COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1978 FASC. 2

REMARKS ON STABLE MEASURES ON BANACH SPACES

BY

Z. JUREK AND K. URBANIK (WROCŁAW)

This study continues the investigations on stable probability measures on Banach spaces started in [6] by Kumar and Mandrekar. Our aim is to give a representation of the characteristic functional of stable measures on Banach spaces analogous to that established by Kuelbs for Hilbert spaces ([5], Theorem 2.1, see also [4]).

Let X denote a real separable Banach space with the norm $\|\cdot\|$ and with the dual space X^*. By $\langle\cdot,\cdot\rangle$ we shall denote the dual pairing between X and X^*. By a *probability measure* μ on X we shall understand a countably additive non-negative set function μ on the class of Borel subsets of X with the property $\mu(X) = 1$. The characteristic functional of μ is defined by the formula

$$\hat{\mu}(y) = \int_X e^{i\langle y,x\rangle}\mu(dx) \qquad (y \in X^*).$$

It is well known that the characteristic functional determines uniquely the probability measure. Further, for two probability measures μ and ν on X, we shall denote by $\mu * \nu$ the convolution of μ and ν. Given μ, by μ^s we shall denote the *symmetrization* of μ, i.e. the probability measure $\mu * \mu'$, where $\mu'(E) = \mu(-E)$ for every Borel subset E of X. Clearly, $\hat{\mu}^s = |\hat{\mu}|^2$. For every $x \in X$, δ_x will denote the probability measure concentrated at the point x. By R and R^+ we shall denote the space of real numbers and positive real numbers, respectively. For any $a \in R^+$, $T_a\mu$ is defined to be the measure on X given by $T_a\mu(E) = \mu(a^{-1}E)$ for every Borel subset E of X. Clearly,

$$\widehat{T_a\mu}(y) = \hat{\mu}(ay) \qquad \text{for } y \in X^*.$$

We say that μ is a *stable probability measure* on X if for each pair $a, b \in R^+$ there exist a number $c \in R^+$ and an element $x \in X$ such that

$$T_a\mu * T_b\mu = T_c\mu * \delta_x.$$

According to Kumar and Mandrekar ([6], Corollary 2.12) the class of stable probability measures on X coincides with the limit laws of normed sums of independent identically distributed X-valued random variables. Every stable probability measure μ on X is infinitely divisible, i.e. for every positive integer n there exists a probability measure μ_n on X such that $\mu = \mu_n^{*n}$, where the power is taken in the sense of convolution ([6], Corollary 2.13). Hence, in particular, it follows that $\hat{\mu}(y) \neq 0$ for all $y \in X^*$. Moreover, there exist a positive constant p and a function z from $R^+ \times R^+$ into X such that

$$(1) \qquad \hat{\mu}(ay)\hat{\mu}(by) = e^{i\langle y, z(a,b)\rangle}\hat{\mu}\big((a^p+b^p)^{1/p}y\big)$$

for all $a, b \in R^+$ and $y \in X^*$ ([6], Lemma 2.6).

The Tortrat representation of infinitely divisible laws is a crucial step in our considerations. We recall that for any bounded non-negative Borel measure F on X vanishing at 0 the *Poisson measure* $e(F)$ associated with F is defined as

$$e(F) = e^{-F(X)}\sum_{k=1}^{\infty}\frac{1}{k!}F^{*k}, \qquad \text{where } F^{*0} = \delta_0.$$

Let M be a not necessarily bounded Borel measure on X vanishing at 0. If there exists a representation

$$M = \sup_n F_n,$$

where F_n are bounded and the sequence $\{e(F_n)\}$ of associated Poisson measures is shift compact, then each cluster point of translates of $\{e(F_n)\}$ will be called a *generalized Poisson measure* and will be denoted by $\tilde{e}(M)$. The measure $\tilde{e}(M)$ is defined uniquely up to a shift transformation, i.e. for two cluster points, say μ_1 and μ_2, of translates of $\{e(F_n)\}$ there exists an element $x \in X$ such that $\mu_1 = \mu_2 * \delta_x$ ([9], p. 313). It is clear that the measure M is finite outside every neighborhood of 0 and $\tilde{e}(M_1) = \tilde{e}(M_2)$ implies $M_1 = M_2$. The set of all measures M for which $\tilde{e}(M)$ exists will be denoted by $\mathscr{M}(X)$. By a *Gaussian measure* on X we mean a measure ϱ such that for every $y \in X^*$ the induced measure $y\varrho$ on R is Gaussian. Tortrat proved in [9], p. 311 (see also [1], p. 22), the following analogue of the Lévy-Khinchine representation of infinitely divisible laws: each infinitely divisible measure μ on X has a unique representation $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure on X and $M \in \mathscr{M}(X)$.

LEMMA. *Let μ be a stable measure on X. Then either μ is Gaussian or $\mu = \tilde{e}(M)$ and there exists a constant p $(0 < p < 2)$ such that $T_a M = a^p M$ for every $a \in R^+$.*

Proof. Given $y \in X^*$, we put $f(a) = \log|\hat{\mu}(a^{1/p}y)|$ $(a \in R^+)$, where the constant p is defined by formula (1). Clearly, f is continuous on R^+

and, by (1), $f(a+b) = f(a)+f(b)$ for every pair $a, b \in R^+$. Thus $f(a) = af(1)$, which implies

$$|\hat{\mu}(ty)|^2 = \exp[c(y)|t|^p] \quad (t \in R), \tag{2}$$

where $c(y) = 2\log|\hat{\mu}(y)|$. Since for measures concentrated at a single point the assertion of the Lemma is obvious, we may assume that $\mu \neq \delta_x$ for all $x \in X$. Then there exists a functional $y_0 \in X^*$ with the property $|\hat{\mu}(y_0)| < 1$ and, consequently, $c(y_0) < 0$. The characteristic function of the symmetrization of the induced measure $y_0\mu$ on R is equal to $|\hat{\mu}(ty_0)|^2$. Taking into account (2) we infer that $(y_0\mu)^s$ is a non-degenerate stable measure on R. Hence it follows that $0 < p \leqslant 2$ ([7], p. 327). If $p = 2$, then, by (2), for any $y \in X^*$ the symmetrization of $y\mu$ is Gaussian. Hence it follows, by the Cramer decomposition theorem ([7], p. 271), that $y\mu$ is Gaussian for every $y \in X^*$. In other words, μ is Gaussian. Suppose now that $0 < p < 2$ and take the Tortrat representation $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure. If $\varrho \neq \delta_0$, then there exists a functional $y_1 \in X^*$ with the property $\hat{\varrho}(y_1) < 1$. Then the symmetrization of $y_1\mu$ contains a non-degenerate Gaussian component, which contradicts (2) since stable laws with exponent $p < 2$ on R have no Gaussian component. Thus $\varrho = \delta_0$ and, consequently, $\mu = \tilde{e}(M)$. Taking into account (1) we obtain the equation

$$T_aM + T_bM = T_cM, \tag{3}$$

where $c = (a^p+b^p)^{1/p}$ and $a, b \in R^+$. Let E be a Borel subset of X such that 0 does not belong to the closure of E. Then $T_aM(E)$ is finite for all $a \in R^+$. Put $g(a) = T_{a^{1/p}}M(E)$ $(a \in R^+)$. The function g is non-negative and, by (3), fulfils the equation

$$g(a+b) = g(a)+g(b) \quad \text{for all } a, b \in R^+.$$

Thus g is monotone non-decreasing and, consequently, $g(a) = ag(1)$, which implies $T_aM(E) = a^pM(E)$. The Lemma is thus proved.

Now we are ready to prove the main result of the present paper.

Theorem 1. *Let μ be a probability measure on a real separable space X. Then μ is stable if and only if either μ is Gaussian or there exist a constant p $(0 < p < 2)$, a finite measure γ on the unit sphere S of X, and an element $x_0 \in X$ such that, for every $y \in X^*$,*

$$\hat{\mu}(y) = \exp\left[i\langle y, x_0\rangle - \int_S \left(1 + \frac{i\langle y, u\rangle}{|\langle y, u\rangle|} h(u, y, p)\right)|\langle y, u\rangle|^p \gamma(du)\right], \tag{4}$$

where

$$h(u, y, p) = \begin{cases} \tan\dfrac{\pi p}{2} & \text{if } p \neq 1, \\[2ex] \dfrac{2}{\pi}\log|\langle y, u\rangle| & \text{if } p = 1. \end{cases}$$

Proof. Suppose that either the characteristic functional of a probability measure μ on X admits representation (4) or μ is Gaussian. Then, after some computation, we have equation (1) which implies the stability of μ. This proves the sufficiency.

To prove the necessity it suffices, by the Lemma, to consider stable measures of the form $\mu = \tilde{e}(M)$, where $T_a M = a^p M$ $(a \in R^+)$ for a certain p $(0 < p < 2)$. We define a finite measure γ on S by

$$\gamma(U) = pM\left\{x: \frac{x}{\|x\|} \in U,\ \|x\| \geqslant 1\right\}$$

for Borel subsets U of S. It is easy to check the formula

$$T_a M\left\{x: \frac{x}{\|x\|} \in U,\ \|x\| \geqslant a\right\} = M\left\{x: \frac{x}{\|x\|} \in U,\ \|x\| \geqslant 1\right\} \quad \text{for all } a \in R^+.$$

Thus

$$M\left\{x: \frac{x}{\|x\|} \in U,\ \|x\| \geqslant a\right\} = \frac{a^{-p}}{p}\gamma(U)$$

and, consequently,

$$M\left\{x: \frac{x}{\|x\|} \in U,\ a \leqslant \|x\| < b\right\} = \gamma(U)\int_a^b \frac{dt}{t^{p+1}}.$$

This shows that

$$M(E) = \int_S \int_0^\infty 1_E(tu)\frac{dt}{t^{p+1}}\gamma(du) \tag{5}$$

for any Borel subset E of X. Here 1_E denotes the indicator of E. Further, from the Dettweiler representation of the characteristic functionals of infinitely divisible measures on X ([1], p. 27) we get the formula

$$\hat{\mu}(y) = \exp\left[i\langle y, x_1\rangle + \int_X K(x, y)\, M(dx)\right],$$

where $x_1 \in X$ and

$$K(x, y) = e^{i\langle y, x\rangle} - 1 - i\langle y, x\rangle 1_B(x),$$

B being the unit ball in X. By a simple computation similar to that in the case of the real line ([7], p. 329-330) for every $u \in S$ and $y \in X^*$ we get the formula

$$\int_0^\infty K(tu, y)\frac{dt}{t^{p+1}} = ic\langle y, u\rangle - \left(1 + \frac{\langle y, u\rangle}{|\langle y, u\rangle|}h(u, y, p)\right)|\langle y, u\rangle|^p,$$

where is a real constant. Combining this with (5) and setting

$$x_0 = x_1 + c \int_S u\gamma(du),$$

where the integral is taken in the Bochner sense, we obtain the required representation (4). The theorem is thus proved.

The measure γ appearing in representation (4) will be called the *representing measure* for μ. Let $\Gamma_p(X)$ denote the set of all representing measures corresponding to stable measures on X with the exponent p $(0 < p < 2)$. Clearly, $\gamma \in \Gamma_p(X)$ if and only if the measure M defined by formula (5) belongs to $\mathscr{M}(X)$. It is easy to check that the set $\mathscr{M}(X)$ has the following property: if N is a non-negative measure on X and $N \leqslant M$, where $M \in \mathscr{M}(X)$, then $N \in \mathscr{M}(X)$. Hence $\gamma \in \Gamma_p(X)$ if and only if the measure γ_0 defined by the formula $\gamma_0(E) = \gamma(E) + \gamma(-E)$ belongs to $\Gamma_p(X)$. This fact reduces the problem of determining $\Gamma_p(X)$ to examining symmetric measures γ. We say that *X is of type (q, r)* $(q \geqslant 0, r > 0)$ whenever there exists a positive constant c such that, for any collection $\xi_1, \xi_2, \ldots, \xi_n$ of independent symmetrically distributed X-valued random variables,

$$\mathrm{E}\Big\|\sum_{j=1}^{n} \xi_i\Big\|^r \leqslant cn^q \sum_{j=1}^{n} \mathrm{E}\,\|\xi_j\|^r.$$

It is obvious that each space X is of type $(0, r)$ with $r \leqslant 1$. Moreover, for $r \geqslant 1$ the spaces of type r considered by Hoffmann-Jørgensen [2] and by Maurey and Pisier [8] are identical with the spaces of type $(0, r)$.

THEOREM 2. *If X is of type (q, r) and $r > p(q+1)$, then $\Gamma_p(X)$ consists of all finite Borel measures on S.*

Proof. To prove the theorem it suffices to show that for each symmetric finite measure γ on X the measure M defined by (5) belongs to $\mathscr{M}(X)$. Put

$$I_0 = [1, \infty), \qquad I_k = [2^{-k}, 2^{-k+1}) \qquad (k = 1, 2, \ldots).$$

Then the measures

$$M_k(E) = \int_S \int_{I_k} 1_E(tu) \frac{dt}{t^{p+1}} \gamma(du) \qquad (k = 0, 1, \ldots)$$

are finite on X and vanish at 0. For simplicity of notation put $\mu_k = e(M_k)$ $(k = 0, 1, \ldots)$. Since

$$M = \sum_{k=0}^{\infty} M_k,$$

we conclude that $M \in \mathcal{M}(X)$ if and only if the sequence $\{\mu_0 * \mu_1 * \ldots * \mu_n\}$ converges to a probability measure on X or, equivalently, the series $\sum_{k=0}^{\infty} \eta_k$ of independent X-valued random variables $\eta_0, \eta_1, \ldots$ with probability distributions $\mu_0, \mu_1, \ldots$, respectively, converges almost surely ([3], Theorem 3.1). To prove the almost sure convergence of $\sum_{k=0}^{\infty} \eta_k$ it suffices, by the Borel-Cantelli Lemma, to show the convergence of the series

$$\sum_{k=0}^{\infty} \mu_k(\{x: \|x\| > a^k\}), \tag{6}$$

where $a = 2^{(r+1)^{-1}(pq-r+p)} < 1$. Setting $a_k = M_k(X)$ and $\nu_k = a_k^{-1} M_k$ $(k = 1, 2, \ldots)$, we have

$$\mu_k = \exp[-a_k] \sum_{n=0}^{\infty} \frac{a_k^n}{n!} \nu_k^{*n} \tag{7}$$

and

$$a_k = \gamma(S) \int_{J_k} \frac{dt}{t^{p+1}} = p^{-1}\gamma(S)\,(2^{kp} - 2^{(k-1)p}). \tag{8}$$

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent symmetrically distributed random variables with the common probability distribution ν_k. Then for a positive constant c we have

$$\begin{aligned} \int_X \|x\|^r \nu_k^{*n}(dx) &= \mathrm{E}\Big\|\sum_{j=1}^{n} \xi_j\Big\|^r \leqslant cn^q \sum_{j=1}^{n} \mathrm{E}\,\|\xi_j\|^r \\ &= cn^{q+1} a_k^{-1} \int_X \|x\|^r M_k(dx) \\ &= cn^{q+1}\gamma(S)\, a_k^{-1}(r-p)^{-1}(2^{-(k-1)(r-p)} - 2^{-k(r-p)}). \end{aligned}$$

Consequently, by (7),

$$\int_X \|x\|^r \mu_k(dx) \leqslant c_1 b_k 2^{-k(r-p)} \qquad (k = 1, 2, \ldots), \tag{9}$$

where c_1 is a positive constant and

$$b_k = \exp[-a_k] \sum_{n=1}^{\infty} \frac{a_k^{n-1} n^q}{(n-1)!}.$$

Since $t^{n-1}n^q \leqslant t^{n-1+q}$ for $t \geqslant n$, and

$$\frac{t^{n-1}n^q}{(n-1)!} \leqslant c_2 \frac{t^{n-[q]-2+q}}{(n-[q]-2)!} \quad \text{for } 0 < t < n,$$

where c_2 is a positive constant and $[q]$ denotes the integral part of q, we infer that

$$b_k \leqslant c_3(1+a_k^q) \quad (k = 1, 2, \ldots)$$

for a certain positive constant c_3. Consequently, by (8), there exists a constant c_4 such that

$$b_k \leqslant c_4 2^{kpq} \quad (k = 1, 2, \ldots).$$

Hence, by virtue of (9), we get the inequality

$$\int_X \|x\|^r \mu_k(dx) \leqslant c_5 a^{k(r+1)} \quad (k = 1, 2, \ldots)$$

with a constant c_5. Consequently,

$$\mu_k(\{x: \|x\| > a^k\}) \leqslant a^{-kr} \int_X \|x\|^r \mu_k(dx) \leqslant c_5 a^k \quad (k = 1, 2, \ldots),$$

which yields the convergence of series (6). This completes the proof of the theorem.

In particular, from Theorem 2 for $p < 1$ and every Banach space X as well for $1 \leqslant p < r$ and Banach spaces X of type r we get the description of $\Gamma_p(X)$. For $p \geqslant 1$ and an arbitrary X the problem whether $\Gamma_p(X)$ consists of all finite measures on S remains still open. (**P 1024**)

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

Reçu par la Rédaction le 9. 9. 1976