_1 \cup K$, and $K \subset B_1$. By Lemma 2.1 $T_1 \cup T_2$ dissects E^3 into three components, A_1, $B_1 \cap A_2$, and $B_1 \cap B_2$. By the same reasoning $T_3 \cup T_1$ dissects E^3 into three components, A_3, $B_3 \cap A_2$, and $B_3 \cap B_1$.

Assume $L \subset B_2$. Then the set $T_2 \cup T_3$, equal to the set $T_2 \cup J$, dissects E^3 into components the A_2, $B_2 \cap A_3$, and $B_2 \cap B_3$.

Since the sets $T_1 \cup T_2$, $T_3 \cup T_1$, $T_2 \cup T_3$ are equal to each other, they have the same components of complements to E^3. Notice $B_1 \cap B_2 = B_3 \cap B_1 = B_2 \cap B_3$. By $A_1 \neq A_2$ and $A_1 \neq B_2 \cap B_3$, $A_1 = B_2 \cap A_3$. From $A_3 \neq A_2$ and $A_3 \neq B_2 \cap B_3$, $A_3 = B_2 \cap A_3$. But now $A_1 = B_2 \cap A_3 = A_3$. This is not possible, because it yields a false result $T_1 = T_3$. Hence $J \not\subset B_2$. But J is connected, $J \subset E^3 \setminus T_2$, and hence $J \subset A_2$.

Now we shall show that the condition $J \subset A_2$ is sufficient for $I \subset B_3$ and $K \subset B_1$.

The set $T_1 \cup T_2$ is equal to the sets $T_2 \cup J$ and $J \subset A_2$. By Lemma 2.1 the set $T_2 \cup T_1$ dissects E^3 into three components, B_2, $A_2 \cap B_1$, and $A_2 \cap A_1$. $T_1 \subset \bar{A}_2$ implies $A_1 \subset A_2$, and hence the components of $E^3 \setminus (T_2 \cup T_1)$ are of the form B_2, $A_2 \cap B_1$, A_1. $K \subset T_1 \cup T_2$ implies $K \cap A_1 = \emptyset$, and since $K \cap T_1 = \emptyset$, we have $K \subset B_1$.

By an analogous reasoning we prove $I \subset B_3$.

LEMMA 2.3. *The Cartesian product of Kuratowski's graph K_1 and the circle S^1 is not embeddable in E^3.*

Proof. Assume that $h\colon K_1 \times S^1 \to E^3$ is a homeomorphic embedding. Denote by S_1, S_2, S_3 the simple closed curves being the subsets of the graph K_1.

$$S_1 \overset{\text{df}}{=} \frown{bd} \cup \frown{dc} \cup \frown{cp} \cup \frown{pb},$$
$$S_2 \overset{\text{df}}{=} \frown{ab} \cup \frown{bp} \cup \frown{pc} \cup \frown{ca},$$
$$S_3 \overset{\text{df}}{=} \frown{ab} \cup \frown{bd} \cup \frown{dc} \cup \frown{ca}.$$

Every set $T_i \overset{\text{df}}{=} h(S_i \times S^1)$, $i = 1, 2, 3$, dissects E^3 into two components, A_i and B_i, B_i being the unbounded component. Let r be a fixed point on the circle S^1.

We can put $p' \overset{\mathrm{df}}{=} h(p, r) \in A_3$; then $q' \overset{\mathrm{df}}{=} h(q, r) \in A_3$ as well, because the arc $\alpha = h(\widehat{pq} \times \{r\})$ is contained in $E^3 \setminus T_3$.

Lemma 2.1 shows that the set $T_3 \cup T_1$ dissects E^3 into the components B_3, $A_3 \cap B_1$, and $A_3 \cap A_1 = A_1$ and $T_3 \cup T_2$ dissects E^3 into the components B_3, $A_3 \cap B_2$, $A_3 \cap A_2 = A_2$. $T_3 \cup T_1$, and $T_3 \cup T_2$ are equal, and thus their complements in E^3 are equal as well. So $T_3 \cup T_1$ dissects E^3 into components B_3, A_2, A_1.

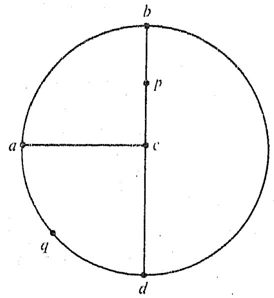

Fig. 5

But $q' \in A_3$, and thus the open arc $\beta = h((\mathring{\widehat{aq}} \cup \{q\} \cup \mathring{\widehat{qd}}) \times \{r\})$ is contained in $A_2 \cup A_1$. The arc β is connected, and thus $\beta \subset A_1$ or $\beta \subset A_2$. Denote

$$J = h((\mathring{\widehat{cp}} \cup \{p\} \cup \mathring{\widehat{pb}}) \times S^1),$$

$$I = h((\mathring{\widehat{ca}} \cup \{a\} \cup \mathring{\widehat{ab}}) \times S^1),$$

$$K = h((\mathring{\widehat{bd}} \cup \{d\} \cup \mathring{\widehat{dc}}) \times S^1).$$

$p' \in A_3$ implies $J \subset A_3$. Lemma 2.2 yields $I \subset B_1$ and $K \subset B_2$. Thus we have the point $a' \overset{\mathrm{df}}{=} h(a, r)$ belongs to B_1 and the point $d' \overset{\mathrm{df}}{=} h(d, r)$ belongs to B_2, which implies $a' \notin \bar{A}_1$, hence $\beta \not\subset A_1$ and $d' \notin \bar{A}_2$, and thus $\beta \not\subset A_2$.

LEMMA 2.4. *The Cartesian product of Kuratowski's graph K_2 and the circle S^1 is not embeddable in E^3.*

Proof. Assume that $h\colon K_2 \times S^1 \to E^3$ is a homeomorphic embedding. Denote:

$$K \overset{\mathrm{df}}{=} h(\mathring{\widehat{ac}} \times S^1),$$

$$L \overset{\mathrm{df}}{=} h((\mathring{\widehat{aq}} \cup \{q\} \cup \mathring{\widehat{qc}}) \times S^1),$$

$$M \overset{\mathrm{df}}{=} h((\mathring{\widehat{ab}} \cup \{b\} \cup \mathring{\widehat{bc}}) \times S^1),$$

$$N \overset{\mathrm{df}}{=} h((\mathring{\widehat{ap}} \cup \{p\} \cup \mathring{\widehat{pc}}) \times S^1),$$

$$P \overset{\mathrm{df}}{=} h((\{a\} \cup \{c\}) \times S^1).$$

$$T_1 \overset{\mathrm{df}}{=} K \cup M \cup P, \quad T_4 \overset{\mathrm{df}}{=} L \cup N \cup P,$$

$$T_2 \overset{\mathrm{df}}{=} K \cup L \cup P, \quad T_5 \overset{\mathrm{df}}{=} N \cup M \cup P,$$

$$T_3 \overset{\mathrm{df}}{=} L \cup M \cup P, \quad T_6 \overset{\mathrm{df}}{=} N \cup K \cup P,$$

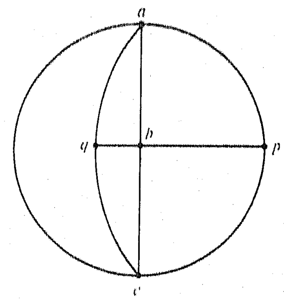

Fig. 6

Every set T_i dissects E^3 into two components, A_i and B_i, B_i being the unbounded component. We may assume that if r is a fixed point on the circle S^1, then $q' \overset{\mathrm{df}}{=} h(q, r) \in A_1$. In this case $p' \overset{\mathrm{df}}{=} h(p, r) \in A_1$ as well, because

$$\alpha = h(\widehat{pq} \times \{r\}) \subset E^3 \setminus T_1.$$

With the help of Lemma 2.1 and Lemma 2.2 we can show, as in Lemma 2.3, that the set $T_1 \cup T_2$, equal to the set $T_1 \cup T_3$, dissects the space E^3 into the components A_2, A_3, B_1. The arc $\beta = h(\mathring{\widehat{pb}} \times \{r\})$ is contained in $E^3 \setminus T_1$. The point p' belongs to A_1, and thus $\beta \subset A_1$. The arc $\gamma = h(\mathring{\widehat{qb}} \times \{r\})$, contained in $E^3 \setminus (T_1 \cup T_2)$, is connected, and thus it is contained in one of the components A_2, A_3, B_1. But $q' \notin B_1$ and from Lemma 2.2 $b' \overset{\mathrm{df}}{=} h(b, r) \in B_2$ (because $L \subset A_1$), hence $b' \notin A_2$, and thus $\gamma \not\subset B_1$ and $\gamma \not\subset A_2$, which gives $\gamma \subset A_3$.

The connected set N is contained in $E^3 \setminus (T_1 \cup T_2)$, and thus N is contained in one of the sets A_2, A_3, B_1. But $p' \in A_1$, hence $p' \notin B_1$ and $p' \in N$, and thus $N \not\subset B_1$. Assume $N \subset A_2$. Then from Lemma 2.1 the set $T_1 \cup T_2 \cup T_4 = T_1 \cup T_2 \cup T_6$ dissects the space E^3 into the components $A_2 \cap A_4$, $A_2 \cap B_4$, A_3, B_1, equal to the components $A_2 \cap A_6$, $A_2 \cap B_6$, A_3, B_1. But $T_4 \subset \bar{A}_2$, and thus $A_4 \subset A_2$ and $A_2 \cap A_4 = A_4$; $T_6 \subset \bar{A}_2$, and thus $A_6 \subset A_2$ and $A_2 \cap A_6 = A_6$. We have $A_2 \neq A_6$ (because $T_2 \neq T_6$), and thus the set $T_1 \cup T_2 \cup T_4$ dissects E^3 into the components A_3, A_4, A_6, B_1. The arc β would have to be contained in one of these components. But $\beta\not\subset B_1$, and thus β is contained in one of the sets A_4, A_6, A_3. The component A_4 lies in A_2, $b'\notin A_2$ and $b'\notin T_4$, and hence $b'\notin \bar{A}_4$ and $\beta\not\subset A_4$. The set N is contained in A_2, and thus $N\cap A_3\neq\emptyset$, whence $p'\notin \bar{A}_3$ and $\beta\not\subset A_3$. We have $N\subset A_2\subset A_1$, and, from Lemma 2.2 $M\subset B_6$; thus $b'\notin A_6$ and $\beta\not\subset A_6$. Thus when $N\subset A_2$, the arc β is not contained in any of the components A_4, A_6, A_3, B_1. Hence $N\not\subset A_2$, that is $N\subset A_3$.

Now, by the same reasoning as in the previous case, we can prove that the set $T_1\cup T_3\cup T_4 = T_1\cup T_3\cup T_5$ dissects E^3 into the components A_4, A_5, A_2, B_1. The arc γ lies in A_3, and thus $\gamma\subset A_4$ or $\gamma\subset A_5$. But $N\subset A_3$ and, from Lemma 2.2, $L\subset B_5$ and $M\subset B_4$; hence $q'\notin A_5$ and $b'\notin A_4$, which implies $\gamma\not\subset A_4$ and $\gamma\not\subset A_5$. This contradiction proves the lemma.

LEMMA 2.5. *The Cartesian product of Kuratowski's curve K_3 and of the circle S^1 is not embeddable in E^3.*

Proof. Assume that $h\colon K_3\times S^1\to E^3$ is the homeomorphic embedding. Denote $\tau = h(\{b_\infty\}\times S^1)$. The uniform continuity of h implies

$$\bigwedge_{\varepsilon>0}\bigvee_{i_0}\bigwedge_{i\geqslant i_0} h(Z_i\times S^1)\subset K(\tau,\varepsilon),$$

where $K(\tau,\varepsilon)$ denote the ball with the centre τ and the radius ε. Let r be the fixed point on the circle S^1 and let p' denote $h(p,r)$. The distance $\varrho(\tau,p')$ of the point p' from the curve τ is greater than zero. Also $\varrho(\tau,\mathscr{Z}_1)>0$, where $\mathscr{Z}_1 \overset{\text{df}}{=} h(Z_1\times S^1)$. Let ε_0 be such that $\varrho(\tau,p')>\varepsilon_0>0$ and $\varrho(\tau,\mathscr{Z}_1)>\varepsilon_0>0$: then there exists an i_0 such that $h(Z_i\times S^1)\subset K(\tau,\varepsilon_0)$ for all $i\geqslant i_0$. Denote

$$K = h\big((\overset{\circ}{\widehat{f_{i_0}b_{i_0-1}}}\cup\{b_{i_0-1}\}\cup\overset{\circ}{\widehat{b_{i_0-1}d_{i_0}}})\times S^1\big),$$
$$I = h\big((\overset{\circ}{\widehat{f_{i_0}c_{i_0}}}\cup\{c_{i_0}\}\cup\overset{\circ}{\widehat{c_{i_0}d_{i_0}}})\times S^1\big),$$
$$J = h\big((\overset{\circ}{\widehat{f_{i_0}e_{i_0}}}\cup\{e_{i_0}\}\cup\overset{\circ}{\widehat{e_{i_0}d_{i_0}}})\times S^1\big),$$
$$P = h\big((\{f_{i_0}\}\cup\{d_{i_0}\})\times S^1\big),$$
$$T_1 = I\cup J\cup P,\qquad T_2 = I\cup K\cup P,\qquad T_3 = J\cup K\cup P.$$

The torus T_1 dissects E^3 into components A_i and B_i one of them, say A_i, is contained in $K(\tau,\varepsilon_0)$ (because the toruses T_i are contained in $K(\tau,\varepsilon_0)$) $i = 1,2,3$. The point $b'_{i_0-1}\overset{\text{df}}{=} h(b_{i_0-1},r)$ can be connected with the set $\mathscr{Z}_1\subset E^3\setminus K(\tau,\varepsilon_0)\subset B_1$ by an arc contained in $h(\bigcup_{i=1}^{i_0-1} Z_i\times S^1)$, and thus disjoint with T_1. Hence $K\subset B_1$.

The set

$$Z \overset{\text{df}}{=} h\Big(\Big(\bigcup_{i=i_0+1}^{\infty} Z_i\cup\widehat{b_\infty p}\Big)\times\{r\}\Big)$$

is connected and disjoint with the sets T_2 and T_3. The point p' does not belong to $K(\tau,\varepsilon_0)$, and therefore $Z\subset B_2$ and $Z\subset B_3$. The point $e'_{i_0}\overset{\text{df}}{=} h(e_{i_0},r)$ can be connected with the set $Z\subset B_2$ by an arc $h\big((\widehat{e_{i_0}a_{i_0}}\cup\widehat{a_{i_0}b_{i_0}})\times\{r\}\big)$ disjoint with T_2, and thus $J\subset B_2$. The point $c'_{i_0}\overset{\text{df}}{=} h(c_{i_0},r)$ can be connected with the set $Z\subset B_3$ by an arc $h\big((\widehat{c_{i_0}a_{i_0}}\cup\widehat{a_{i_0}b_{i_0}})\times\{r\}\big)$ disjoint with T_3; therefore $I\subset B_3$.

But from Lemma 2.2 it follows that the inclusions $K\subset B_1$, $J\subset B_2$, $I\subset B_3$ are mutually exclusive. This contradiction proves the lemma.

LEMMA 2.6. *The Cartesian product of Kuratowski's curve K_4 and of the circle S^1 is not embeddable in E^3.*

Proof. Assume that $h\colon K_4\times S^1\to E^3$ is a homeomorphic embedding. Denote $\tau\overset{\text{df}}{=} h(\{a_\infty\}\times S^1)$. The uniform continuity of h implies $\bigwedge_{\varepsilon>0}\bigvee_{i_0}\bigwedge_{i\geqslant i_0} h(R_i\times S^1)\subset K(\tau,\varepsilon)$.

Let r be a fixed point on the circle S^1; by p' we denote the point $h(p,r)$. The distance between the point p and the curve τ is greater than zero. $\varrho(\tau,\mathscr{Z}_1)>0$, where $\mathscr{Z}_1\overset{\text{df}}{=} h(R_1\times S^1)$. Let ε_0 be such that $\varrho(\tau,p')>\varepsilon_0>0$ and $\varrho(\tau,\mathscr{Z}_1)>\varepsilon_0>0$: then there exists an i_0 such that for $i\geqslant i_0$, $h(R_i\times S^1)\subset K(\tau,\varepsilon_0)$. We denote:

$$K\overset{\text{df}}{=} h\big((\overset{\circ}{\widehat{b_{i_0}a_{i_0}}}\cup\{a_{i_0}\}\cup\overset{\circ}{\widehat{a_{i_0}c_{i_0}}})\times S^1\big),$$
$$L\overset{\text{df}}{=} h\big((\overset{\circ}{\widehat{b_{i_0}e_{i_0}}}\cup\{e_{i_0}\}\cup\overset{\circ}{\widehat{e_{i_0}c_{i_0}}})\times S^1\big),$$
$$M\overset{\text{df}}{=} h\big((\overset{\circ}{\widehat{b_{i_0}d_{i_0}}}\cup\{d_{i_0}\}\cup\overset{\circ}{\widehat{d_{i_0}c_{i_0}}})\times S^1\big),$$
$$N\overset{\text{df}}{=} h(\overset{\circ}{\widehat{b_{i_0}c_{i_0}}}\times S^1),$$
$$P\overset{\text{df}}{=} h\big((\{b_{i_0}\}\cup\{c_{i_0}\})\times S^1\big),$$
$$T_1\overset{\text{df}}{=} K\cup N\cup P,\qquad T_4\overset{\text{df}}{=} N\cup M\cup P,$$
$$T_2\overset{\text{df}}{=} K\cup M\cup P,\qquad T_5\overset{\text{df}}{=} N\cup L\cup P,$$
$$T_3\overset{\text{df}}{=} K\cup L\cup P,\qquad T_6\overset{\text{df}}{=} M\cup L\cup P.$$

The torus $T_i\subset K(\tau,\varepsilon_0)$, for $i = 1,\dots,6$, dissects E^3 into the two components A_i and B_i and one of them, say A_i, is contained in $K(\tau,\varepsilon_0)$. The connected set

$$R = h\Big(\Big(\bigcup_{i=i_0+1}^{\infty} R_i\cup\widehat{a_\infty p}\Big)\times\{r\}\Big)$$

is disjoint with the toruses T_i. The point p' does not belong to $K(\tau,\varepsilon_0)$ and thus $B_i\supset R$, for $i = 1,\dots,6$.

The set M is contained in B_1 because it can be connected with the set R by an arc $h(\widehat{d_{i_0}a_{i_0+1}}\times\{r\})$ disjoint with T_1. The set K is contained in B_4 because it can be connected with the set $\mathscr{L}_1 \subset E^3 \setminus K(\tau, \varepsilon_0) \subset B_4$ by an arc contained in $h(\bigcup_{i=1}^{i_0-1} R_i \times S^1)$, and thus disjoint with T_4. Therefore we have from Lemma 2.2 the set $N \subset A_2$. From Lemma 2.1 we know that the set $T_1 \cup T_2 = T_1 \cup T_4 = T_2 \cup T_4$ dissects E^3 into three components $A_1 \cap A_2$, $B_1 \cap A_2$, B_2 equal to A_1, $B_1 \cap A_2$, $B_1 \cap B_2$, respectively and to A_4, $B_2 \cap B_4$, $A_2 \cap B_4$, respectively. These are obviously the components A_1, A_4, B_2.

Consider the arc $\alpha = h\big((\widehat{a_{i_0} e_{i_0}} \cup \widehat{e_{i_0} d_{i_0}}) \times \{r\}\big)$. The arc $\mathring{\alpha}$ is disjoint with $T_1 \cup T_2$; therefore α is contained in the closure of one the components A_1, A_4, B_2. But $a'_{i_0} \overset{\text{df}}{=} h(a_{i_0}, r) \notin \bar{A}_4$ (because $K \subset B_4$) and $d'_{i_0} \overset{\text{df}}{=} h(d_{i_0}, r) \notin \bar{A}_1$ (because $M \subset B_1$), and thus $\alpha \subset B_2$. But $\alpha \cap L = \{h(e_{i_0}, r)\}$. Therefore $L \subset B_2$. We can easily notice that $K \subset B_6$ and $M \subset B_3$. But in this case Lemma 2.2 yields $L \subset A_2$. This contradiction proves the lemma.

According to Lemmas 2.3, 2.4, 2.5, 2.6 the sets $K_i \times S^1$ are not embeddable in E^3 for $i = 1, 2, 3, 4$. The space $X \times S^1$ is embeddable in E^3, and hence the space X cannot contain any of the Kuratowski curves K_i. From Claytor's Theorem it follows that $X \underset{\text{top}}{\subset} S^2$, but $X \underset{\text{top}}{\neq} S^2$, and hence $X \underset{\text{top}}{\subset} E^2$.

3. The case of an arc being a one-dimensional factor.

THEOREM 3.1. *Let Y be a locally connected continuum embeddable in E^3. If Y is homeomorphic to the Cartesian product $X \times \langle 0, 1\rangle$ and Čech cohomology groups with integer coefficients $\check{H}^1(Y)$ and $\check{H}^2(Y)$ are trivial, then X is a flat space.*

Proof. From $Y \underset{\text{top}}{=} S^2 \times \langle 0, 1\rangle$ we obtain $\check{H}^2(S^2 \times \langle 0, 1\rangle) \approx Z \neq 0$ where Z stands for the group of integers. Hence Y fails to satisfy the assumptions of our theorem. If none of the Kuratowski curves were a subset of X, we would obtain by the Claytor theorem $X \underset{\text{top}}{\subset} S^2$. Since $X \underset{\text{top}}{\neq} S^2$, we see that $X \underset{\text{top}}{\subset} E^2$.

Now let us consider the case where for a certain K_i, $i = 1, 2, 3, 4$, we have $K_i \subset X$ and $X \times \langle 0, 1\rangle \underset{\text{top}}{\subset} E^3$. When h denotes this embedding, the following inclusion holds:

$$h\big((X \times \{0\}) \cup (K_i \times \langle 0, 1\rangle) \cup (X \times \{1\})\big) \subset E^3 .$$

It is sufficient to show that this is false.

Let A be a compact subset of E^n. The connection between the reduced singular homology groups of $E^n \setminus A$ and the Čech cohomology groups of A is established by the Duality theorem ([3] 6, 2, Th. 16), namely $\tilde{H}_q(E^n \setminus A)$ is isomorphic to $\check{H}^{n-q-1}(A)$.

Adopting the assumption of Theorem 3.1 we shall prove several lemmas.

LEMMA 3.1. *If $S \underset{\text{top}}{=} S^1$, $S \subset X$ and h is a homeomorphic embedding of $X \times \langle 0, 1\rangle$ into E^3, then the set $T = h\big((X \times \{0\}) \cup (S \times \langle 0, 1\rangle) \cup (X \times \{1\})\big)$ dissects E^3 into two components and the closure of both of them contains $h(S \times \langle 0, 1\rangle)$.*

Proof. Let us define the following sets:

$$Z_0 \overset{\text{df}}{=} h(S \times \{\tfrac{1}{2}\}) ,$$

$$Z_1 \overset{\text{df}}{=} h\big((X \times \{0\}) \cup (S \times \langle 0, \tfrac{1}{2}\rangle)\big) ,$$

$$Z_2 \overset{\text{df}}{=} h\big((X \times \{1\}) \cup (S \times \langle \tfrac{1}{2}, 1\rangle)\big) .$$

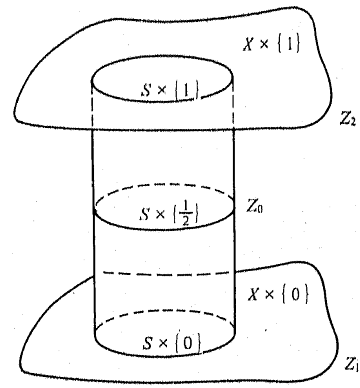

Fig. 7

It is easily noticed that $T = Z_1 \cup Z_2$ and $Z_0 = Z_1 \cap Z_2$. The Čech cohomology groups $\check{H}^2(Z_1)$, $\check{H}^2(Z_2)$, $\check{H}^1(Z_1)$, $\check{H}^1(Z_2)$ are trivial, as X is homeomorphic to deformation retracts $h(X \times \{0\})$ of the set Z_1 and $h(X \times \{1\})$ of Z_2, therefore $\check{H}^1(X) \approx \check{H}^1(X \times \langle 0, 1\rangle) \approx \check{H}^1(Y) \approx 0$ and $\check{H}^2(X) \approx \check{H}^2(X \times \langle 0, 1\rangle) \approx \check{H}^2(Y) \approx 0$

The Duality theorem yields $\tilde{H}_0(E^3 \setminus Z_1) \approx 0$, $\tilde{H}_0(E^3 \setminus Z_2) \approx 0$, $\tilde{H}_1(E^3 \setminus Z_1) \approx 0$ and $\tilde{H}_1(E^3 \setminus Z_2) \approx 0$.

Since the sets $E^3 \setminus Z_1$ and $E^3 \setminus Z_2$ are open in E^3, we may write the Mayer–Vietoris exact sequence:

$$\ldots \to \tilde{H}_1(E^3 \setminus Z_1) \oplus \tilde{H}_1(E^3 \setminus Z_2) \to \tilde{H}_1\big((E^3 \setminus Z_1) \cup (E^3 \setminus Z_2)\big) \to$$
$$\to \tilde{H}_0\big((E^3 \setminus Z_1) \cap (E^3 \setminus Z_2)\big) \to \tilde{H}_0(E^3 \setminus Z_1) \oplus \tilde{H}_0(E^3 \setminus Z_2) .$$

This sequence takes the form

$$0 \oplus 0 \to \tilde{H}_1(E^3 \setminus Z_0) \to \tilde{H}_0(E^3 \setminus T) \to 0 \oplus 0 .$$

Hence we infer that $\tilde{H}_1(E^3 \setminus Z_0)$ is isomorphic to $\tilde{H}_0(E^3 \setminus T)$. But, since $\tilde{H}_1(E^3 \setminus Z_0) \approx Z$, we have $\tilde{H}_0(E^3 \setminus T) \approx Z$ and $H_0(E^3 \setminus T) \approx Z \oplus Z$. This means that $E^3 \setminus T$ consists of exactly two components, A and B. This proves the first part of our lemma.

Let $x \in h(S\times(0,1))$ and let U be a neighbourhood of x in E^3. Then $U \cap h(S\times(0,1))$ is a neighbourhood of X in $h(S\times(0,1))$. There exists such an open neighbourhood $D \subset U \cap h(S\times(0,1))$ of X that $h^{-1}(D)$ is an open disc in $S\times(0,1)$. The set $(S\times(0,1))\setminus h^{-1}(D)$ may be deformation retracted to the set $(S\times\{1\}) \cup (\langle a\rangle\times\langle 0,1\rangle) \cup (S\times\{0\})$ where $a\in S$. It follows that $(X\times\{0\})\cup(\{a\}\times\langle 0,1\rangle)\cup(X\times\langle 1\rangle)$ is a deformation retract of $h^{-1}(T)\setminus h^{-1}(D) \underset{\text{top}}{=} T\setminus D$. Since the appriopriate pairs of sets are open, we may write the Mayer–Vietoris exact sequence:

$$\ldots\to\tilde{H}_1\Big(\big(E^3\setminus h((X\times\{0\})\cup(X\times\{1\}))\big)\cup\big(E^3\setminus h(\{a\}\times\langle 0,1\rangle)\big)\Big)\to$$

$$\to\tilde{H}_0\Big(\big(E^3\setminus h((X\times\{0\})\cup(X\times\{1\}))\big)\cap\big(E^3\setminus h(\{a\}\times\langle 0,1\rangle)\big)\Big)\to$$

$$\to\tilde{H}_0\big(E^3\setminus h((X\times\{0\})\cup(X\times\{1\}))\big)\oplus\tilde{H}_0\big(E^3\setminus h(\{a\}\times\langle 0,1\rangle)\big)\to\ldots$$

This sequence takes the following form:

$$0\approx\tilde{H}_1(E^3\setminus\{h(a,0),h(a,1)\})\to$$
$$\to\tilde{H}_0\big(E^3\setminus h((X\times\{0\})\cup(\{a\}\times\langle 0,1\rangle)\cup(X\times\{1\}))\big)\to 0\oplus 0\,.$$

From the above we see that

$$\tilde{H}_0\big(E^3\setminus h((X\times\{0\})\cup(\{a\}\times\langle 0,1\rangle)\cup(X\times\{1\}))\big)\approx 0 \quad\text{and}\quad \tilde{H}_0(E^3\setminus(T\setminus D))\approx 0\,,$$

and so $E^3\setminus(T\setminus D)$ is connected.

Let us take $p\in A$, $q\in B$, and join them by an arc $\widehat{qp}\subset E^3\setminus(T\setminus D)$. Since p, q belong to different components of $E^3\setminus T$, we notice that $\widehat{qp}\not\subset E^3\setminus T$. We see that $\widehat{pq}\cap D\neq\emptyset$. This implies that $\bar{A}\cap D\neq\emptyset$ and $\bar{B}\cap D\neq\emptyset$, so $\bar{A}\cap U\neq\emptyset$ and $\bar{B}\cap U\neq\emptyset$. But as we have taken U arbitrarily, we have $x\in\bar{A}$ and $x\in\bar{B}$. This completes the proof of the lemma.

LEMMA 3.2. *Let $S_i \underset{\text{top}}{=} S^1$ for $i=1,2,\ldots,n$, $\bigcup_{i=1}^{k} S_i\cap S_{k+1} \underset{\text{top}}{=} \langle 0,1\rangle$ for $1\leqslant k\leqslant n-1$ and let h be a homeomorphic embedding of the set*

$$(X\times\{0\})\cup\Big(\bigcup_{i=1}^{n} S_i\times\langle 0,1\rangle\Big)\cup(X\times\{1\})$$

into E^3. Then the set

$$Z=h\Big((X\times\{0\})\cup\Big(\bigcup_{i=1}^{n} S_i\times\langle 0,1\rangle\Big)\cup(X\times\{1\})\Big)$$

dissects E^3 into exactly $n+1$ components. Moreover, if

$$Z'=h\Big((X\times\{0\})\cup\Big(\bigcup_{i=1}^{n-1} S_i\times\langle 0,1\rangle\Big)\cup(X\times\{1\})\Big)$$

dissects E^3 into the components $A_1,\ldots,A_n$ and $h\big((X\times\{0\})\cup(S_n\times\langle 0,1\rangle)\cup(X\times\{1\})\big)$ dissects E^3 into the components B_1 and B_2 and $Z\setminus Z'\subset A_n$, then the set Z dissects E^3 into the components $A_1,\ldots,A_n$, $A_n\cap B_1$, $A_n\cap B_2$.

Proof. We proceed by induction as in the proof of Lemma 2.1.

LEMMA 3.3. *Let I', J', K' be arcs in X with end-points a and b and with disjoint interiors. Let h be an embedding of the set $(X\times\{0\})\cup((I'\cup J'\cup K')\times\langle 0,1\rangle)\cup(X\times\{1\})$ into E^3. Denote:*

$$X_0=h(X\times\{0\}),$$
$$X_1=h(X\times\{1\}),$$
$$T_1=X_0\cup X_1\cup h((I'\cup J')\times\langle 0,1\rangle),$$
$$T_2=X_0\cup X_1\cup h((I'\cup K')\times\langle 0,1\rangle),$$
$$T_3=X_0\cup X_1\cup h((J'\cup K')\times\langle 0,1\rangle).$$
$$I=h(\mathring{I}'\times(0,1)),$$
$$J=h(\mathring{J}'\times(0,1)),$$
$$K=h(\mathring{K}'\times(0,1)).$$

For every $i=1,2,3$, the set T_i dissects E^3 into the bounded component A_i and the unbounded component B_i. Then $I\subset B_3$ and $K\subset B_1$ if and only if $J\subset A_2$.

Proof. We proceed as in the proof of Lemma 2.2.

LEMMA 3.4. *The sets $\underline{K}_i \overset{\text{df}}{=} (X\times\{0\})\cup(K_i\times\langle 0,1\rangle)\cup(X\times\{1\})$ where $K_i\subset X$, $i=1,2,3,4$ are not embeddable in E^3.*

Proof. The proofs of these facts are similar to the proofs of Lemmas 2.3, 2.4, 2.5, 2.6.

From Lemma 3.4 it follows that the set X cannot contain any of the Kuratowski curves K_i, $i=1,2,3,4$ because that would imply $\underline{K}_i\subset X\times\langle 0,1\rangle$ and this set is embeddable in E^3. Therefore from the Claytor theorem it follows that $X\underset{\text{top}}{\subset}S^2$. But $X\underset{\text{top}}{\neq}S^2$, and hence $X\underset{\text{top}}{\subset}E^2$. This proves the theorem.

Remark. The assumptions $\check{H}^1(Y)=0$ and $\check{H}^2(Y)=0$ are necessary, as is shown in the following examples:

Let us consider $Y=S^2\times\langle 0,1\rangle$. Y is a locally connected continuum embeddable in E^3 and $\check{H}^1(S^2\times\langle 0,1\rangle)=0$ but $\check{H}^2(S^2\times\langle 0,1\rangle)\neq 0$ and $S^2\underset{\text{top}}{\not\subset}E^2$.

Let T be a torus lacking an open disc. Then $T\times\langle 0,1\rangle$ is a locally connected continuum embeddable in E^3 and $\check{H}^2(T\times\langle 0,1\rangle)=0$, but $\check{H}^1(T\times\langle 0,1\rangle)\neq 0$ and $T\underset{\text{top}}{\not\subset}E^2$.

Theorem 3.1 yields the following

COROLLARY. *Every proper topological factors of an AR embeddable in E^3 is flat.*

Remark. If Y satisfies assumptions of Theorem 3.1, then Y is AR, because X is a flat locally connected continuum and X does not dissect a plane.

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