from H_j, K_n is partition realizable from each of the graphs formed below. Further, each of the following graphs has $\binom{n}{2}$ lines and hence each is n-minimal.

Consider a graph G_i. Let $P = \{V_1, V_2, \dots, V_{n-1}\}$ be a complete partition of $V(G_i)$. It is easy to see that each H_j has at least $n-1$ points of degree one, say u_r, for $r = 1, 2, \dots, n-1$.

Let G be a graph obtained from G_i and H_j by identifying some, all or none of the points u_r with the points of G_i such that, no two points u_r are identified with the points of the same set $V_i \in P$. We claim that any n-minimal graph is isomorphic to a graph obtained above. For, let G be n-minimal and $P(G) = K_n$. Then as K_{n-1} is an induced subgraph of K_n, there exists an induced subgraph of G_i', of G such that K_{n-1} is partition realizable from G_i' and $K_{1,n-1}$ is partition realizable from $G-G_i'$, by Corollary 4.1. Therefore, G_i' has $\binom{n-1}{2}$ lines and hence it is $(n-1)$-minimal and $G-G_i'$ has $n-1$ lines. Therefore G is isomorphic to one of the graphs obtained above.

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Continuous extenders in normal and collectionwise normal spaces

by

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Abstract. For a Banach space B and a space Z denote by $C^*(Z, B)$ the space of continuous, bounded mappings $\varphi\colon Z \to B$ with the sup-norm topology and by $P^*(Z)$ (resp. $P^*_\lambda(Z)$) the space of continuous, bounded (resp. bounded and λ-separable) pseudometrics $\varrho\colon Z\times Z \to \boldsymbol{R}$ on Z with the topology of the subspace of $C^*(Z\times Z) = C^*(Z\times Z, \boldsymbol{R})$.

THEOREM 1. *Let F be a closed subset of a λ-collectionwise normal space X and B a Banach space of weight $\leqslant \lambda$. There exist continuous extenders*:

$$e\colon C^*(F, B) \to C^*(X, B) \quad \text{and} \quad E\colon P^*_\lambda(F) \to P^*_\lambda(X)\,.$$

COROLLARY 1. *Let F be a closed subset of a collectionwise normal space X and let B be a Banach space. There exist continuous extenders*:

$$e\colon C^*(F, B) \to C^*(X, B) \quad \text{and} \quad E\colon P^*(F) \to P^*(X)\,.$$

COROLLARY 2. *Let F be a closed subset of a normal space X. There exist continuous extenders*:

$$e\colon C^*(F) \to C^*(X) \quad \text{and} \quad E\colon P^*_{\omega_0}(F) \to P^*_{\omega_0}(X)\,.$$

The above extenders are homeomorphic (but, in general, neither linear nor isometric) embeddings.

§ 1. Introduction. The symbol λ will always denote infinite cardinal number and $\boldsymbol{R}$ stands for the real line. A T_1-space X is *λ-collectionwise normal* if each discrete collection of cardinality $\leqslant\lambda$ of subsets of X can be separated by disjoint open sets. A space is *normal* if and only if it is ω_0-collectionwise normal (cf. [E]; Theorem 2.1.14).

For a Banach space B and a topological space Z, $C^*(Z, B)$ will denote the Banach space of all continuous, bounded functions $\varphi\colon Z \to B$ with the sup-norm $\|\varphi\| = \sup_{z\in Z} \|\varphi(z)\|$. If $B = \boldsymbol{R}$, then we write $C^*(Z)$ instead of $C^*(Z, \boldsymbol{R})$. A pseudo-

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metric ϱ on a space Z is *λ-separable* if the topology induced by ϱ on Z has a dense subset of cardinality $\leqslant\lambda$ or — equivalently — if its weight is $\leqslant\lambda$. By $P^*(Z)$ (resp. $P^*_\lambda(Z)$) we shall denote the space of all continuous, bounded (resp. bounded and λ-separable) pseudometrics $\varrho\colon Z\times Z\to \boldsymbol{R}$ on Z topologized as a subspace of $C^*(Z\times Z)$. One easily checks that $P^*(Z)$ and $P^*_\lambda(Z)$ are closed and convex subsets of $C^*(Z\times Z)$.

If A is a subset of X and if $T(A)$ and $T(X)$ are two sets of functions defined on A and X respectively, then we say that the function $e\colon T(A)\to T(X)$ is an *extender* if for each $\varphi\in T(A)$ we have $e(\varphi)|A=\varphi$, i.e. if $e(\varphi)$ is an extension of φ onto X.

THEOREM 1. *Let F be a closed subset of a λ-collectionwise normal space X and let B be a Banach space of weight $\leqslant\lambda$. There exist continuous extenders*:

$$e\colon C^*(F,B)\to C^*(X,B) \quad \textit{and} \quad E\colon P^*_\lambda(F)\to P^*_\lambda(X).$$

Moreover, the above extenders are homeomorphic embeddings onto closed subsets (in fact, retracts) of $C^*(X,B)$ and $P^*_\lambda(X)$, respectively.

COROLLARY 1. *Let F be a closed subset of a collectionwise normal space X and let B be a Banach space. There exist continuous extenders*:

$$e\colon C^*(F,B)\to C^*(X,B) \quad \textit{and} \quad E\colon P^*(F)\to P^*(X).$$

The following corollary gives an affirmative answer to the question — raised by R. W. Heath, D. J. Lutzer and P. L. Zenor in [HLZ] — whether there exists a continuous extender $e\colon C(\beta N\setminus N)\to C(\beta N)$.

COROLLARY 2. *Let F be a closed subset of a normal space X. There exist continuous extenders*:

$$e\colon C^*(F)\to C^*(X) \quad \textit{and} \quad E\colon P^*_{\omega_0}(F)\to P^*_{\omega_0}(X).$$

Though the above extenders are homeomorphic embeddings, they are, in general, neither linear nor isometric functions (See [DLP] for more information). The extenders e can be extended continuously onto the space of all continuous (not necessarily bounded) mappings considered with the topology of uniform convergence (see Section 3 for details). Also, instead of a Banach space B, one can consider an arbitrary complete absolute retract for metric spaces (see Section 3). Moreover, one can always assume that the norms of the extended mappings and pseudometrics are preserved.

Theorem 1 will be a consequence of a more general Theorem 2. Before stating Theorem 2 we have to recall the definition of a P^λ-embedded subset.

DEFINITION ([A], [S]). A subset A of a space X is *P-embedded* (resp. *P^λ-embedded*) if every continuous (resp. continuous λ-separable) pseudometric on A can be continuously extended onto X.

The following facts are known (see [P] for more information):

1. ([AS], [P]). A subset A of X is P-embedded (resp. P^λ-embedded) if and only if every continuous mapping $\varphi\colon A\to B$ into a Banach space (resp. a Banach space of weight $\leqslant\lambda$) can be continuously extended onto X.

2. ([G_1], [S]). A T_1-space X is collectionwise normal (resp. λ-collectionwise normal) if and only if its every closed subset F is P-embedded (resp. P^λ-embedded).

3. ([G]). A subset A of X is P^{ω_0}-embedded if and only if it is C-embedded, i.e. if every continuous real-valued function on A can be continuously extended onto X.

4. ([AS_1]). If A is P-embedded (resp. P^λ-embedded) in X and if Z is a compact space (resp. compact space of weight $\leqslant\lambda$), then $A\times Z$ is P-embedded (resp. P^λ-embedded) in $X\times Z$. In particular, if F is a closed subset of a collectionwise normal space X and Z is compact, then $F\times Z$ is P-embedded in $X\times Z$, even though the last space need not be, in general, even normal.

For a Banach space B and a space Z denote by $C^*_\lambda(Z,B)$ the subspace of $C^*(Z,B)$ consisting of all those functions $\varphi\colon Z\to B$ for which the weight of $\varphi(Z)$ is $\leqslant\lambda$. One can easily check that $C^*_\lambda(Z,B)$ is a Banach subspace of $C^*(Z,B)$.

Theorem 1 is an immediate consequence of the following more general Theorem 2, which will be proved in Section 2.

THEOREM 2. *Let A be a P^λ-embedded subset of a space X and let B be a Banach space. There exist continuous extenders*:

$$e\colon C^*_\lambda(A,B)\to C^*_\lambda(X,B) \quad \textit{and} \quad E\colon P^*_\lambda(A)\to P^*_\lambda(X).$$

Moreover, the above extenders are homeomorphic embeddings onto closed subsets (in fact, retracts) of $C^*_\lambda(X,B)$ and $P^*_\lambda(X)$, respectively.

COROLLARY 3. *Let A be a P-embedded subset of a space X and let B be a Banach space. There exist continuous extenders*:

$$e\colon C^*(A,B)\to C^*(X,B) \quad \textit{and} \quad E\colon P^*(A)\to P^*(X).$$

COROLLARY 4. *Let A be a C-embedded subset of a space X. There exist continuous extenders*:

$$e\colon C^*(A)\to C^*(X) \quad \textit{and} \quad E\colon P^*_{\omega_0}(A)\to P^*_{\omega_0}(X).$$

§ 2. **Proof of Theorem 2.** The existence of the extender e is a simple consequence of the Bartle–Graves theorem [BG]. Let $\Omega\colon C^*_\lambda(X,B)\to C^*_\lambda(A,B)$ be a continuous linear mapping defined by

$$\Omega(\psi)=\psi|A \quad \text{for} \quad \psi\in C^*_\lambda(X,B).$$

The mapping Ω is a surjection. Indeed, if $\varphi\colon A\to B$ belongs to $C^*_\lambda(A,B)$, then the weight of $\psi(A)$ is $\leqslant\lambda$ and $\varphi(A)$ is contained in a Banach subspace B^* of B of weight $\leqslant\lambda$. Since A is P^λ-embedded in X the mapping $\varphi\colon A\to B^*$ can be extended to a continuous mapping $\psi\colon X\to B^*\subset B$. Clearly $\psi\in C^*_\lambda(X,B)$ and $\Omega(\psi)=\varphi$.

Since Ω is a continuous linear surjection between Banach spaces, hence it follows from the Bartle–Graves theorem [BG] that there exists a continuous mapping (a cross-section)

$$e\colon C^*_\lambda(A, B)\to C^*_\lambda(X, B)$$

such that $e(\varphi)\in \Omega^{-1}(\varphi)$ for every $\varphi\in C^*_\lambda(A, B)$. Clearly e is a continuous extender.

Let us prove that e is a homeomorphic embedding (¹). Since e is an extender, it is one-to-one. If $\varphi_n\in C^*_\lambda(A, B)$ for $n<\omega$ and $e(\varphi_n)\underset{n}{\to} e(\varphi_0)$, then also

$$\varphi_n = e(\varphi_n)|A\underset{n}{\to} e(\varphi_0)|A = \varphi_0\,,$$

which means that the inverse mapping is also continuous. Moreover, $e\big(C^*_\lambda(A, B)\big)$ is a retract of $C^*_\lambda(X, B)$: the retraction $r\colon C^*_\lambda(X, B)\to e\big(C^*_\lambda(A, B)\big)$ can be defined by putting

$$r(\psi) = e(\psi|A)\,.$$

If $e(\varphi)\in e\big(C^*_\lambda(A, B)\big)$, then $r\big(e(\varphi)\big) = e\big(e(\varphi)|A\big) = e(\varphi)$. Therefore, e is a homeomorphic embedding onto a closed subset of $C^*_\lambda(X, B)$.

It remains to show that there exists a continuous extender $E\colon P^*_\lambda(A)\to P^*_\lambda(X)$.

Let $\Lambda\colon P^*_\lambda(A)\to C^*_\lambda(A, C^*(A))$ be a mapping defined by

$$[\Lambda(\varrho)(x)](y) = \varrho(x, y) \quad \text{for} \quad \varrho\in P^*_\lambda(A) \text{ and } x, y\in A\,.$$

The mapping is well-defined, because for each $\varrho\in P^*_\lambda(A)$ the weight of $\Lambda(\varrho)(A)$ is $\leqslant\lambda$. Indeed, if $\{x_\alpha\}_{\alpha<\lambda}$ is a dense subset of A considered with the topology induced by ϱ, then the set $\{\Lambda(\varrho)(x_\alpha)\}_{\alpha<\lambda}$ is a dense subset of $\Lambda(\varrho)(A)$. To show this, let $x\in A$, $\varepsilon>0$ and choose α such that $\varrho(x, x_\alpha)<\varepsilon$. Then

$$\begin{aligned}\|\Lambda(\varrho)(x_\alpha)-\Lambda(\varrho)(x)\| &= \sup_{y\in A}|[\Lambda(\varrho)(x_\alpha)](y)-[\Lambda(\varrho)(x)](y)|\\ &= \sup_{y\in A}|\varrho(x_\alpha, y)-\varrho(x, y)|\leqslant\varrho(x, x_\alpha)<\varepsilon\,.\end{aligned}$$

We shall prove that Λ is an isometric embedding. Let ϱ, $\delta\in P^*_\lambda(A)$. We have

$$\begin{aligned}\|\Lambda(\varrho)-\Lambda(\delta)\| &= \sup_{x\in A}\|\Lambda(\varrho)(x)-\Lambda(\delta)(x)\|\\ &= \sup_{x\in A}\sup_{y\in A}|[\Lambda(\varrho)(x)](y)-[\Lambda(\delta)(x)](y)|\\ &= \sup_{x\in A}\sup_{y\in A}|\varrho(x, y)-\delta(x, y)| = \|\varrho-\delta\|\,.\end{aligned}$$

We infer that the mapping Λ is continuous.

By the first part of the proof, there exists a continuous extender $e\colon C^*_\lambda(A, C^*(A))\to C^*_\lambda(X, C^*(A))$. Let $\Sigma\colon C^*_\lambda(X, C^*(A))\to P^*_\lambda(X)$ be defined by putting

$$\Sigma(\psi)(x, y) = \|\psi(x)-\psi(y)\| \quad \text{for} \quad \psi\in C^*_\lambda(X, C^*(A)) \text{ and } x, y\in X\,.$$

(¹) Since e is a cross-section it follows immediately from ([B], p. 7) that e is a homeomorphic embedding and that the mapping $e\circ\Omega$ is a retraction (referee's remark).

To verify that Σ is well-defined, we have to check, that for each $\psi\in C^*_\lambda(X, C^*(A))$ the pseudometric $\Sigma(\psi)$ is λ-separable. Let the set $\{\psi(x_\alpha)\}_{\alpha<\lambda}$ be dense in $\psi(X)$. We will show that the set $\{x_\alpha\}_{\alpha<\lambda}$ is dense in X considered with the topology introduced by $\Sigma(\psi)$. Let $x\in X$ and $\varepsilon>0$ and choose $\alpha<\lambda$ such that

$$\|\psi(x)-\psi(x_\alpha)\|<\varepsilon\,.$$

Then $\Sigma(\psi)(x, x_\alpha) = \|\psi(x)-\psi(x_\alpha)\|<\varepsilon$. Hence the definition of Σ is correct.

To show that Σ is continuous it suffices to prove that for each $\psi, \psi'\in C^*_\lambda(X, C^*(A))$ we have

$$\|\Sigma(\psi)-\Sigma(\psi')\|\leqslant 2\|\psi-\psi'\|\,.$$

Now,

$$\|\Sigma(\psi)-\Sigma(\psi')\| = \sup_{x,y\in X}|\Sigma(\psi)(x, y)-\Sigma(\psi')(x, y)|\,,$$

but

$$\begin{aligned}|\Sigma(\psi)(x, y)-\Sigma(\psi')(x, y)| &= \big|\|\psi(x)-\psi(y)\|-\|\psi'(x)-\psi'(y)\|\big|\\ &\leqslant \|\psi(x)-\psi(y)-\psi'(x)+\psi'(y)\|\\ &= \|\psi(x)-\psi'(x)+\psi'(y)-\psi(y)\|\\ &\leqslant \|\psi(x)-\psi'(x)\|+\|\psi(y)-\psi'(y)\|\\ &\leqslant 2\|\psi-\psi'\|\,.\end{aligned}$$

Define $E\colon P^*_\lambda(A)\to P^*(X)$ by

$$E = \Sigma\circ e\circ\Lambda\,.$$

It remains to show, that E is an extender. Let $\varrho\in P^*_\lambda(A)$. We will show that for each $x, y\in A$

$$E(\varrho)(x, y) = \varrho(x, y)\,.$$

$$\begin{aligned}E(\varrho)(x, y) &= \Sigma[e(\Lambda(\varrho))](x, y)\\ &= \|e(\Lambda(\varrho))(x)-e(\Lambda(\varrho))(y)\|\\ &= \|\Lambda(\varrho)(x)-\Lambda(\varrho)(y)\|\\ &= \sup_{z\in A}|[\Lambda(\varrho)(x)](z)-[\Lambda(\varrho)(y)](z)|\\ &= \sup_{z\in A}|\varrho(x, z)-\varrho(y, z)| = \varrho(x, y)\,.\end{aligned}$$

The proof, that the extender E is a homeomorphic embedding onto a retract of the space $P^*_\lambda(X)$ is similar to the proof of the analogous property of e and therefore will be omitted. This completes the proof of Theorem 2 (²).

(²) It does not seem obvious that the restriction operator $r\colon P^*_\lambda(X)\to P^*_\lambda(A)$ is open, though with some effort one can prove that this is the case. Having the openess of r the existence of a continuous extender $E\colon P^*_\lambda(A)\to P^*_\lambda(X)$ follows from Michael's selection theory (cf. [DLP]). Moreover, as the referee points out, every continuous extender $E'\colon F\to P^*_\lambda(X)$ defined on a closed subset F of $P^*_\lambda(A)$ can be then extended continuously onto $P^*_\lambda(A)$.

§ 3. Final remarks.

1. The existence of a continuous extender $e\colon C^*(F)\to C^*(X)$, where F is a closed subset of a normal space X (or, more generally, if F is a C^*-embedded subset of X), can be proved directly, without using of the Bartle–Graves theorem. For details see [DLP].

2. One can always assume that extenders e and E constructed in the proof of Theorem 2 preserve norms of mappings and pseudometrics. If e and E are arbitrary extenders, then one can define modified extenders $\tilde{e}$ and $\tilde{E}$ by putting:

$$\tilde{e}(\varphi)(x) = \begin{cases} e(\varphi)(x), & \text{if } \|e(\varphi)(x)\| \leqslant \|\varphi\|, \\ \dfrac{e(\varphi)(x)}{\|e(\varphi)(x)\|}\cdot\|\varphi\|, & \text{otherwise;} \end{cases}$$

and $\tilde{E}(\varrho)(x, y) = \min\{E(\varrho)(x, y), \|\varrho\|\}$, where $\varphi \in C^*(A, B)$, $\varrho \in P^*(A)$ and $x, y \in X$. The newly defined continuous extenders satisfy

$$\|\tilde{e}(\varphi)\| = \|\varphi\| \quad \text{and} \quad \|\tilde{E}(\varrho)\| = \|\varrho\| .$$

3. The space $C^*(Z, B)$ is a subspace of a metric space $C(Z, B)$ of *all* continuous mappings $\varphi\colon Z\to B$, considered with the topology of uniform convergence (see [E], Sec. 4.2). It turns out that an arbitrary continuous extender

$$e\colon\ C^*(A, B)\to C^*(X, B) .$$

can be extended continuously onto $C(A, B)$, giving rise to a continuous extender

$$\hat{e}\colon\ C(A, B)\to C(X, B) .$$

Let us first introduce an equivalence relation $\sim$ in $C(A, B)$ defined by

$$\varphi\sim\varphi' \quad \text{iff} \quad \varphi-\varphi' \in C^*(A, B) .$$

One easily checks that equivalence classes of $\sim$ are open and closed subsets of $C(A, B)$ and each class has the form $\varphi+C^*(A, B)$, where $\varphi \in C(A, B)$. Let $\{\varphi_s+C^*(A, B)\}_{s\in S}$ be the set of all different equivalence classes of $\sim$ and for each $s \in S$ choose an arbitrary extension $\tilde{\varphi}_s\colon X\to B$ of φ_s onto X. Define $\hat{e}$ by

$$\hat{e}(\varphi_s+\varphi) = \tilde{\varphi}_s+e(\varphi) \quad \text{for} \quad s\in S \text{ and } \varphi\in C^*(A, B) .$$

One can proceed similarly in case of extensors between $C^*_\lambda(A, B)$ and $C^*_\lambda(X, B)$.

4. Instead of a Banach space B in Theorem 2 one can take an arbitrary complete absolute retract for metric spaces M. It is known, that M can be embedded as a retract of a Banach space B of the same weight. Let $r\colon B\to M$ be a retraction and let $e\colon C^*_\lambda(A, B)\to C^*_\lambda(X, B)$ be an arbitrary continuous extender. One can define a continuous extender

$$\hat{e}\colon\ C^*_\lambda(A, M)\to C^*_\lambda(X, M)$$

by putting

$$\hat{e}(\varphi) = r\circ e(\varphi) \quad \text{for} \quad \varphi\in C^*_\lambda(A, M) .$$

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