References

[1] L. B. Ciric, *Generalized contractions and fixed-point theorems*, Publ. L'Inst. Math. (Beograd) 12 (26) (1971), pp. 19–26.

[2] D. E. Daykin and J. K. Dugdale, *Triangle contractive self maps of a Hilbert space*, Fund. Math. 83 (1974), pp. 187–195.

[3] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977), pp. 257–290.

INDIANA UNIVERSITY

Accepté par la Rédaction le 24. 11. 1975

Partitions of pairs of reals

by

Alan D. Taylor (Schenectady, N. Y.)

Abstract. We prove that there is essentially only one simple counterexample to the partition relation $2^{\aleph_0}\to(2^{\aleph_0})^2_{\aleph_0}$. The partition relation $2^{\aleph_0}\to(2^{\aleph_0})^2_{\aleph_1}$ is also considered, and some independence results concerning it are derived from some known independence results in set theory.

1. Introduction. Despite the title of this paper, we will primarily work with the set ${}^{\omega}2$ of all functions from the set $\omega=\{0,1,2,\ldots\}$ into the two element set $2=\{0,1\}$, rather than work with the real line R itself. The set ${}^{\omega}2$ can be endowed with a topology and a measure in a natural way be regarding it as a countable product of the two element set 2 where 2 is equipped with both the discrete topology and the probability measure that assigns both $\{0\}$ and $\{1\}$ measure one-half. By considering the binary expansion of a real number, it will be clear that all our results stated in terms of ${}^{\omega}2$ carry over to the real line R. We will also identify $[{}^{\omega}2]^2$ with $\{(x,y)\in{}^{\omega}2\times{}^{\omega}2\colon x<y\}$ where $<$ denotes the usual lexicographic ordering. This not only equips $[{}^{\omega}2]^2$ with a topology and a measure, but gives meaning to assertions such as "$A\times B\subseteq[X]^2$". For all relevant topological notions (e.g. analytic set, restricted property of Baire) we refer the reader to [4] or [5].

Our starting point is the following observation of Sierpiński. If we let $<$ be the usual ordering of R and $\circledless$ be a well ordering of R of type $2^{\aleph_0}$ and define $f\colon[R]^2\to 2$ by declaring that $f(\{x,y\})=0$ iff the two orderings agree on $\{x,y\}$, then there is no uncountable set $X\subseteq R$ that is *homogeneous* for f (i.e. such that f is constant on $[X]^2$). Thus, using the arrow notation of Erdös-Rado [2], this example shows that $2^{\aleph_0}\not\to(\aleph_1)^2_2$. Since this counterexample makes heavy use of the axiom of choice, it is natural to ask if it can be replaced by a constructive counterexample. Silver observed that by combining a special case of a theorem of Mycielski [7] with a special case of a theorem of Galvin (unpulished) one obtains the following.

THEOREM 1.1 (Galvin, Mycielski, Silver). *Suppose $f\colon[{}^{\omega}2]^2\to 2$ and $f^{-1}(\{i\})$ has the property of Baire for all $i<2$. Then there exists a perfect set $P\subseteq{}^{\omega}2$ that is homogeneous for f.*

This theorem was first brought to our attention by Baumgartner, who rediscovered it independently of the work of Galvin, Mycielski and Silver. It has since been rediscovered by Burgess [1] and probably by others as well. Actually, the Galvin–

Mycielski-Silver theorem is somewhat stronger than what we have stated in Theorem 1.1 in that it shows that there is essentially only one simple counterexample showing $2^{\aleph_0}\nrightarrow(2^{\aleph_0})^3_2$. However our concern here is with partitions of pairs, not triples.

Two immediate corollaries of Theorem 1.1 should be noted. First of all, if $f\colon [{}^\omega 2]^2\to 2$ is analytic (i.e. $f^{-1}(\{0\})$ is an analytic set) then there is a perfect set homogeneous for f. Secondly, in the Lévy-Solovay model [9] for ZF$+\sim$AC, (where all sets are measurable and have the property of Baire) every partition $f\colon [{}^\omega 2]^2\to 2$ is constant on all pairs from some perfect set $P\subseteq {}^\omega 2$. Although the proof of Theorem 1.1 requires no more choice than is known to be available in the Lévy–Solovay model, this last assertion can also be verified by using some fairly simple absoluteness techniques. These results suggest that there is no "simple" counterexample showing that $2^{\aleph_0}\nrightarrow(2^{\aleph_0})^2_2$.

On the other hand, there is a simple constructive counterexample showing that $2^{\aleph_0}\nrightarrow(2^{\aleph_0})^2_{\aleph_0}$. In fact, if one defines $\delta\colon [{}^\omega 2]^2\to\omega$ by $\delta(\{x,y\}) = \min(\{n\in\omega\colon x(n)\neq y(n)\})$ (i.e. the *discrepency* of x and y) then there is no homogeneous set for δ of cardinality three. The purpose of this paper is to show that this is essentially the only simple counterexample showing that $2^{\aleph_0}\nrightarrow(2^{\aleph_0})^2_{\aleph_0}$.

In Section 2 we consider "property of Baire partitions" of $[{}^\omega 2]^2$ and prove both Theorem 1.1 and a similar result concerning the partition relation $2^{\aleph_0}\to(2^{\aleph_0})^2_{\aleph_0}$. In Section 3 we consider "restricted property of Baire partitions", and obtain here a result that is then combined in Section 4 with another theorem of Mycielski yielding results analogous to those of Section 2 but for "measurable partitions" instead of "property of Baire partitions". In Section 5, we consider the possibility of extending these results to partitions of $[{}^\omega 2]^2$ into, say, $\aleph_1$ pieces where $2^{\aleph_0}>\aleph_1$. It turns out that, for example, the results of Section 2 generalize in this case iff the real line is not the union of $\aleph_1$ nowhere dense sets. Hence, the validity of these extensions is independent of ZFC$+2^{\aleph_0}>\aleph_1$. Section 6 contains a conjecture related to these results.

The results in this paper are from Chapter 3 of the author's doctoral dissertation [10] written under the supervision of Professor James E. Baumgartner, to whom we are grateful.

2. A property of Baire version of $2^{\aleph_0}\to(2^{\aleph_0})^2_{\aleph_0}$. Our goal in this section is to show that if $f\colon [{}^\omega 2]^2\to\omega$ and $f^{-1}(\{i\})$ has the property of Baire for all $i\in\omega$ then there is a perfect set $P\subseteq{}^\omega 2$ such that either f is constant on $[P]^2$ or else f induces the same equivalence relation on $[P]^2$ as does the discrepency partition $\delta\colon [{}^\omega 2]^2\to\omega$. The proof of this will require a series of lemmas. The first lemma is the well known sequential lemma (or fusion lemma) of Sacks, and an easy proof can be found in [3]. The next two lemmas are the special cases of Mycielski's theorem and Galvin's theorem needed to prove Theorem 1.1. For completeness, we include Baumgartner's proofs of these two lemmas. We will let Seq denote the set of all finite sequences of zeros and ones and if $s=\langle x_0,\dots,x_{n-1}\rangle$ then $s^\frown\varepsilon=\langle x_0,\dots,x_{n-1},\varepsilon\rangle$.

LEMMA 2.1 (Sacks). *Suppose that for each $s\in$ Seq P_s is a perfect set and that this "tree" $\{P_s\colon s\in \mathrm{Seq}\}$ of perfect sets satisfies the following:*

(a) *The diameter of P_s tends to 0 with increasing length of s.*

(b) *$P_{s^\frown 0}\cap P_{s^\frown 1}=0$ and $P_{s^\frown\varepsilon}\subseteq P_s$ for $\varepsilon\in\{0,1\}$ and $s\in$ Seq.*

Then the fusion $P=\bigcup_{f\in{}^\omega 2}\bigcap_{n\in\omega}P_{f\restriction n}$ of the tree $\{P_s\colon s\in\mathrm{Seq}\}$ is a perfect set.

LEMMA 2.2 (Mycielski [7]). *Suppose $A\subseteq[{}^\omega 2]^2$ and A is meager. Then there is a perfect set $P\subseteq{}^\omega 2$ such that $[P]^2\cap A=0$.*

Proof. Let $A=\bigcup_{i\in\omega}N_i$ where each N_i is nowhere dense and $N_i\subseteq N_{i+1}$ for all $i\in\omega$. Then it is easy to construct a tree $\{P_s\colon s\in\mathrm{Seq}\}$ of perfect sets satisfying (i)-(iii):

(i) For each $s\in$ Seq, p_s is a *Baire interval* $[p_s]=\{f\in{}^\omega 2\colon f\supseteq p_s\}$.

(ii) If $\mathrm{length}(s)=\mathrm{length}(t)=i$ and s lexicographically precedes t (i.e. $s(\delta(\{s,t\}))=0$) then $(P_s\times P_t)\cap N_i=0$.

(iii) Conditions (a) and (b) of Lemma 2.1.

If we let P be the fusion of the tree, then P is perfect by Lemma 2.1 and $[P]^2\cap A=0$.

LEMMA 2.3 (Galvin). *Let $P\subseteq{}^\omega 2$ be perfect and let $A\subseteq[{}^\omega 2]^2$ be such that A is open in the relative topology of $[P]^2$. Then there is a perfect set $Q\subseteq P$ such that either $[Q]^2\subseteq A$ or $[Q]^2\cap A=0$.*

Proof. Assume that there is no perfect set $Q\subseteq P$ such that $[Q]^2\cap A=0$. We will produce a tree $\{P_s\colon s\in\mathrm{Seq}\}$ of perfect sets satisfying (i)-(iii).

(i) $P_0=P$.

(ii) If s lexicographically precedes t and s is incomparable with t (i.e. $\exists x\in\mathrm{domain}(s)\cap\mathrm{domain}(t)\ (s(x)\neq t(x))$) then $P_s\times P_t\subseteq A$.

(iii) Conditions (a) and (b) of Lemma 2.1.

Given P_s we must produce $P_{s^\frown 0}$ and $P_{s^\frown 1}$. By hypothesis, there exists $\{x,y\}\in[P_s]^2\cap A$. Since A is open relative to $[P_s]^2$, there are sequences u and v in Seq such that $x\in[u]$, $y\in[v]$, $[u]\cap[v]=0$, length (u), length $(v)\geqslant$length(s), and $[P_s]^2\cap([u]\times[v])\subseteq A$. It is easy to see that this works and that the desired set Q can be realized as the fusion of this tree.

Notice that Theorem 1.1 is an immediate consequence of Lemmas 2.2 and 2.3. To see this, let $B=f^{-1}(\{0\})$ and choose $A\subseteq[{}^\omega 2]^2$ such that A is open and $A\Delta B$ is meager. By Lemma 2.2 there is a perfect set $P\subseteq{}^\omega 2$ such that $[P]^2\cap(A\Delta B)=0$, and by Lemma 2.3 there is a perfect set $Q\subseteq P$ such that $[Q]^2\subseteq A$ or $[Q]^2\cap A=0$. But then either $[Q]^2\subseteq B$ or $[Q]^2\cap B=0$ and Theorem 1.1 is proved.

LEMMA 2.4. *Suppose $P\subseteq{}^\omega 2$ is perfect and $f\colon [P]^2\to\omega$ is such that for each $i\in\omega$, $f^{-1}(\{i\})$ is open in the relative topology of $[P]^2$. Then there exists a perfect set $Q\subseteq P$ such that either (1) or (2) holds.*

(1) *If $\{x,y\},\{x',y'\}\in[Q]^2$ then $f(\{x,y\})=f(\{x',y'\})$.*

(2) *If $\{x,y\},\{x',y'\}\in[Q]^2$ then $f(\{x,y\})=f(\{x',y'\})$ iff $\delta(\{x,y\})=\delta(\{x',y'\})$.*

Proof. Assume that for every perfect set $Q \subseteq P$ there exists $\{x, y\}, \{x', y'\} \in [Q]^2$ such that $f(\{x, y\}) \neq f(\{x', y'\})$. We will simultaneously construct a tree $\{P_s: s \in \mathrm{Seq}\}$ of perfect sets and a tree $\{(n_s, \delta_s): s \in \mathrm{Seq}\}$ of pairs of natural numbers satisfying the following:

(i) If $\mathrm{length}(s) < \mathrm{length}(t)$ or $\mathrm{length}(s) = \mathrm{length}(t)$ and s lexicographically precedes t then $n_s < n_t$ and $\delta_s < \delta_t$.

(ii) If $x \in P_{s^\frown 0}$ and $y \in P_{s^\frown 1}$ then $f(\{x, y\}) = n_s$ and $\delta(\{x, y\}) = \delta_s$.

(iii) $\{P_s: s \in \mathrm{Seq}\}$ satisfies (a) and (b) of Lemma 2.1.

We let $P_0 = P$ and proceed inductively. Thus, assume that $P_t, P_{t^\frown 0}, P_{t^\frown 1}, n_t$ and δ_t have been defined for all t such that either $\mathrm{length}(t) < \mathrm{length}(s)$ or $\mathrm{length}(t) = \mathrm{length}(s)$ and t lexicographically precedes s. We will produce $P_{s^\frown 0}, P_{s^\frown 1}, n_s$ and δ_s. Let $\delta' = \max(\{\delta_t: \delta_t \text{ already defined}\})$. Choose $x \in P_s$ and let $P_s' = P_s \cap [x \restriction \delta'+1]$. Let $n' = \max(\{n_t: n_t \text{ already defined}\})$. Choose n_s such that $n_s > n_t$ and $f(\{x', y'\}) = n_s$ for some $\{x', y'\} \in [P_s']^2$. This is possible since otherwise we would have

$$[P_s']^2 \subseteq \bigcup \{f^{-1}(\{i\}): i < n'\}$$

and then repeated applications of Lemma 2.3 would yield a perfect set $Q \subseteq P_s'$ such that f is constant on $[Q]^2$ in contradiction to our assumption. Choose

$$\{x, y\} \in [P_s']^2 \cap f^{-1}(\{n_s\})$$

and choose sequences u and v such that

$$x \in [u], \quad y \in [v], \quad [u] \cap [v] = 0$$

and

$$[P]^2 \cap ([u] \times [v]) \subseteq f^{-1}(\{n_s\}).$$

Let $P_{s^\frown 0} = P_s' \cap [u]$ and let $P_{s^\frown 1} = P_s' \cap [v]$ and set $\delta_s = \delta(\{u, v\})$. Notice that $\delta_s > \delta'$ since $x \in [u]$, $y \in [v]$ and $y \in [x \restriction \delta+1]$. Thus we have $P_{s^\frown 0}, P_{s^\frown 1}, n_s$, and δ_s completing the construction.

Let Q be the fusion of the tree $\{P_s: s \in \mathrm{Seq}\}$. Notice that if $\{x, y\} \in [Q]^2$ then there is some $s = s(x, y) \in \mathrm{Seq}$ such that $x \in P_{s^\frown 0}$ and $y \in P_{s^\frown 1}$, so $\delta(\{x, y\}) = \delta_s$ and $f(\{x, y\}) = n_s$. Thus if $\{x, y\}, \{x', y'\} \in [Q]^2$ then $n_s = f(\{x, y\}) = f(\{x', y'\})$ iff $s(x, y) = s = s(x', y')$ iff $\delta(\{x, y\}) = \delta_s = \delta(\{x', y'\})$.

THEOREM 2.5. *If $f: [{}^\omega 2]^2 \to \omega$ and $f^{-1}(\{i\})$ has the property of Baire for all $i \in \omega$ then there is a perfect set $Q \subseteq {}^\omega 2$ such that either (1) or (2) holds.*

(1) *If $\{x, y\}, \{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x' y'\})$.*

(2) *If $\{x, y\}, \{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$.*

Proof. Choose open sets $\{A_n: n \in \omega\}$ such that $A_n \Delta f^{-1}(\{n\})$ is meager for all $n \in \omega$. Let $A = \bigcup \{A_n \Delta f^{-1}(\{n\}): n \in \omega\}$. Then A is meager so Lemma 2.2 guarantees the existence of a perfect set $P \subseteq {}^\omega 2$ such that $[P]^2 \cap A = 0$. Thus $f \restriction [P]^2$ is such that for each $n \in \omega$, $(f \restriction [P]^2)^{-1}(\{n\})$ is open in the relative topology of $[P]^2$. Choose $Q \subseteq P$ as guaranteed to exist by Lemma 2.4. Then clearly Q is the desired perfect set.

3. The restricted property of Baire. Notice that although Lemma 2.4 deals with a more restrictive class of partitions of $[{}^\omega 2]^2$ than does Theorem 2.5, it nevertheless has a stronger conclusion. Namely, one can not only find a perfect set of the desired kind, but can, in fact, find one contained in any preassigned perfect set. In general, this stronger conclusion does not hold for "property of Baire partitions".

Our goal in this section is to extend Lemma 2.4 to the case where $f^{-1}(\{i\})$ has the restricted property of Baire (i.e. $f^{-1}(\{i\}) \cap T$ has the property of Baire relative to the subspace T for every $T \subseteq [{}^\omega 2]^2$). This result will allow us to extend the proof of Theorem 2.5 so as to consider "measurable partitions" rather than "property of Baire partitions".

We begin with two technical lemmas necessary for the proof of Theorem 3.3.

LEMMA 3.1. *Let $P \subseteq {}^\omega 2$ be perfect. Then there is a perfect set $P' \subseteq P$ and a homeomorphism $\psi: {}^\omega 2 \to P'$ such that if $\delta(\{x, y\}) = n$, $x' \in [x \restriction n+1]$ and $y' \in [y \restriction n+1]$ then $\delta(\{\psi(x), \psi(y)\}) = \delta(\{\psi(x'), \psi(y')\})$.*

Proof. We construct a tree $\{P_s: s \in \mathrm{Seq}\}$ of perfect sets and a tree $\{n_s: s \in \mathrm{Seq}\}$ of natural numbers simultaneously by induction, starting with $P_0 = P$. Suppose $P_s = P \cap [t]$ has been constructed. We construct n_s, $P_{s^\frown 0}$, and $P_{s^\frown 1}$. Choose $\{x, y\} \in [P_s]^2$ and let $n_s = \delta(\{x, y\})$. Let

$$P_{s^\frown 0} = P_s \cap [x \restriction n_s+1] \quad \text{and} \quad P_{s^\frown 1} = P_s \cap [y \restriction n_s+1].$$

Notice that this tree of perfect sets satisfies conditions (a) and (b) of Lemma 2.1. Let P' be the fusion of the tree $\{P_s: s \in \mathrm{Seq}\}$, and define $\psi: {}^\omega 2 \to P'$ by letting $\psi(x)$ be the unique element in $\bigcap \{P_{x \restriction n}: n \in \omega\}$. To see that this works, suppose $\delta(\{x, y\}) = n$ and $x' \in [x \restriction n+1]$ and $y' \in [y \restriction n+1]$. Then $\psi(x), \psi(x') \in P_{x \restriction n+1}$ and $\psi(y), \psi(y') \in P_{y \restriction n+1}$. Let $s = x \restriction n = y \restriction n$. Then

$$\delta(\{\psi(x), \psi(y)\}) = n_s = \delta(\{\psi(x'), \psi(y')\}),$$

since $\delta(\{x, y\}) = n_s$ for any $x \in P_{s^\frown 0}$ and $y \in P_{s^\frown 1}$. Thus P' and ψ are as desired.

LEMMA 3.2. *Let $\psi: {}^\omega 2 \to P'$ be as in Lemma 3.1, and let Q' be an arbitrary perfect subset of ${}^\omega 2$. Then there exists a perfect set $Q'' \subseteq Q'$ such that if $\{x, y\}, \{x', y'\} \in [Q'']^2$ then $\delta(\{x, y\}) = \delta(\{x', y'\})$ iff*

$$\delta(\{\psi(x), \psi(y)\}) = \delta(\{\psi(x'), \psi(y')\}).$$

Proof. Define $g: [Q']^2 \to \omega$ by $g(\{x, y\}) = \delta(\{\psi(x), \psi(y)\})$. Notice that Lemma 3.1 guarantees that $g^{-1}(\{i\})$ is open for every $i \in \omega$. Hence Lemma 2.4 applies to yield a perfect set $Q'' \subseteq Q'$ such that either g is constant on $[Q'']^2$ or else $g(\{x, y\}) = g(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$ for all $\{x, y\}, \{x', y'\} \in [Q'']^2$. The first case can clearly not occur and the second case is exactly what was desired.

THEOREM 3.3. *Suppose $P \subseteq {}^\omega 2$ is perfect and $f: [{}^\omega 2]^2 \to \omega$ is such that $f^{-1}(\{i\})$ has the restricted property of Baire for all $i \in \omega$. Then there is a perfect set $Q \subseteq P$ such that either (1) or (2) holds.*

(1) *If $\{x, y\}, \{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$.*

(2) *If $\{x, y\}, \{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$.*

Proof. Choose $P' \subseteq P$ and ψ: ${}^{\omega}2 \to P'$ as in Lemma 3.1. Define ψ': $[{}^{\omega}2]^2 \to [P']^2$ by $\psi'(\{x, y\}) = \{\psi(x), \psi(y)\}$, and notice that ψ' is also a homeomorphism. Now consider the partition f': $[{}^{\omega}2]^2 \to \omega$ where f' is the composition of f and ψ'. Then $(f')^{-1}(\{i\}) = (f \circ \psi')^{-1}(\{i\}) = (\psi')^{-1} \circ f^{-1}(\{i\})$, so $(f')^{-1}(\{i\})$ has the property of Baire for all $i \in \omega$ (since $f^{-1}(\{i\})$ has the property of Baire with respect to the subspace P'). By Theorem 2.5 there exists a perfect set $Q' \subseteq {}^{\omega}2$ such that either (i) or (ii) holds.

(i) If $\{x, y\}$, $\{x', y'\} \in [Q']^2$ then $f'(\{x, y\}) = f'(\{x', y'\})$.

(ii) If $\{x, y\}, \{x', y'\} \in [Q']^2$ then $f'(\{x, y\}) = f'(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$.

Choose $Q'' \subseteq Q'$ as guaranteed to exist in Lemma 3.2. Finally, let $Q = \psi(Q'')$. Then $Q \subseteq P$ and Q is a perfect set in P and hence a perfect set in ${}^{\omega}2$.

If f' is constant on $[Q']^2$ (i.e. condition (i) holds) then clearly f is constant on $[Q]^2$ and we are done. On the other hand, suppose f' and Q' satisfy (ii). Let $\{\psi(x), \psi(y)\}$, $\{\psi(x'), \psi(y')\} \in [Q]^2$. Then

$$f(\{\psi(x), \psi(y)\}) = f(\{\psi(x'), \psi(y')\})$$

iff $f \circ \psi'(\{x, y\}) = f \circ \psi'(\{x', y'\})$ iff $f'(\{x, y\}) = f'(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$ iff $\delta(\{\psi(x), \psi(y)\}) = \delta(\{\psi(x'), \psi(y')\})$, where the last equivalence follows from the fact that $\{x, y\}$, $\{x', y'\} \in [Q'']^2$. Hence Q satisfies Condition (2) of the theorem and the proof is complete.

4. A measurable version of $2^{\aleph_0} \to (2^{\aleph_0})^2_{\aleph_0}$. The first step in extending (the proof of) Theorem 2.5 to the case where f: $[{}^{\omega}2]^2 \to \omega$ is a "measurable partition" (i.e. $f^{-1}(\{i\})$ is measurable for all $i \in \omega$) is to obtain the suitable analogue of Lemma 2.2. Fortunately, Mycielski has provided us with this also.

Lemma 4.1. (Mycielski [6]). *Suppose $A \subseteq [{}^{\omega}2]^2$ and A has measure zero. Then there is a perfect set $P \subseteq {}^{\omega}2$ such that $[P]^2 \cap A = 0$.*

The next step is to obtain an extension of Lemma 2.4 that considers partitions f: $[{}^{\omega}2]^2 \to \omega$ where $f^{-1}(\{i\})$ is, say, a G_δ subset of $[{}^{\omega}2]^2$ for every $i \in \omega$. However, Theorem 3.3 will certainly suffice for this since every Borel set has the restricted property of Baire. Thus, we obtain the following.

Theorem 4.2. *If f: $[{}^{\omega}2]^2 \to \omega$ and $f^{-1}(\{i\})$ is measurable for every $i \in \omega$, then there is a perfect set $Q \subseteq {}^{\omega}2$ such that either (1) or (2) holds.*

(1) *If $\{x, y\}$, $\{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$.*

(2) *If $\{x, y\}, \{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$.*

Proof. Choose Borel sets $\{A_n: n \in \omega\}$ such that $A_n \Delta f^{-1}(\{n\})$ is of measure zero for each $n \in \omega$. Let $A = \bigcup \{A_n \Delta f^{-1}(\{n\}): n \in \omega\}$. Then A is of measure zero so Lemma 4.1 guarantees the existence of a perfect set $P \subseteq {}^{\omega}2$ such that $[P]^2 \cap A = 0$. Thus $f \restriction [P]^2$ is such that for each $i \in \omega$ $(f \restriction [P]^2)^{-1}(\{i\})$ is Borel, and hence has the restricted property of Baire. Let Q be a perfect subset of P as guaranteed to exist in Theorem 3.3. Then clearly Q is the desired set.

5. Some versions of $2^{\aleph_0} \to (2^{\aleph_0})^2_{\varkappa}$ for $\aleph_0 < \varkappa < 2^{\aleph_0}$. A natural question at this point is whether or not Theorems 2.5 and 4.2 can be generalized to hold for partitions of $[{}^{\omega}2]^2$ into $\varkappa$ pieces where $\varkappa < 2^{\aleph_0}$. (Of course this is interesting only in the absence of the continuum hypothesis.) To facilitate this discussion, we will let $B(\varkappa)$ and $M(\varkappa)$ denote the following assertions:

$B(\varkappa)(M(\varkappa))$: If f: $[{}^{\omega}2]^2 \to \varkappa$ and $f^{-1}(\{i\})$ has the property of Baire (is measurable) for all $i \in \omega$, then there is a perfect set $Q \subseteq {}^{\omega}2$ such that either (1) or (2) holds:

(1) If $\{x, y\}$, $\{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$.

(2) If $\{x, y\}, \{x', y'\} \in [Q]^2$ then $f(\{x, y\}) = f(\{x', y'\})$ iff $\delta(\{x, y\}) = \delta(\{x', y'\})$.

Our goal in this section is to show that $B(\varkappa)$ holds iff the real line is not the union of $\varkappa$ meager sets and $M(\varkappa)$ holds iff the real line is not the union of $\varkappa$ sets of measure zero. We will consider $B(\varkappa)$ and $M(\varkappa)$ simultaneously, since this is most expedient and should cause no confusion. The proof requires two lemmas.

Lemma 5.1. (i) *${}^{\omega}2$ is the union of $\varkappa$ meager sets ($\varkappa$ sets of measure zero) iff $[{}^{\omega}2]^2$ is the union of $\varkappa$ meager sets ($\varkappa$ sets of measure zero).*

(ii) *If ${}^{\omega}2$ is not the union of $\varkappa$ meager sets ($\varkappa$ sets of measure zero) then the union of $\varkappa$ meager sets ($\varkappa$ sets of measure zero) is either meager (of measure zero) or does not posses the property of Baire (is not measurable).*

(iii) *Suppose ${}^{\omega}2$ is not the union of $\varkappa$ meager sets ($\varkappa$ sets of measure zero). Let $A \subseteq [{}^{\omega}2]^2$ be such that A has the property of Baire (is measurable) and can be written as the union of $\varkappa$ meager sets (sets of measure zero). Then A is meager (of measure zero).*

The facts listed in Lemma 5.1 are generally well known and easy to verify. Detailed proofs can be found in [10].

Lemma 5.2. *Suppose f: $[{}^{\omega}2]^2 \to 2$ and for every perfect set $P \subseteq {}^{\omega}2$ there exists $\{x, y\}$, $\{x', y'\} \in [P]^2$ such that $f(\{x, y\}) \neq f(\{x', y'\})$. Then for every perfect set $P \subseteq {}^{\omega}2$ there exists $\{x, y\}, \{x', y'\} \in [P]^2$ such that $f(\{x, y\}) \neq f(\{x', y'\})$ but $\delta(\{x, y\}) = \delta(\{x', y'\})$.*

Proof. Suppose f: $[{}^{\omega}2]^2 \to 2$ and $P \subseteq {}^{\omega}2$ is a perfect set such that whenever $\{x, y\}$, $\{x', y'\} \in [P]^2$ and $\delta(\{x, y\}) = \delta(\{x', y'\})$ then $f(\{x, y\}) = f(\{x', y'\})$. Then $A = f^{-1}(\{0\})$ is open in the relative topology of $[P]^2$, so we can appeal to Lemma 2.3 and get a perfect set $Q \subseteq P$ which is homogeneous for f. This completes the proof.

Theorem 5.3. *For any cardinal $\varkappa$, we have the following:*

(i) *$B(\varkappa)$ holds iff ${}^{\omega}2$ is not the union of $\varkappa$ meager sets.*

(ii) *$M(\varkappa)$ holds iff ${}^{\omega}2$ is not the union of $\varkappa$ sets of measure zero.*

Proof. We will prove both (i) and (ii) simultaneously. Supose first that ${}^{\omega}2$ is the union of $\varkappa$ meager sets (sets of measure zero). Then by Lemma 5.1 (i), $[{}^{\omega}2]^2 = \bigcup \{M_\alpha: \alpha < \varkappa\}$ where each M_α is meager (of measure zero). Let $[{}^{\omega}2]^2 = S_0 \cup S_1$ be Sierpiński's counterexample showing that $2^{\aleph_0} \nrightarrow (\aleph_1)^2_2$. Let f: $[{}^{\omega}2]^2 \to \varkappa$ be any function which induces the partition $[{}^{\omega}2]^2 = \bigcup \{M_\alpha \cap S_i: \alpha < \varkappa$ and $i < 2\}$. We claim that f is a counterexample to $B(\varkappa)$ $(M(\varkappa))$. Notice first that for all $i < \varkappa$, $f^{-1}(\{i\})$ is contained in some M_β so it is meager (of measure zero) and hence has the property of Baire (is measurable). If $P\subseteq{}^\omega 2$ is any perfect set then it is clear that f is not constant on $[P]^2$ since we clearly cannot have $[P]^2\subseteq S_i$. Suppose that, on the other hand, $P\subseteq{}^\omega 2$ is a perfect set such that

$$f(\{x,y\})=f(\{x',y'\}) \quad \text{iff} \quad \delta(\{x,y\})=\delta(\{x',y'\})$$

whenever $\{x,y\},\{x',y'\}\in[P]^2$. Then, by Lemma 5.2, there exists $\{x,y\},\{x',y'\}\in[P]^2$ such that $\{x,y\}\in S_0$ and $\{x',y'\}\in S_1$ but $\delta(\{x,y\})=\delta(\{x',y'\})$. Thus $f(\{x,y\})\neq f(\{x',y'\})$ but $\delta(\{x,y\})=\delta(\{x',y'\})$, and this is a contradiction.

Conversely, suppose that ${}^\omega 2$ is not the union of $\varkappa$ meager sets (sets of measure zero). Let $f\colon [{}^\omega 2]^2\to\varkappa$. For each $\alpha<\varkappa$ let A_α be a Borel set such that $A_\alpha\Delta f^{-1}(\{\alpha\})$ is meager (of measure zero). Let $A=\{\alpha<\varkappa\colon A_\alpha$ is not meager (not of measure zero)$\}$. Notice that $A\neq 0$ since

$$[{}^\omega 2]^2=\bigcup_{\alpha<\varkappa}(A_\alpha\Delta f^{-1}(\{\alpha\}))\cup\bigcup_{\alpha<\varkappa}A_\alpha$$

and Lemma 5.1 (i) shows that $[{}^\omega 2]^2$ is not the union of $\varkappa$ meager sets (sets of measure zero). But it is also easy to see that A is at most countable since $A_\alpha\cap A_\beta$ is meager (of measure zero) if $\alpha\neq\beta$. Let $B=\bigcup\{A_\alpha\colon \alpha\in A\}$. Then B is a Borel set so $[{}^\omega 2]^2-B$ has the property of Baire (is measurable). But

$$[{}^\omega 2]^2-B\subseteq\bigcup\{A_\alpha\Delta f^{-1}(\{\alpha\})\colon \alpha\in A\}\cup\bigcup\{f^{-1}(\{\alpha\})\colon \alpha\notin A\}$$

and so it is a union of $\varkappa$ meager sets (sets of measure zero). Thus, by Lemma 5.1 (iii), $[{}^\omega 2]^2-B$ is meager (of measure zero), so Lemma 2.2 (Lemma 4.1) guarantees the existence of a perfect set $P\subseteq{}^\omega 2$ such that $[P]^2\subseteq B$. The result now follows immediately from Theorem 3.3.

Corollary 5.4. *The following assertions are all consistent with* $\mathrm{ZFC}+2^{\aleph_0}=\aleph_2$:

(i) $B(\aleph_1)$ *and* $M(\aleph_1)$,
(ii) $\neg B(\aleph_1)$ *and* $M(\aleph_1)$,
(iii) $B(\aleph_1)$ *and* $\neg M(\aleph_1)$,
(iv) $\neg B(\aleph_1)$ *and* $\neg M(\aleph_1)$.

Proof. These follow immediately from the known independence results concerning the equivalent assertions given in Theorem 5.3. In particular, if M is a countable transitive model of $\mathrm{ZFC}+\mathrm{V}=\mathrm{L}$ then the appropriate models to consider are certain generic extensions $M[G]$ of M where $M[G]$ is (respectively);

(i) a Martin's axiom extension of M where $2^{\aleph_0}=\aleph_2$,
(ii) the result of adding $\aleph_2$ random reals to M,
(iii) the result of adding $\aleph_2$ Cohen reals to M,
(iv) the result of adding $\aleph_2$ Sacks reals to M.

6. Conjecture. In closing, we mention one problem that is very closely connected with the problems considered in this paper (especially Section 5), but which does not seem to be very amenable to the same techniques that handled the others. We state this precisely as a conjecture.

Conjecture. *If E is an equivalence relation on $[{}^\omega 2]^2$ and E is an analytic subset of $[{}^\omega 2]^2\times[{}^\omega 2]^2$ and E has fewer than $2^{\aleph_0}$ equivalence classes, then there exists a perfect set $P\subseteq{}^\omega 2$ such that either* (1) *or* (2) *holds.*

(1) *If* $\{x,y\},\{x',y'\}\in[P]^2$ *then* $\{x,y\}E\{x',y'\}$.
(2) *If* $\{x,y\},\{x',y'\}\in[P]^2$ *then* $\{x,y\}E\{x',y'\}$ *iff* $\delta(\{x,y\})=\delta(\{x',y'\})$.

Of course this is consistent with ZFC (or $\mathrm{ZFC}+2^{\aleph_0}>\aleph_1$). Moreover, if "analytic" is replaced by "coanalytic" then the resulting proposition follows from Theorem 3.3 and the theorem of Silver [8] that asserts that any coanalytic equivalence relation on ${}^\omega 2$ with fewer than $2^{\aleph_0}$ equivalence classes has at most countably many equivalence classes.

References

[1] J. Burgess, *A Ramsey theorem*, Notices Amer. Math Soc. 161 (1975), A391.
[2] P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. 62 (1956), pp. 427–489.
[3] T. Jech, *Lectures in set theory with particular emphasis on the method of forcing*, Lecture Notes in Math. 217, Berlin 1971.
[4] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[5] — and A. Mostowski, *Set Theory*, Amsterdam 1968.
[6] J. Mycielski, *Algebraic independence and measure*, Fund. Math. 61 (1967), pp. 165–169.
[7] — *Independent sets in topological algebras*, Fund. Math. 55 (1964), pp. 139–147.
[8] J. Silver, *On π_1^1 equivalence relations*, handwritten notes.
[9] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), pp. 1–56.
[10] A. Taylor, *Some results in partition theory*, Doctoral Dissertation, Dartmouth College 1975.

DEPARTMENT OF MATHEMATICS
UNION COLLEGE
Schenectady, New York

Accepté par la Rédaction le 19. 12. 1975