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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

On spaces which have the shape of compact metric spaces

by

Tadashi Watanabe (Yamaguchi)

Abstract. In this paper we shall discuss topological spaces which have the shape of compact metric spaces. We shall give characterizations of such spaces and apply them to one of the questions raised by Mardešić.

§ 0. Introduction. The notion of shape was originally introduced by K. Borsuk with respect to compact metric spaces and was extended to arbitrary topological spaces by Mardešić [10] and Morita [12]. Throughout this paper we shall use the notions of Mardešić [10] and Morita [12] without any specifications.

The purpose of this paper is to discuss topological spaces which have the shape of compact metric spaces. We say that a topological space X has *compact metric shape* if there exists a compact metric space Y such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. In this paper we shall give characterizations of such spaces. For our purpose we shall introduce the notions of shape density, shape length and M-condition. These three are shape invariants.

We shall prove the following.

(1) A topological space has compact metric shape if and only if it is shape dominated by a compact metric space.

(2) A compact space has compact metric shape if and only if shape density of the compact space is not greater than $\aleph_0$.

(3) A compact space X has compact metric shape if and only if the set of homotopy classes of all continuous maps from X to P is countable for each finite simplicial complex P.

(4) A compact space with M-condition has compact metric shape.

(5) A compact space is strongly movable if and only if it is an ANSR.

The assertion (5) gives a solution to a question raised by Mardešić [11]. In [3] Dydak makes assertion (5), but there is a gap in his proof. Because his proof depends on the result of Edwards and Geoghegan [6], but their theorem is still open for the unpointed case (cf. Dydak [4] and [5]). His proof, however, is true for the pointed case.

The author thanks to the referee for his valuable suggestion which helped to simplify exposition.

§ 1. Preliminaries. In this section we recapitulate basic notions and theorems in [10] and [12].

Throughout this paper we use the following. $k(A)$ denotes the cardinality of a set A. Spaces and maps denote topological spaces and continuous functions, respectively. $\simeq$, $[f]$ and $[X, Y]$ denote the homotopic relation, the homotopy class of a map f, and the set of homotopy classes of all maps from a space X to a space Y, respectively. $\mathscr{CW}$ and $\mathscr{FCW}$ denote the category of CW complexes and maps, and the category of finite CW complexes and maps, respectively. $\mathscr{HCW}$ and $\mathscr{HFCW}$ denote the category of CW complexes and homotopy classes of maps, and the category of finite CW complexes and homotopy classes of maps, respectively.

Let $\mathscr{C}$ be any category. $\mathrm{Ob}\mathscr{C}$, $\mathscr{C}(X, Y)$ and $\mathrm{Mor}\mathscr{C}$ denote the collection of all objects in $\mathscr{C}$, the set of all morphisms in $\mathscr{C}$ from an object X to an object Y, and the collection of all morphisms in $\mathscr{C}$, respectively. Pro-$\mathscr{C}$ denotes the pro-category of $\mathscr{C}$ (see [4] and [7]).

Let $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ be an inverse system in $\mathscr{HCW}$. We say that $\mathfrak{X}$ is *associated with* a space X if there exist maps $p_a\colon X \to X_a$ for $a \in A$ satisfying the following conditions:

(1.1) $$p_{aa'}p_{a'} \simeq p_a \quad \text{for} \quad a \leqslant a',$$

(1.2) for any map $f\colon X \to Q$ with $Q \in \mathrm{Ob}\mathscr{CW}$ there exist $a \in A$ and a map $f_a\colon X_a \to Q$ such that $f \simeq f_a p_a$,

(1.3) for $a \in A$ and for two maps $f_a, g_a\colon X_a \to Q$ with $Q \in \mathrm{Ob}\mathscr{CW}$ such that $f_a p_a \simeq g_a p_a$ there exists $a' \in A$ with $a' \geqslant a$ satisfying $f_a p_{aa'} \simeq g_a p_{aa'}$.

Here $\leqslant$ is the directed order in A. We say that a set $\{p_a\colon a \in A\}$, where $p_a\colon X \to X_a$ for $a \in A$, is a *projection* from X to $\mathfrak{X}$ if it satisfies conditions (1.1)-(1.3). These two notions are due to Morita [12].

In [12] Morita has proved the following lemma.

LEMMA 1. *For any space X there exists an inverse system in $\mathscr{HCW}$ associated with X.*

In [10] Mardešić has proved the following lemma.

LEMMA 2. *For any compact space X there exists an inverse system in $\mathscr{HFCW}$ associated with X. Moreover, if X is compact metric then there exists an inverse sequence in $\mathscr{HFCW}$ associated with X.*

LEMMA 3. *Let $\{X_a, p_{aa'}, A\}$ be an inverse system in $\mathscr{FCW}$ and let X be the inverse limit space of it. Then the inverse system $\{X_a, [p_{aa'}], A\}$ in $\mathscr{HFCW}$ is associated with X.*

In this paper $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ means that a space X is shape equivalent to a space Y in the sense of Mardešić [10], and $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ means that X is shape dominated by Y.

In [12] Morita has proved the following lemma.

LEMMA 4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be inverse systems in $\mathscr{HCW}$ associated with a space X and a space Y, respectively. Then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ if and only if $\mathfrak{X}$ is isomorphic in pro-$\mathscr{HCW}$ to $\mathfrak{Y}$, and also for shape domination.*

We can easily show the following lemma.

LEMMA 5. *Let $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ be an inverse system in $\mathscr{HCW}$, that is, an object in pro-$\mathscr{HCW}$. If B is a cofinal subset of A, then $\{X_b, [p_{bb'}], B\}$ is isomorphic in pro-$\mathscr{HCW}$ to $\mathfrak{X}$.*

§ 2. Associated inverse systems. In this section we shall discuss associated inverse systems.

Let X be a space, and let $\{f_a\colon a \in A\}$ be a set of maps $f_a\colon X \to Q_a$ with $Q_a \in \mathrm{Ob}\mathscr{CW}$ for $a \in A$. We say that $\{f_a\colon a \in A\}$ is a *semi-projection* of X if it satisfies the following condition:

(2.1) for any map $g\colon X \to Q$ with $Q \in \mathrm{Ob}\mathscr{CW}$ there exist $a \in A$ and a map $g_a\colon Q_a \to Q$ such that $g_a f_a \simeq g$.

We say that $\{f_a\colon a \in A\}$ is a *compact semi-projection* of X if it is a semi-projection of X and each Q_a is a finite CW complex.

By Lemmas 1 and 2 we obtain that

(2.2) *all projections are semi-projections, and hence all compact space have compact semi-projections.*

LEMMA 6. *Let X and Y be spaces and let $\{f_a\colon a \in A\}$ be a semi-projection of Y. If $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$, then there is a semi-projection $\{g_a\colon a \in A\}$ of X. Moreover, if $\{f_a\colon a \in A\}$ is a compact semi-projection of Y, then $\{g_a\colon a \in A\}$ forms a compact semi-projection of X.*

Proof. Let $\boldsymbol{f}\colon X \to Y$ and $\boldsymbol{g}\colon Y \to X$ be shapings such that $\boldsymbol{gf} = \mathbf{1}_X$. Let $g_a\colon X \to Q_a$ be a map, where Q_a is the range of f_a, defined by $[g_a] = \boldsymbol{f}([f_a])$.

We show that $\{g_a\colon a \in A\}$ forms a semi-projection of X. To prove this fact let $h\colon X \to Q$ with $Q \in \mathrm{Ob}\mathscr{CW}$ be any map. Let $h'\colon Y \to Q$ be a map defined by $[h'] = \boldsymbol{g}([h])$. Since $\{f_a\}$ is a semi-projection of Y there exist $a \in A$ and a map $h'_a\colon Q_a \to Q$ such that $h' \simeq h'_a f_a$. Thus we obtain that $[h] = \mathbf{1}_X([h]) = (\boldsymbol{gf})([h]) = \boldsymbol{f}(\boldsymbol{g}([h])) = \boldsymbol{f}([h']) = \boldsymbol{f}([h'_a f_a]) = [h'_a]\boldsymbol{f}([f_a]) = [h'_a][g_a]$. This means that $\{g_a\colon a \in A\}$ is a semi-projection of X.

This completes the proof of Lemma 6.

The following theorem is essential in this paper.

THEOREM 1. *Let X be a space, and let $\{f_a\colon a \in A\}$ be a semi-projection of X. Then there exist an inverse system $\mathfrak{X} = \{X_b, [p_{bb'}], B\}$ in $\mathscr{HCW}$ associated with X and a projection $\{p_b\colon b \in B\}$ from X to $\mathfrak{X}$ such that each p_b is equal to some f_a. Moreover, if $\{f_a\colon a \in A\}$ is a compact semi-projection of X, then we can achieve $\boldsymbol{k}(B) \leqslant \aleph_0 \times \boldsymbol{k}(A)$.*

Proof. Let Q_a be the range of f_a for $a \in A$. Let $\mathscr{I}$ be the category defined as follows: $\mathrm{Ob}\mathscr{I} = A$, $\mathscr{I}(a, c) = \{[h]\colon h$ is a map from Q_a to Q_c such that $hf_a \simeq f_c\}$ for $a, c \in A$, and the composition in $\mathscr{I}$ is equal to the composition in $\mathscr{HCW}$.

We show that $\mathscr{I}$ has the following two properties:

(2.3) For any two objects a, c of $\mathscr{I}$ there exist $d \in \mathrm{Ob}\,\mathscr{I}$ and morphisms $[g] \in \mathscr{I}(d, a)$ and $[h] \in \mathscr{I}(d, c)$.

(2.4) If $h, g\colon Q_a \to Q$ with $Q \in \mathrm{Ob}\,\mathscr{CW}$ and $a \in \mathrm{Ob}\,\mathscr{I}$ are maps such that $hf_a \simeq gf_a$, then there exist $d \in \mathrm{Ob}\,\mathscr{I}$ and a morphism $[r] \in \mathscr{I}(d, a)$ such that $hr \simeq gr$.

To prove that $\mathscr{I}$ has these properties let $\mathfrak{K} = \{K_u, [q_{uu'}], U\}$ be an inverse system in $\mathscr{HCW}$ associated with X and let $\{q_u\colon u \in U\}$ be a projection from X to $\mathfrak{K}$ (see Lemma 1). To prove (2.3) let a and c be arbitrary objects of $\mathscr{I}$. Since $\mathfrak{K}$ is associated with X, there exist $u \in U$ and maps $i\colon K_u \to Q_a$, $j\colon K_u \to Q_c$ such that $iq_u \simeq f_a$ and $jq_u \simeq f_c$. By (2.1) there exist $d \in \mathrm{Ob}\,\mathscr{I}$ and a map $k\colon Q_d \to K_u$ such that $kf_d \simeq q_u$. Let $h = ik$ and $g = jk$. It is easy to show that d, $[h]$ and $[g]$ satisfy the required conditions. Thus (2.3) holds. Next, to prove (2.4) let $g, h\colon Q_a \to Q$ with $a \in \mathrm{Ob}\,\mathscr{I}$ and $Q \in \mathrm{Ob}\,\mathscr{CW}$ be maps such that $gf_a \simeq hf_a$. Since $\mathfrak{K}$ is associated with X, there exist $u \in U$ and a map $m\colon K_u \to Q_a$ such that $mq_u \simeq f_a$. Since $hmq_u \simeq gmq_u$, by (1.3) there exists $v \in U$ such that $hmq_{uv} \simeq gmq_{uv}$. By (2.1) there exist $d \in \mathrm{Ob}\,\mathscr{I}$ and a map $n\colon Q_d \to K_v$ such that $nf_d \simeq q_v$. Let $r = mq_{uv}n$. It is easy to show that d and $[r]$ satisfy the required conditions. Thus (2.4) holds.

Since $\mathscr{I}$ has properties (2.3) and (2.4), $\mathscr{I}$ forms a left filtering category (see [7]). Thus we can use the trick in Edwards and Hastings [7, pp. 6–7]. In this proof we use the following terms: A *finite diagram* D over $\mathscr{I}$ means an ordered set (E, M), where E is a finite subset of $\mathrm{Ob}\,\mathscr{I}$ and M is a finite subset of $\mathrm{Mor}\,\mathscr{I}$ whose domains and ranges lie in E. Let $D = (E, M)$ be a finite diagram over $\mathscr{I}$. We say that an object $e_0 \in E$ is an *initial object* of D if $M \cap \mathscr{I}(e_0, e)$ is a singleton set for $e \in E$ with $e \neq e_0$ and $M \cap \mathscr{I}(e, e_0) = \emptyset$ for $e \in E$.

Let B be the set of all finite diagrams over $\mathscr{I}$ with initial objects. Let us define the order $\leqslant$ in B as follows: $D = (E, M) \leqslant D' = (E', M')$ if and only if $E \subset E'$ and $M \subset M'$. Since $\mathscr{I}$ is a left filtering category, we can easily show that $(B, \leqslant)$ forms a directed set.

Let $I\colon B \to A = \mathrm{Ob}\,\mathscr{I}$ be a function such that $I(b)$ is an initial object of b for $b \in B$. Let $X_b = Q_{I(b)}$ and $p_b\colon X \to X_b$ be the map $f_{I(b)}$ for $b \in B$. For $b, b' \in B$ with $b \leqslant b'$ let $[p_{bb'}]$ be the unique morphism in b' from $I(b')$ to $I(b)$. Thus $\mathfrak{X} = \{X_b, [p_{bb'}], B\}$ forms an inverse system in $\mathscr{HCW}$. Moreover, by using (2.3) and (2.4) we can easily show that $\{p_b\colon b \in B\}$ forms a projection from X to $\mathfrak{X}$. Hence $\mathfrak{X}$ is associated with X.

If $\{f_a\colon a \in A\}$ is a compact semi-projection of X, then each Q_a is a finite CW complex for $a \in A$. Then $[Q_a, Q_{a'}]$ is countable for each $a, a' \in A$. Therefore, we can easily show that $k(B) \leqslant \aleph_0 \times k(A)$.

This completes the proof of Theorem 1.

§ 3. Shape density and shape length. In this section we shall introduce concepts of shape density and shape length.

Let X be a space. Let $I(X)$ and $\mathrm{SP}(X)$ be the collection of all inverse systems in $\mathscr{HCW}$ associated with X, and the collection of all semi-projections of X, respectively.

Now, we define shape density of a space X (in notation: $\mathrm{sd}(X)$) and shape length of X (in notation: $\mathrm{sl}(X)$) as follows:

(3.1) $$\mathrm{sd}(X) = \mathrm{Min}\{k(A)\colon \{f_a\colon a \in A\} \in \mathrm{SP}(X)\},$$

(3.2) $$\mathrm{sl}(X) = \mathrm{Min}\{k(A)\colon \{X_a, [p_{aa'}], A\} \in I(X)\}.$$

By Lemma 1 and the definitions we can easily show the following relations; For a space X,

(3.3) $\mathrm{sd}(X) \leqslant \mathrm{sl}(X)$,

(3.4) if $\mathrm{sd}(X) < \aleph_0$, then $\mathrm{sd}(X) = 1$,

(3.5) if $\mathrm{sl}(X) < \aleph_0$, then $\mathrm{sl}(X) = 1$,

(3.6) $\mathrm{sl}(X) = 1$ if and only if there exists a CW complex P such that $\mathrm{Sh}(X) = \mathrm{Sh}(P)$.

For shape density we have the following theorem.

THEOREM 2. *Let X and Y be spaces. If $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$, then $\mathrm{sd}(X) \leqslant \mathrm{sd}(Y)$.*

Theorem 2 is a consequence of Lemma 6.

COROLLARY 1. *The notion of shape density is shape invariant.*

For shape length we have the following theorem.

THEOREM 3. *The notion of shape length is shape invariant.*

We can easily prove Theorem 3 by using the following fact: Let X and Y be spaces such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and let $\mathfrak{Z}$ be an inverse system in $\mathscr{HCW}$. Then $\mathfrak{Z}$ is associated with X if and only if $\mathfrak{Z}$ is associated with Y.

§ 4. Compact metric shape. In this section we shall give characterizations of spaces having compact metric shape.

We say that a space X has *compact metric shape* if there exists a compact metric space Y such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Now, we show the following theorem.

THEOREM 4. *Let X be a space. Then the following conditions are equivalent:*

(A) *X has compact metric shape,*

(B) *X is shape dominated by a compact metric space.*

Proof. It is a trivial fact that (A) implies (B). Suppose that the condition (B) holds. Let Y be a compact metric space such that $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. Since Y is compact metric, by Lemma 2 there is a compact semi-projection $\{f_i\colon i \in N\}$ of Y, where N is the set of all positive integers. Then by Lemma 6 there is a compact semi-projection $\{g_i\colon i \in N\}$ of X. Thus by applying Theorem 1 to $\{g_i\}$ we obtain an inverse system $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ in $\mathscr{HFCW}$ associated with X such that $k(A) \leqslant \aleph_0 \times k(N) = \aleph_0$.

Since $k(A)\leqslant\aleph_0$, we can easily construct an increasing function $f\colon N\to A$ such that $f(N)$ is a cofinal subset of A. Let $Z_i = X_{f(i)}$ and $r_{i,i+1} = p_{f(i)f(i+1)}$ for each $i\in N$, and let $r_{ij} = r_{i,i+1}r_{i+1,i+2}\dots r_{j-1,j}$ for $i,j\in N$ with $i\leqslant j$. Thus $\{Z_i, r_{ij}, N\}$ forms an inverse sequence in $\mathscr{FCW}$. Let Z be the inverse limit space of $\{Z_i\}$. Since each Z_i is a finite CW complex, Z is a compact metric space.

We show that $\mathrm{Sh}(X) = \mathrm{Sh}(Z)$. By Lemma 3, $\mathfrak{Z} = \{Z_i, [r_{ij}], N\}$ is associated with Z. Since $f(N)$ is cofinal in A, by Lemma 5 $\mathfrak{Z}$ is isomorphic in pro-$\mathscr{HCW}$ to $\mathfrak{X}$. Hence by Lemma 4 $\mathrm{Sh}(X) = \mathrm{Sh}(Z)$.

This completes the proof of Theorem 4.

COROLLARY 2. *A space which is shape dominated by a finite* CW *complex has compact metric shape.*

Next, we show the following theorem.

THEOREM 5. *Let X be a compact space. Then the following conditions are equivalent:*

(A) *X has compact metric shape,*

(B) $\mathrm{sl}(X)\leqslant\aleph_0$,

(C) $\mathrm{sd}(X)\leqslant\aleph_0$,

(D) *$[X, P]$ is a countable set for each finite simplicial complex P.*

Proof. We show that (A) implies (B). Let Y be a compact metric space such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. Then by Theorem 3, $\mathrm{sl}(X) = \mathrm{sl}(Y)$. Since Y is compact metric, by Lemma 2 there is an inverse sequence in $\mathscr{HCW}$ associated with Y. Hence $\mathrm{sl}(Y)\leqslant\aleph_0$.

We can show by (3.3) that (B) implies (C).

We show that (C) implies (D). To do so let $\{f_a\colon a\in A\}$ be a semi-projection of X such that $k(A) = \mathrm{sd}(X)\leqslant\aleph_0$. Let Q_a be the range of f_a for $a\in A$. Since X is compact, by Lemma 2 there are an inverse system $\mathfrak{X} = \{X_b, [p_{bb'}], B\}$ in $\mathscr{HFCW}$ associated with X and a projection $\{p_b\colon b\in B\}$ from X to $\mathfrak{X}$. Since $\mathfrak{X}$ is associated with X, for each $a\in A$ there exist $u(a)\in B$ and a map $h_a\colon X_{u(a)}\to Q_a$ such that $f_a\simeq h_a p_{u(a)}$. Let Q be an arbitrary finite simplicial complex, and let

$$K = \bigcup\{[X_{u(a)}, Q]\colon a\in A\},$$

where union means disjoint union. Since Q and all $X_{u(a)}$ are finite CW complexes and $k(A)\leqslant\aleph_0$, then K becomes a countable set. Now, let $\Psi\colon K\to[X, Q]$ be the function defined by $\Psi([r]) = [rp_{u(a)}]$ for $[r]\in[X_{u(a)}, Q]$. We show that Ψ is an onto function. To prove this fact let $g\colon X\to Q$ be any map. Since $\{f_a\colon a\in A\}$ is a semi-projection of X, there exist $a\in A$ and a map $g_a\colon Q_a\to Q$ such that $g\simeq g_a f_a$. Then $g\simeq g_a f_a\simeq g_a h_a p_{u(a)}$, and hence $\Psi([g_a h_a]) = [g]$. Thus Ψ is onto. Since K is countable, $[X, Q]$ is countable.

We show that (D) implies (A). Let $\mathscr{W}_f$ be the collection of all finite simplicial complexes (more precisely homeomorphism classes of such complexes). Thus $k(\mathscr{W}_f)\leqslant\aleph_0$. Let $L = \bigcup\{[X, P]\colon P\in\mathscr{W}_f\}$. Then by condition (D) L is countable.

Since X is compact, by Lemma 2 there exist an inverse system $\mathfrak{X} = \{X_{a'}, [p_{aa'}], A\}$ in $\mathscr{HFCW}$ associated with X and a projection $\{p_a\colon a\in A\}$ from X to $\mathfrak{X}$. We can assume that each X_a is a member of $\mathscr{W}_f$. Then $M = \{[p_a]\colon a\in A\}$ is a subset of L. Thus M is countable. Let $\Phi\colon M\to A$ be a function such that $[p_{\Phi(m)}] = m$ for each $m\in M$. Since $\{p_a\colon a\in A\}$ is a projection, we can easily show that $\{p_{\Phi(m)}\colon m\in M\}$ is a compact semi-projection of X. Now, we can apply Theorem 1 to $\{p_{\Phi(m)}\colon m\in M\}$. Then we obtain an inverse system $\mathfrak{Z} = \{Z_c, [q_{cc'}], C\}$ in $\mathscr{HFCW}$ associated with X such that $k(C)\leqslant\aleph_0\times k(M) = \aleph_0$. Since $\mathfrak{Z}$ is an inverse system in $\mathscr{HFCW}$ and $k(C)\leqslant\aleph_0$, in the same way as in the proof of Theorem 4 we can construct a compact metric space Z such that $\mathrm{Sh}(X) = \mathrm{Sh}(Z)$.

This completes the proof of Theorem 5.

§ 5. A condition for compact metric shape.

In this section we shall introduce M-condition (= metric condition) and discuss it's properties.

Let $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ be an object of pro-$\mathscr{HCW}$, that is, an inverse system in $\mathscr{HCW}$. We say that $\mathfrak{X}$ satisfies M-*condition* if for each $a\in A$ there exists $a'\in A$ with $a'\geqslant a$ such that for each $a''\in A$ with $a''\geqslant a$ there exist $a^*\in A$ with $a^*\geqslant a', a''$ and a map $r^{a'a''}\colon X_{a'}\to X_{a''}$ satisfying the following condition:

$$r^{a'a''}p_{a'a^*}\simeq p_{a''a^*}. \tag{5.1}$$

If, in addition, the following condition:

$$p_{aa''}r^{a'a''}\simeq p_{aa'} \tag{5.2}$$

is satisfied, then we say that $\mathfrak{X}$ is *strongly movable* (see [11]).

In [11] Mardešić has proved that if $\mathfrak{X}$ is dominated in pro-$\mathscr{HCW}$ by $\mathfrak{Y}$ and $\mathfrak{Y}$ is strongly movable, then $\mathfrak{X}$ is also strongly movable. Similarly we can prove the following theorem.

THEOREM 6. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be objects of* pro-$\mathscr{HCW}$. *If $\mathfrak{X}$ is dominated in* pro-$\mathscr{HCW}$ *by $\mathfrak{Y}$ and $\mathfrak{Y}$ satisfies* M-*condition, then $\mathfrak{X}$ satisfies* M-*condition.*

We say that a space X satisfies M-*condition* if there is an inverse system $\mathfrak{X}$ in $\mathscr{HCW}$ associated with X such that $\mathfrak{X}$ satisfies M-condition.

By Lemma 4 and Theorem 6 we can easily prove the following condition.

(5.3) If a space X satisfies M-condition, then every inverse system in $\mathscr{HCW}$ associated with X satisfies M-condition.

By Lemma 4 and Theorem 6 we can easily prove the following theorem.

THEOREM 7. *Let X and Y be spaces. If* $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$ *and Y satisfies* M-*condition, then X satisfies* M-*condition.*

COROLLARY 3. *The notion of* M-*condition is shape invariant.*

The main result of this section is the following theorem.

THEOREM 8. *Let X be a compact space. If X satisfies* M-*condition, then there exists a compact metric space Z with finite dimension such that* $\mathrm{Sh}(X) = \mathrm{Sh}(Z)$, *that is, X has compact metric shape.*

Proof. Since X is compact, by Lemma 2 there exist an inverse system $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ in $\mathscr{HFCW}$ associated with X and a projection $\{p_a\colon a \in A\}$ from X to $\mathfrak{X}$. Since X satisfies M-condition, by (5.3) $\mathfrak{X}$ satisfies M-condition.

Let a_0 be an arbitrary element of A. Since $\mathfrak{X}$ satisfies M-condition, there exists $a_1 \in A$ with $a_1 \geqslant a_0$ such that for each $a'' \in A$ with $a'' \geqslant a_0$ there exist $a^* \in A$ with $a^* \geqslant a_0$, a_1 and a map $r^{a_1 a''}\colon X_{a_1} \to X_{a''}$ satisfying $r^{a_1 a''} p_{a_1 a^*} \simeq p_{a'' a^*}$. This condition implies the following condition:

(5.4) $$r^{a_1 a''} p_{a_1} \simeq r^{a_1 a''} p_{a_1 a^*} p_{a^*} \simeq p_{a'' a^*} p_{a^*} \simeq p_{a''}\,.$$

We show that the singleton set $\{p_{a_1}\}$ is a compact semi-projection of X. To do so let $h\colon X \to Q$ with $Q \in \mathrm{Ob}\,\mathscr{CW}$ be any map. Since $\mathfrak{X}$ is associated with X, there exist $a_2 \in A$ with $a_2 \geqslant a_0$ and a map $h_{a_2}\colon X_{a_2} \to Q$ such that $h_{a_2} p_{a_2} \simeq h$. Thus by applying (5.4) to $a'' = a_2$ we obtain the following: $h \simeq h_{a_2} p_{a_2} \simeq h_{a_2} r^{a_1 a_2} p_{a_1}$. Thus $\{p_{a_1}\}$ forms a compact semi-projection of X.

By applying Theorem 1 to $\{p_{a_1}\}$ we obtain an inverse system $\mathfrak{Z} = \{Z_c, [q_{cc'}], C\}$ in $\mathscr{HFCW}$ associated with X such that $k(C) \leqslant \aleph_0$ and each Z_c is equal to X_{a_1}. Then in the same way as in the proof of Theorem 4 we can construct a compact metric space Z such that $\mathrm{Sh}(X) = \mathrm{Sh}(Z)$ and $\dim Z \leqslant \dim X_{a_1} < \infty$.

This completes the proof of Theorem 8.

In [2] Borsuk introduced the notion of strong movability for compact metric spaces and proved that this notion is equivalent to the notion of FANR (= ANSR). In [11] Mardešić defined the notion of strong movability for compact spaces and proved that his notion is equivalent to Borsuk's for compact metric spaces. He raised the following question: Is every strongly movable compact space an ANSR? Recently Dydak [3] provided an answer to this question, but there is a gap in his proof. Because the result of Edwards and Geoghegan [6] on which he depends is still open for the unpointed case (see Dydak [4] and [5]). His proof, however, holds for pointed case. We can answer this question in the following fashion without using Edwards and Geoghegan's result.

COROLLARY 4. *Let X be a compact space. Then X is strongly movable if and only if it is an* ANSR.

Proof. First, we assume that X is strongly movable. Then X satisfies M-condition. By Theorem 8 X has compact metric shape. Since strong movability is shape invariant, our assertion can be reduced to the compact metric case. In the case of compact metric, however, our assertion has already been proved by Mardešić [11]. The converse assertion is trivial (see [11]).

This completes the proof of Corollary 4.

Remark 1. Analogously we can introduce the concepts of shape density, shape length and M-condition for pointed spaces. Obviously we can extend all results in this paper to the pointed case.

Remark 2. Recently K. Tsuda has proved that a compact metric space satisfies M-condition if and only if it is an AWNR (see Bogatyĭ [1] for the definition of AWNR spaces), and hence by Corollary 3 the notion of AWNR is shape invariant.

§ 6. **$\mathscr{P}$-like continua.** In this section we shall discuss $\mathscr{P}$-like continua.

Let $\mathscr{P}$ be a collection of finite connected simplicial complexes. We say that a continuum X (= compact connected space) is $\mathscr{P}$-like if for each open covering $\mathscr{U}$ of X there exists an onto map $f\colon X \to P$ with $P \in \mathscr{P}$ such that for each point p of P, $f^{-1}(p)$ is contained in some member of $\mathscr{U}$.

For $\mathscr{P}$-like continua we show the following theorem.

THEOREM 9. *Let X be a $\mathscr{P}$-like continuum. Then there exists a $\mathscr{P}$-like metric continuum Y such that* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *if and only if* $[X, P]$ *is countable for each* $P \in \mathscr{P}$.

Proof. Let $\mathscr{P}'$ be the homeomorphism classes of complexes in $\mathscr{P}$. We can assume that $\mathscr{P}'$ is a subset of $\mathscr{W}_f$ (see the proof of Theorem 5). Since $\mathscr{W}_f$ is countable, $\mathscr{P}'$ is also countable. Since X is $\mathscr{P}$-like, X is also $\mathscr{P}'$-like. Thus by Proposition 1 of [13] there is an inverse system $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ in $\mathscr{HCW}$ associated with X such that each X_a is a member of $\mathscr{P}'$.

Now, we suppose that $[X, P]$ is countable for each $P \in \mathscr{P}'$. Let $K = \bigcup \{[X, P]\colon P \in \mathscr{P}'\}$. From the assumption it follows that K is countable. Let $K = \{[f_i]\colon i \in N\}$, where N is the set of integers. We can easily show, by using $\mathfrak{X}$, that $\{f_i\colon i \in N\}$ is a compact semi-projection of X. Then by applying Theorem 1 to $\{f_i\colon i \in N\}$ we obtain an inverse system $\mathfrak{X}' = \{X'_b, [q_{bb'}], B\}$ in $\mathscr{HFCW}$ associated with X such that each X'_b is a member of $\mathscr{P}'$ and $k(B) \leqslant \aleph_0$. Thus in the same way as in the proof of Theorem 5 we obtain an inverse sequence $\{Z_i, u_{ij}, N\}$ in $\mathscr{CW}$ such that $\mathfrak{Z} = \{Z_i, [u_{ij}], N\}$ is isomorphic in pro-$\mathscr{HCW}$ to $\mathfrak{X}'$ and each Z_i is a member of $\mathscr{P}'$. By Proposition 2 of [13] there is an onto map $s_{i,i+1}\colon Z_{i+1} \to Z_i$ such that $s_{i,i+1} \simeq u_{i,i+1}$ for each i. Thus we obtain an inverse sequence $\mathfrak{Z}' = \{Z_i, s_{i,i+1}, N\}$ in $\mathscr{CW}$. Let Y be the inverse limit space of $\mathfrak{Z}'$. Since all bonding maps in $\mathfrak{Z}'$ are onto and each Z_i is a member of $\mathscr{P}$, Y is a $\mathscr{P}$-like metric continuum. By Lemma 3 $\mathfrak{Z}'' = \{Z_i, [s_{i,i+1}], N\}$ is associated with Y. Since $\mathfrak{Z}'' = \{Z_i, [s_{ij}], N\} = \{Z_i, [u_{ij}], N\}$ is isomorphic in pro-$\mathscr{HCW}$ to $\mathfrak{X}'$, by Lemma 4 $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

The converse assertion follows from Theorem 5.

This completes the proof of Theorem 9.

We say that $\mathscr{P}$ is monomorphic if every map $f\colon P_2 \to P_1$ with $P_1, P_2 \in \mathscr{P}$, which is not null homotopic, satisfies the following condition:

(6.1) if $h, g\colon P_3 \to P_2$ with $P_3 \in \mathscr{P}$ are maps such that $fh \simeq fg$, then $h \simeq g$.

For monomorphic $\mathscr{P}$ we show the following theorem.

THEOREM 10. *Let $\mathscr{P}$ be monomorphic. Then for every $\mathscr{P}$-like continuum X there exists a $\mathscr{P}$-like metric continuum Y such that* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

Proof. Since X is $\mathscr{P}$-like, by Proposition 1 of [13] there is an inverse system $\mathfrak{X} = \{X_a, [p_{aa'}], A\}$ in $\mathscr{HFCW}$ associated with X such that each X_a is a member of $\mathscr{P}$. Now, we consider the following two cases:

(6.2) for each $a \in A$ there exists $a' \in A$ with $a' \geqslant a$ such that $p_{aa'}$ is null homotopic,

(6.3) there exists $a' \in A$ such that for $a_1, a_2 \in A$ with $a_2 \geqslant a_1 \geqslant a'$, $p_{a_1 a_2}$ is not null homotopic.

In the case of (6.2) it is obvious that X is of trivial shape. Let P be a member of $\mathscr{P}$ and p_0 be a point of P. Let $f\colon P \to P$ be the map defined by $f(P) = p_0$. Let $\mathfrak{Z} = \{Z_i, [q_{ij}], N\}$ be an inverse sequence such that $Z_i = P$ and $q_{i,i+1} = f$ for each i. It is easy to show that $\mathfrak{Z}$ is associated with the one point space, that is, associated with X. In the same way as in the proof of Theorem 9 we can construct a $\mathscr{P}$-like metric continuum Y such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

In the case of (6.3) let $A_1 = \{a \in A\colon a \geqslant a'\}$ and $\mathfrak{X}_1 = \{X_a, [p_{aa'}], A_1\}$. Since A_1 is a cofinal subset of A, by Lemma 5 $\mathfrak{X}_1$ is also associated with X. Let $\{p_a\colon a \in A_1\}$ be a projection from X to $\mathfrak{X}_1$. Let $B = \{[p_a]\colon a \in A_1\}$ and let $f\colon B \to A_1$ be a function such that $[p_{f(b)}] = b$ for each $b \in B$. We can easily by using $\mathfrak{X}_1$ show that $\{p_{f(b)}\colon b \in B\}$ is a compact semi-projection of X.

We show that $k(B) \leqslant \aleph_0$. To do so let $L = \bigcup \{[P, X_{a'}]\colon P \in \mathscr{P}'\}$ (see the proof of Theorem 9 for $\mathscr{P}'$). Thus L is a countable set. Let $d\colon B \to L$ be the function defined by $d(b) = [p_{a'f(b)}]$ for each $b \in B$. We show that d is injective. To prove this let $d(b_1) = d(b_2)$ for $b_1, b_2 \in B$. Then $p_{a'f(b_1)} \simeq p_{a'f(b_2)}$. Let a'' be an element of A with $a'' \geqslant f(b_1), f(b_2)$. Then we obtain that $p_{a'f(b_1)} p_{f(b_1)a''} p_{a''} \simeq p_{a'} \simeq p_{a'f(b_2)} p_{f(b_2)a''} p_{a''}$. Since $\mathfrak{X}_1$ is associated with X, there exists $a^* \in A$ with $a^* \geqslant a''$ such that $p_{a'f(b_1)} p_{f(b_1)a^*} \simeq p_{a'f(b_2)} p_{f(b_2)a^*}$. Since $\mathscr{P}$ is monomorphic, $p_{f(b_1)a^*} \simeq p_{f(b_2)a^*}$. Thus $p_{f(b_1)} \simeq p_{f(b_1)a^*} p_{a^*} \simeq p_{f(b_2)a^*} p_{a^*} \simeq p_{f(b_2)}$. Hence $b_1 = [p_{f(b_1)}] = [p_{f(b_2)}] = b_2$. Thus d is injective. Since L is countable, B becomes a countable set.

Then $\{p_{f(b)}\colon b \in B\}$ is a compact semi-projection of X such that $k(B) \leqslant \aleph_0$ and each range of $p_{f(b)}$ is a member of $\mathscr{P}$. Therefore, in the same way as in the proof of Theorem 9 we can construct a $\mathscr{P}$-like metric continuum Y such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

This completes the proof of Theorem 10.

Let S^n and CP^n be n-dimensional sphere and n-dimensional complex projective space, respectively. Since $[S^n, S^n]$ and $[\mathrm{CP}^n, \mathrm{CP}^n]$ are monomorphic, we obtain the following by Theorem 10.

COROLLARY 5. *Every S^n-like or CP^n-like continuum has compact metric shape.*

Remark 3. In [8] Gordh has proved Theorem 10 by using reduced inverse systems. Theorem 10 is, however, essentially due to Gordh and Mardešić [9]. Corollary 5 was proved by many authors (see Introduction of [13]).

References

[1] S. Bogatyĭ, *Approximate and fundamental retracts*, Math. Sbornik 93 (135), 1974, pp. 90–102.
[2] K. Borsuk, *A note on the theory of shape of compacta*, Fund. Math. 67 (1970), pp. 265–278.
[3] J. Dydak, *On the Whitehead theorem in pro-homotopy and on a question of Mardešić*, Bull. Acad. Polon. Sci. 23 (1975), pp. 775–779.
[4] — *The Whitehead theorem and Smale theorem in shape theory*, Polish Acad. Sciences, Preprint 87, 1976.
[5] — *A simple proof that pointed connected* FANR*-spaces are regular fundamental retracts of* ANR*'s*, to appear.
[6] D. A. Edwards and R. Geoghgan, *Shapes of complexes, end of manifolds, homotopy limits and Wall obstruction*, Ann. of Math. 10 (1975), pp. 521–535.
[7] — and H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Springer-Verlag Lecture Note, No. 542.
[8] G. D. Gordh, Jr., *On inverse limits of* ANR*'s with metric shape*, Math. Balkanica 38 (4) (1974), pp. 225–228.
[9] — and S. Mardešić, *Reducing inverse systems of monomorphisms*, Colloquium Math. 33 (1975), pp. 83–90.
[10] S. Mardešić, *Shape for topological spaces*, Gen. Top. Appl. 3 (1973), pp. 265–282.
[11] — *Strongly movable compacta and shape retracts*, Proc. Int. Sym. on Top. and its Appl. (Budva 1972), 1973, pp. 163–166.
[12] K. Morita, *On shapes of topological spaces*, Fund. Math. 86 (1975), pp. 251–259.
[13] T. Watanabe, *Shape classifications for complex projective space-like and wedges of n-spheres-like continua*, Sci. Report of Tokyo Kyoiku Daigaku, Sec. A, 12 (1974), pp. 233–245.

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
UNIVERSITY OF YAMAGUCHI
Yamaguchi City, Japan

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