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On fine shape theory

by

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Dedicated to Professor Karol Borsuk for his 70th birthday

Abstract. A fine shape category $\mathcal{C}_f$ is defined. The shape category $\mathcal{C}$ introduced by Borsuk is a quotient category of $\mathcal{C}_f$. $\mathcal{C}$ and $\mathcal{C}_f$ are not isomorphic. It is proved that $\mathcal{C}_f$ is isomorphic to the proper homotopy category of complements of compacta in the Hilbert cube and to a certain full subcategory of the proper shape category introduced by Ball and Sher.

1. Introduction. The concept of shape for compacta was first introduced by K. Borsuk [2]. T. Chapman [4] defined a weak proper homotopy category of complements of compacta in the Hilbert cube Q and proved that this category is isomorphic to Borsuk's shape category. As D. A. Edwards [6] asserted, it is natural to introduce a new shape category corresponding to a proper homotopy category of complements of compacta in Q. D. A. Edwards called it a strong shape category, but this terminology was already used by Borsuk [3] for a different concept, so we call it a fine shape category.

In this paper, first we shall define a fine shape category after a manner of Borsuk's fundamental sequences, and prove the equivalence to the proper homotopy category of complements of compacta in Q. Next, we shall give another characterization of this category in terms of the proper shape category introduced by Ball and Sher [1].

Throughout the paper all spaces are metrizable and maps are continuous. AR and ANR mean those for metric spaces.

2. Fine shape category. Let X be a compactum. We denote by $\mathcal{A}(X)$ the family of AR's M containing X as a subset. Let R_+ ($=[0,\infty)$) be the space of non negative reals. For compacta X, Y and for $M\in\mathcal{A}(X)$, $N\in\mathcal{A}(Y)$, a continuous map $F\colon M\times R_+\to N$ is said to be a *fundamental map from X to Y in M, N* if for every neighborhood V of Y in N there exist a neighborhood U of X in M and a number $t_0\in R_+$ such that

$$F(U\times[t_0,\infty))\subset V. \tag{1.1}$$

We write $F\colon X\to Y$ in M, N. Two fundamental maps $F, G\colon X\to Y$ in M, N are said to be *fine homotopic* (notation: $F \underset{f}{\simeq} G$) if there exists a homotopy $H\colon M\times R_+\times I\to N$ such that

(1.2) $$H(p,0) = F(p), \quad H(p,1) = G(p) \quad \text{for} \quad p\in M\times R_+,$$

(1.3) for every neighborhood V of Y in N there exist a neighborhood U of X in M and a number $t_0\in R_+$ such that $H(U\times[t_0,\infty)\times I)\subset V$.

Note that the relation of the fine homotopy of fundamental maps from X to Y in M, N is an equivalence relation. We call the equivalence classes the *fine homotopy classes*. The fine homotopy class represented by F is denoted by $[F]$.

Suppose that fundamental maps $F\colon X\to Y$ in M, N and $G\colon Y\to Z$ in N, P are given, where Z is a compactum and $P\in\mathscr{A}(Z)$. A *composition* $G*F$ of F and G is a fundamental map from X to Z in M, P defined by

(1.4) $$G*F(x,t) = G(F(x,t),t) \quad \text{for} \quad (x,t)\in M\times R_+.$$

The identity fundamental map $\mathrm{Id}_{X,M}$ from X to X in M, M is defined by

(1.5) $$\mathrm{Id}_{X,M}(x,t) = x \quad \text{for} \quad (x,t)\in M\times R_+.$$

Let $M, M'\in\mathscr{A}(X)$ and let $\varphi\colon M\to M'$ and $\psi\colon M'\to M$ be maps such that $\varphi|X = \psi|X = 1_X$ (= the identity on X). Define $\Phi\colon M\times R_+\to M'$ and $\Psi\colon M'\times R_+\to M$ by $\Phi(x,t) = \varphi(x)$, $(x,t)\in M\times R_+$, and $\Psi(x,t) = \psi(x)$, $(x,t)\in M'\times R_+$. Obviously

(1.6) $$\Phi\colon X\to X \quad \text{in } M, M' \quad \text{and} \quad \Psi\colon X\to X \quad \text{in } M', M,$$

(1.7) $$\Psi*\Phi \underset{t}{\simeq} \mathrm{Id} \quad \text{and} \quad \Phi*\Psi \underset{t}{\simeq} \mathrm{Id}_{X,M'}.$$

Let $M, M'\in\mathscr{A}(X)$ and $N, N'\in\mathscr{A}(Y)$. Fundamental maps $F\colon X\to Y$ in M, N and $G\colon X\to Y$ in M', N' are said to be *equivalent* (notation: $F\equiv G$) if there exist maps $\varphi\colon M\to M'$ and $\psi\colon N'\to N$ such that $\varphi|X = 1_X$ and $\psi|Y = 1_Y$ and

(1.8) $$F \underset{f}{\simeq} \psi G\Phi,$$

where $\Phi\colon M\times R_+\to M'\times R_+$ is defined by $\Phi(x,t) = (\varphi(x),t)$ for $(x,t)\in M\times R_+$. It is easy to see that the relation "$\equiv$" is an equivalence. The equivalence class is said to be the *f-class*.

Obviously we obtain a category $\mathscr{C}_f$ if we consider the collections consisting of every compactum as objects and the f-classes as morphisms. We call $\mathscr{C}_f$ the *fine shape category*.

Let X, Y be compacta and $M\in\mathscr{A}(X)$, $N\in\mathscr{A}(Y)$. We say that X is *fine equivalent* to Y or simply *f-equivalent* to Y rel. M, N if there exist fundamental maps $F\colon X\to Y$ in M, N and $G\colon Y\to X$ in N, M such that

(1.9) $$G*F \underset{f}{\simeq} \mathrm{Id}_{X,M} \quad \text{and} \quad F*G \underset{f}{\simeq} \mathrm{Id}_{Y,N}.$$

If only the first relation in (1.9) holds, then Y is said to be *f-dominate* X rel. M, N. By (1.6), (1.7) (or Borsuk [3, Chap. III]) the relations of the f-equivalence and the f-domination do not depend on the choice of AR's $M\in\mathscr{A}(X)$ and $N\in\mathscr{A}(Y)$. Thus we can say that X is *f-equivalent* to Y (resp. *f-dominates* Y) if for every (or equivalently some) AR's $M\in\mathscr{A}(X)$ and $N\in\mathscr{A}(Y)$ X is f-equivalent to Y (resp. f-dominates Y) rel. M, N.

By the *fine shape* $\mathrm{Sh}_f(X)$ of a compactum X we understand the collection consisting of all compacta Y which are f-equivalent to X. If X f-dominates Y, then we write $\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(Y)$.

In the above construction of $\mathscr{C}_f$, if we replace fundamental maps by fundamental sequences and fine homotopies by fundamental homotopies, then the shape category $\mathscr{C}$ in the sense of Borsuk [3], Chap. VII, is obtained. (Exactly, $\mathscr{C}$ is a quotient category of Borsuk's shape category under a certain equivalence like the relation "$\equiv$". Cf. Borsuk [3], p. 55).

Let X, Y be compacta and $M\in\mathscr{A}(X)$, $N\in\mathscr{A}(Y)$. Every fundamental map $F\colon X\to Y$ in M, N determines a fundamental sequence $f_F = \{f_k, X, Y\}_{M,N}$ as follows (see for notations Borsuk [3], Chap. VIII).

(1.10) $$f_k(x) = F(x,k) \quad \text{for} \quad x\in M \text{ and } k = 0,1,2,\ldots$$

That $f_F = \{f_k\}$ forms a fundamental sequence follows from the property (1.1) of a fundamental map F. Conversely, a fundamental sequence $f = \{f_k, X, Y\}_{M,N}$ defines a fundamental map $F\colon X\to Y$ in M, N such that f and f_F are fundamentally homotopic. Such an F is constructed from $f = \{f_k\}$ by applying repeatedly Borsuk's homotopy extension theorem such that for some neighborhood basis $\{U_k\colon k = 0,1,2,\ldots\}$ of X in M

$$F(x,k) = f_k(x) \quad \text{for} \quad x\in U_k,\ k = 0,1,2,\ldots$$

Also, if F and G are fundamental maps from X to Y in M, N such that $F \underset{f}{\simeq} G$ then f_F and f_G are fundamentally homotopic by (1.3). Thus we have

LEMMA 1. *There exists a covariant functor Θ from $\mathscr{C}_f$ onto $\mathscr{C}$ such that $\Theta(X) = X$ for a compactum X and $\Theta([F]) = [f_F]$ for a fundamental map F, where $[F]$ is the f-class of F.*

By the example in the next section it is known that Θ is not an isomorphism.

COROLLARY 1. *For compacta X and Y, $\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(Y)$ (resp. $\mathrm{Sh}_f(X) = \mathrm{Sh}_f(Y)$) implies $\mathrm{Sh}(X)\geqslant\mathrm{Sh}(Y)$ (resp. $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$), where $\mathrm{Sh}(X)$ is the shape of X in the sense of Borsuk* [3].

We do not know whether $\mathrm{Sh}(X)\geqslant\mathrm{Sh}(Y)$ implies $\mathrm{Sh}_f(X)\geqslant\mathrm{Sh}_f(Y)$.

3. Main theorems. A closed subset X of a metrizable space M is said to be *unstable* in M (Sher [11], p. 346) if there exists a homotopy $H\colon M\times I\to M$ such that

(3.1) $$H(y,0) = y \text{ for } y\in M \text{ and } H(y,t)\in M-X \text{ for } y\in M \text{ and } 0<t\leqslant 1.$$

By Lemma 4.1 of Chapman [4] it is known that every Z-set in the Hilbert cube Q is unstable in Q. Also, it is shown by [8], Theorem 1, that every metrizable space X is unstably imbedded into an AR $M(X)$ with $\dim M(X) = \dim X+1$ and $w(M(X)) = w(X)$.

For a compactum X, we denote by $\mathscr{M}(X)$ the family of compact AR's M containing X as an unstable subset. The following was proved in [7], Lemma 3.

Lemma 2. *Let X be a compactum and $M, M' \in \mathscr{M}(X)$. Then there exists a map $\xi\colon M\to M'$ such that*

$$\xi|X = 1_X \quad \text{and} \quad \xi(M-X)\subset M'-X. \tag{3.2}$$

If $\xi, \eta\colon M\to M'$ satisfy condition (3.2), *then there exists a homotopy $H\colon M\times I\to M'$ such that*

$$\begin{gathered} H(y,0) = \xi(y) \quad \text{and} \quad H(y,1) = \eta(y) \quad \text{for} \quad y\in M, \\ H(x,t) = x \quad \text{for} \quad x\in X, \\ H((M-X)\times I)\subset M'-X. \end{gathered} \tag{3.3}$$

This is shown by an argument similar to Lemma 2.1 of Sher [11].

Suppose that proper maps $f\colon M-X\to N-Y$ and $f'\colon M'-X\to N'-Y$ are given, where $M, M'\in\mathscr{M}(X)$ and $N, N'\in\mathscr{M}(Y)$. Then f is said to be *equivalent* to f' (notation: $f\equiv f'$) if there exist maps $\xi\colon M\to M'$ and $\eta\colon N'\to N$ such that $\xi|X = 1_X$, $\xi(M-X)\subset M'-X$, $\eta|Y = 1_Y$ and $\eta(N'-Y)\subset N-Y$, and $f \underset{p}{\simeq} \eta f'\xi|M-X$ in $N-Y$, where $\underset{p}{\simeq}$ means properly homotopic. From Lemma 2 it follows that the relation "$\equiv$" is an equivalence relation. The equivalence class is said to be the *p-class*.

Now, we shall define a category $\mathscr{P}$ as follows. The collection of objects in $\mathscr{P}$ consists of all compacta. For compacta X and Y morphisms from X to Y consist of the p-classes of all proper maps of $M-X$ into $N-Y$, where $M\in\mathscr{M}(X)$ and $N\in\mathscr{M}(Y)$.

Theorem 1. *There exists a category isomorphism $\Phi\colon \mathscr{P}\to\mathscr{C}_f$ such that $\Phi(X) = X$ for every compactum X.*

Next, let us remind the proper shape category $\mathscr{C}_p$ introduced by Ball and Sher [1]. For a locally compact separable metrizable space X, we denote by $\mathscr{B}(X)$ the family of locally compact AR's M containing X as a closed subset. Let X and Y be locally compact separable metrizable spaces and let $M\in\mathscr{B}(X)$, $N\in\mathscr{B}(Y)$. A *proper fundamental net* $\underline{f} = \{f_\lambda\colon \lambda\in\Lambda\}$, Λ a directed set, from X to Y in M, N (denoted by $\{\underline{f}, X, Y\}_{M,N}$) is a family of maps $f_\lambda\colon M\to N$ indexed by Λ provided that for every closed neighborhood V of Y in N, there exist a closed neighborhood U of X in M and an index $\lambda_0\in\Lambda$ such that for every $\lambda\geqslant\lambda_0$

$$f_\lambda|U \underset{p}{\simeq} f_{\lambda_0}|U \quad \text{in } V. \tag{3.4}$$

Two proper fundamental nets $\{\underline{f}, X, Y\}_{M,N}$ and $\{\underline{g}, X, Y\}_{M,N}$, where $\underline{f} = \{f_\lambda\colon \lambda\in\Lambda\}$ and $\underline{g} = \{g_\mu\colon \mu\in\Omega\}$, are said to be *properly homotopic* (denoted by $\underline{f} \underset{p}{\simeq} \underline{g}$), if for every closed neighborhood V of Y, there exists a closed neighborhood U of X and indices $\lambda_0\in\Lambda$, $\mu_0\in\Omega$ such that if $\lambda\geqslant\lambda_0$, $\mu\geqslant\mu_0$

$$f_\lambda|U \underset{p}{\simeq} g_\mu|U \quad \text{in } V. \tag{3.5}$$

Let $M, M'\in\mathscr{B}(X)$ and $N, N'\in\mathscr{B}(Y)$. Proper fundamental nets $\{\underline{f}, X, Y\}_{M,N}$ and $\{\underline{g}, X, Y\}_{M',N'}$ are said to be *equivalent* (notation: $\underline{f}\equiv\underline{g}$) if there exist maps $\varphi\colon M\to M'$ and $\psi\colon N'\to N$ such that $\varphi|X = 1_X$, $\psi|Y = 1_Y$ and

$$\underline{f} \underset{p}{\simeq} \psi\underline{g}\varphi, \tag{3.6}$$

where $\psi\underline{g}\varphi$ is the proper fundamental net of X to Y in M, N consisting of maps $\psi g_\mu\varphi$, $g_\mu\in\underline{g}$. The relation "$\equiv$" is an equivalence. The equivalence class is said to be the *pn-class*. We obtain the proper shape category $\mathscr{C}_p$ if we consider the collection of every locally compact separable metrizable spaces as objects and the pn-classes as morphisms.

Theorem 2. *Let $\tilde{\mathscr{C}}_p$ be the full subcategory of $\mathscr{C}_p$ whose objects consist of spaces of the form $X\times R_+$, where X is any compactum. Then there exists a category isomorphism $\Psi\colon \mathscr{C}_f\to\tilde{\mathscr{C}}_p$ such that $\Psi(X) = X\times R_+$ for every object X of $\mathscr{C}_f$.*

The proofs of Theorems 1 and 2 are given in the next section.

Corollary 2. *For compacta X and Y the followings are equivalent.*

(1) $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

(2) $\mathrm{Sh}_f(X) = \mathrm{Sh}_f(Y)$.

(3) $\mathrm{Sh}_p(X\times R_+) = \mathrm{Sh}_p(Y\times R_+)$.

Here $\mathrm{Sh}_p(Z)$ means the proper shape of Z in the sense of Ball and Sher [1].

The equivalence of (1) and (2) is a consequence of Chapman ([4], Theorem 2) and Theorem 1. Also, the equivalence of (2) and (3) follows from Theorem 2.

Example. Let X be a one point space and let S be a dyadic solenoid (a solenoid to the sequence $2, 2, \ldots$; cf. [3, p. 154]). Since S is connected, $|\mathrm{Mor}_{\mathscr{C}}(X, S)| = 1$, where $\mathrm{Mor}_{\mathscr{C}}(X, S)$ means the set of morphisms from X to Y in the category $\mathscr{C}$ and $|Z|$ is the cardinal number of Z. On the other hand $|\mathrm{Mor}_{\mathscr{C}_f}(X, S)| = \mathfrak{c}$. Because, by Theorem 1 and [7], Example 2, it is easy to prove that $|\mathrm{Mor}_{\mathscr{C}_f}(X, <)|$ is equal to the cardinal number of the arc-components of S and the latter equals $\mathfrak{c}$. Therefore $\mathscr{C}$ and $\mathscr{C}_f$ are different. The functor Θ defined in Lemma 1 is onto but not an isomorphism.

4. Proofs of theorems.

Proof of Theorem 1. We need the following lemma.

Lemma 3. *Let M be a space and X an unstable subset of M. Then there exists an imbedding $j\colon M-X\to M\times R_+$ such that $j(M-X)$ is a strong deformation retract of $M\times R_+$.*

Proof. Since X is unstable in M, there exists a homotopy $\xi\colon M\times I\to M$ such that

$$\xi(x,0)=x,\ x\in M,\quad \text{and}\quad \xi(M\times(0,1])\subset M-X. \tag{4.1}$$

Choose a map $\alpha\colon M\to I$ such that $\alpha^{-1}(0)=X$. Put $K=\{(x,1/\alpha(x))\colon x\in M-X\}$. Then K is a closed subset of $M\times R_+$. Since the map $j\colon M-X\to K$ defined by $j(x)=(x,1/\alpha(x))$ for $x\in M-X$ is a homeomorphism onto, it is enough to prove that K is a strong deformation retract of $M\times R_+$. Define a map $r\colon M\times R_+\to K$ and a homotopy $h\colon M\times R_+\times I\to M\times R_+$ by

$$\begin{aligned}
r(x,t)&=(x,1/\alpha(x)),\quad t\geqslant 1/\alpha(x),\ x\in M-X,\\
&=\Big(\xi\big(x,(1-t\cdot\alpha(x))/1+t\big),\ 1/\alpha\big(\xi\big(x,(1-t\cdot\alpha(x))/1+t\big)\big)\Big),\\
&\qquad\qquad 0\leqslant t<1/\alpha(x),\ x\in M-X,\\
&=\big(\xi(x,1/1+t),\ 1/\alpha(\xi(x,1/1+t))\big),\quad (x,t)\in X\times R_+;\\
h(x,t,s)&=\big(x,\ s(1-t\cdot\alpha(x))/\alpha(x)+t\big),\quad t\geqslant 1/\alpha(x),\ s\in I,\ x\in M-X,\\
&=\Big(\xi\big(x,s(1-t\cdot\alpha(x))/1+t\big),\ \Big(1/\alpha\big(\xi\big(x,(1-t\cdot\alpha(x))/1+t\big)\big)-t\Big)s+t\Big),\\
&\qquad\qquad 0\leqslant t<1/\alpha(x),\ s\in I,\ x\in M-X,\\
&=\Big(\xi(x,s/1+t),\big(1/\alpha(\xi(x,1/1+t))-t\big)s+t\Big),\quad (x,t,s)\in X\times R_+\times I.
\end{aligned}\tag{4.2}$$

Obviously r is a retraction, $h(x,t,0)=(x,t)$ and $h(x,t,1)=r(x,t)$ for $(x,t)\in M\times R_+$, and $h(x,1/\alpha(x),s)=(x,1/\alpha(x))$ for $x\in M-X$ and $s\in I$. This completes the proof.

Let X and Y be compacta and let $M\in\mathscr{M}(X)$ and $N\in\mathscr{M}(Y)$. Since X and Y are unstable in M and N respectively, there exist homotopies $\xi\colon M\times I\to M$ and $\eta\colon N\times I\to N$ such that

$$\xi(x,0)=x,\ x\in M,\quad\text{and}\quad \xi(M\times(0,1]\subset M-X, \tag{4.3}$$

$$\eta(y,0)=y,\ y\in N,\quad\text{and}\quad \eta(N\times(0,1])\subset N-Y. \tag{4.4}$$

Let $f\colon M-X\to N-Y$ be a proper map. Consider the subset $K=\{(x,1/\alpha(x))\colon x\in M-X\}$ of $M\times R_+$ and a homeomorphism $j\colon M-X\to K$ defined by $j(x)=(x,1/\alpha(x))$, $x\in M-X$. Let $i\colon N-Y\to N$ be the inclusion. Define a map $\varphi(f)\colon M\times R_+\to N$ by

$$\varphi(f)=i\cdot f\cdot j^{-1}\cdot r, \tag{4.5}$$

where r is the strong deformation retraction from $M\times R_+$ into K defined in Lemma 3 (cf. (4.2)). Obviously $\varphi(f)$ is a fundamental map from X to Y in M,N. Let $g\colon M-X\to N-Y$ be a proper map. Then

$$f\underset{p}{\simeq}g\quad\text{if and only if}\quad \varphi(f)\underset{p}{\simeq}\varphi(g). \tag{4.6}$$

Indeed, if $H\colon(M-X)\times I\to N-Y$ is a proper homotopy connecting f and g, then the map $H'\colon M\times R_+\times I\to N$ defined by $H'(x,t,s)=iH(j^{-1}r(x,t),s)$, $(x,t,s)\in M\times R_+\times I$, is a fine homotopy connecting $\varphi(f)$ and $\varphi(g)$. Conversely, suppose $\varphi(f)\underset{f}{\simeq}\varphi(g)$ and let $H\colon M\times R_+\times I\to N$ be a fine homotopy connecting $\varphi(f)$ and $\varphi(g)$. Define a map $H'\colon M\times R_+\times I\to N-Y$ by

$$H'(x,t,s)=\eta\big(H(x,t,s),s(1-s)/1+t\big)\quad\text{for}\quad (x,t,s)\in M\times R_+\times I. \tag{4.7}$$

Note that iH' is a fine homotopy connecting $\varphi(f)$ and $\varphi(g)$, where i is the inclusion: $N-Y\to N$. Moreover it is easy to prove that $H'|K\times I\colon K\times I\to N-Y$ is a proper map. The map $H''\colon(M-X)\times I\to N-Y$ defined by $H''(x,s)=H'(j(x),s)$, $(x,s)\in(M-X)\times I$, gives a proper homotopy connecting f and g.

Finally, let $F\colon X\to Y$ in M,N be a fundamental map. We shall prove that

$$\text{there exists a proper map } f\colon M-X\to N-Y \text{ such that } \varphi(f)\underset{f}{\simeq}F. \tag{4.8}$$

Define $F'\colon M\times R_+\to N-Y$ by

$$F'(x,t)=\eta(F(x,t),1/1+t)\quad\text{for}\quad (x,t)\in M\times R_+.$$

Note that $iF'\underset{f}{\simeq}F$ and $F'|K\colon K\to N-Y$ is a proper map. Let us define $f\colon M-X\to N-Y$ by $f(x)=F'j(x)$, $x\in M-X$. Then f is a proper map. To prove $\varphi(f)\underset{f}{\simeq}F$, it is enough to prove that $\varphi(f)\underset{f}{\simeq}iF'$. Since $\varphi(f)=iF'r$, we have $\varphi(f)\underset{f}{\simeq}iF'$. A fine homotopy connecting iF' and $iF'r$ is given by $iF'h$, where h is the homotopy defined in (4.2).

To complete the proof of Theorem 1, for a morphism ξ from X to Y in the category $\mathscr{P}$, take a proper map $f\colon M-X\to N-Y$ representing ξ. Define $\Phi(\xi)$ as the f-class determined by the map $\varphi(f)\colon M\times R_+\to N\times R_+$. It is obvious that Φ is a functor. (4.6) and (4.8) show that Φ is an isomorphism. This completes the proof.

Proof of Theorem 2. Let X and Y be compacta and let M and N be compact AR's containing X and Y respectively. Throughout the proof we use the following notations.

$J=$ the set of non negative integers,

$\Delta=$ the set of all increasing functions $\delta\colon J\to J$,

$\{U_i\colon i\in J\}=$ a neighborhood basis of X in M such that each U_i is closed and $U_i\supset U_{i+1}$ for $i\in J$,

$\{V_i\colon i\in J\}=$ a neighborhood basis of Y in N such that each V_i is open and $V_i\supset V_{i+1}$ for $i\in J$,

$U_\delta=\bigcup_{i\in J}U_{\delta(i)}\times[i,i+1]$ for $\delta\in\Delta$,

$V_\delta=\bigcup_{i\in J}V_{\delta(i)}\times[i,i+1]$ for $\delta\in\Delta$.

Note that $\{U_\delta\colon\delta\in\Delta\}$ and $\{V_\delta\colon\delta\in\Delta\}$ form neighborhood bases of $X\times R_+$ and $Y\times R_+$ in $M\times R_+$ and $N\times R_+$, respectively. For $\delta,\delta'\in\Delta$, if $\delta(i)\leqslant\delta'(i)$ for each $i \in J$, we write $\delta \leqslant \delta'$. Obviously Δ forms a directed set under the relation $\leqslant$. Note that

(4.9) if $\delta \leqslant \delta'$ then $U_\delta \supset U_{\delta'}$ and $V_\delta \supset V_{\delta'}$.

For each $\delta \in \Delta$, let $\varrho_\delta\colon R_+ \to R_+$ be a map defined by

(4.10) $$\varrho_\delta(t) = (t-i)\big(\delta(i+1)-\delta(i)\big)+\delta(i)\,, \quad i \leqslant t \leqslant i+1,\ i \in J.$$

Let F be a fundamental map from X to Y in M, N. We shall define a proper fundamental net $\psi(F) = \{f_\delta\colon \delta \in \Delta\}$ as follows. By the definition of a fundamental map (cf. (1.1)), we can find a $\delta_F \in \Delta$ such that

(4.11) $$F\big(U_{\delta_F(i)} \times [\delta_F(i), \infty)\big) \subset V_i \quad \text{for} \quad i \in J.$$

Define $f_\delta\colon M \times R_+ \to M \times R_+$ for $\delta \in \Delta$ by

(4.12) $$f_\delta(x, t) = \big(F(x, \varrho_{\delta_F \cdot \delta}(t)), t\big) \quad \text{for} \quad (x, t) \in M \times R_+\,,$$

where $\delta_F \delta$ is the composition of δ and δ_F, that is, $\delta_F \cdot \delta(i) = \delta_F(\delta(i))$, $i \in J$ (see (4.9) for $\varrho_{\delta_F\delta}$). Put $\psi(F) = \{f_\delta\colon \delta \in \Delta\}$.

(4.13) $\psi(F)$ is a proper fundamental net from $X \times R_+$ to $Y \times R_+$ in $M \times R_+$, $N \times R_+$ and its proper fundamental class does not depend on the choice of a δ_F satisfying (4.11).

Indeed, let W be a neighborhood of $Y \times R_+$ in $N \times R_+$. There exists a $\delta_0 \in \Delta$ such that $V_{\delta_0} \subset W$. Let $\delta \geqslant \delta_0$, $\delta \in \Delta$. Define $H\colon U_{\delta_F \delta_0} \times I \to V_{\delta_0} (\subset W)$ by

$$H(x, t, s)\ \big(F(x, (1-s)\varrho_{\delta_F \delta_0}(t) + s\varrho_{\delta_F \delta}(t)), t\big) \quad \text{for} \quad (x, t, s) \in U_{\delta_F \delta_0} \times I.$$

Then H is a proper homotopy connecting $f_\delta | U_{\delta_F \delta_0}$ and $f_{\delta_0} | U_{\delta_F \delta_0}$. Therefore $\psi(F)$ forms a proper fundamental net. The latter half of (4.13) is proved similarly. Next, we shall prove

(4.14) if F and G are fundamental maps from X to Y in M, N then $F \underset{f}{\simeq} G$ iff $\psi(F) \underset{p}{\simeq} \psi(G)$.

Indeed, let $F \underset{f}{\simeq} G$ and let $L\colon M \times R_+ \times I \to N$ be a fine homotopy connecting F and G. In the light of (1.3) we can find a $\delta_L \in \Delta$ such that

$$\delta_L \geqslant \delta_F, \quad \delta_L \geqslant \delta_G \quad \text{and} \quad L\big(U_{\delta_L(i)} \times [\delta_L(i), \infty) \times I\big) \subset V_i \ \text{for}\ i \in J.$$

Let W be any neighborhood of $Y \times R_+$ in $N \times R_+$. By (4.12) and (4.13) there exists a $\delta_0 \in \Delta$ such that

$$f_{\delta_0} | U_{\delta_L \delta_0} \underset{p}{\simeq} f_\delta | U_{\delta_L \delta_0} \quad \text{in } W, \quad \delta \geqslant \delta_0\,,$$

$$g_{\delta_0} | U_{\delta_L \delta_0} \underset{p}{\simeq} g_\delta | U_{\delta_L \delta_0} \quad \text{in } W, \quad \delta \geqslant \delta_0\,.$$

Define $\xi\colon U_{\delta_L \delta_0} \times I \to W$ by

$$\begin{aligned}\xi(x, t, s) &= \big(F(x, (1-3s)\varrho_{\delta_F \delta_0}(t) + 3s\varrho_{\delta_L \delta_0}(t)), t\big) \quad 0 \leqslant s \leqslant \tfrac{1}{3}\,,\\ &= \big(L(x, \varrho_{\delta_L \delta_0}(t), 3s-1), t\big)\,, \quad \tfrac{1}{3} \leqslant s \leqslant \tfrac{2}{3}\,,\\ &= \big(G(x, (3s-2)\varrho_{\delta_L \delta_0}(t) + (3-3s)\varrho_{\delta_L \delta_0}(t)), t\big)\,, \quad \tfrac{2}{3} \leqslant t \leqslant 1,\ (x, t) \in U_{\delta_L \delta_0}.\end{aligned}$$

Then ξ is a proper homotopy connecting $f_{\delta_0} | U_{\delta_L \delta_0}$ and $g_{\delta_0} | U_{\delta_L \delta_0}$.

Conversely, suppose that $\psi(F) \underset{p}{\simeq} \psi(G)$. Consider the neighborhood $W_0 = V_{\mathrm{id}} = \bigcup_{i \in J} V_i \times [i, i+1]$ of $Y \times R_+$ in $N \times R_+$. Since $\psi(F) \underset{p}{\simeq} \psi(G)$, there exists a $\delta_0 \in \Delta$ such that

$$f_\delta | U_{\delta_0} \underset{p}{\simeq} g_{\delta'} | U_{\delta_0} \quad \text{in } W_0\,, \quad \delta, \delta' \geqslant \delta_0\,.$$

Let $H\colon U_{\delta_0} \times I \to W_0$ be a proper homotopy connecting $f_{\delta_0} | U_{\delta_0}$ and $g_{\delta_0} | U_{\delta_0}$. Since H is a proper map, there exists a $\delta_H \in \Delta$ such that $\delta_H \geqslant \delta_F$, δ_G, δ_0 and

(4.15) $$H\big((U_{\delta_0} \cap M \times [\delta_H(i), \infty)) \times I\big) \subset V_i \times [i, \infty) \quad \text{for} \quad i \in J.$$

Let $q\colon N \times R_+ \to N$ be the projection. Since V_i is an ANR, by making use of Borsuk's homotopy extension theorem repeatedly qH is extended to a homotopy $H'\colon M \times \times R_+ \times I \to N$ such that

(4.16) $$\begin{aligned} &H'(x, t, 0) = qf_{\delta_0}(x, t)\,, \quad H'(x, t, 1) = qg_{\delta_0}(x, t) \ \text{for}\ (x, t) \in M \times R_+\,,\\ &H'\big(U_{\delta_H(i)} \times [\delta_H(i), \infty) \times I\big) \subset V_i \quad \text{for} \quad i \in J.\end{aligned}$$

Define $L\colon M \times R_+ \times I \to N$ by

$$\begin{aligned} L(x, t, s) &= F\big(x, (1-3s)t + 3s\varrho_{\delta_F \delta_0}(t)\big)\,, \quad 0 \leqslant s \leqslant \tfrac{1}{3}\,,\\ &= H'(x, t, 3s-1)\,, \quad \tfrac{1}{3} \leqslant s \leqslant \tfrac{2}{3},\ (x, t) \in M \times R_+\,,\\ &= G\big(x, (3s-2)t + (3-3s)\varrho_{\delta_G \delta_0}(t)\big)\,, \quad \tfrac{2}{3} \leqslant s \leqslant 1\,.\end{aligned}$$

By (4.15) and (4.16), $L\big(U_{\delta_H(i)} \times [\delta_H(i), \infty) \times I\big) \subset V_i$ for each $i \in J$. Hence L is a fine homotopy connecting F and G. This completes the proof of (4.14).

Finally, we shall prove that

(4.17) for every proper fundamental net $\underline{f}$ from $X \times R_+$ to $Y \times R_+$ in $M \times R_+$, $N \times R_+$ there exists a fundamental map $G\colon X \to Y$ in M, N such that $\psi(G) \underset{p}{\simeq} \underline{f}$.

Let $\underline{f} = \{f_\lambda\colon \lambda \in \Lambda\}$, $f_\lambda\colon M \times R_+ \to N \times R_+$, where Λ is a directed set. There exist a $\lambda_0 \in \Lambda$ and a $\delta_0 \in \Delta$ such that

(4.18) $$f_\lambda | U_{\delta_0} \underset{p}{\simeq} f_{\lambda_0} | U_{\delta_0} \quad \text{in } W_0 \quad \text{for every } \lambda \geqslant \lambda_0\,,$$

where $W_0 = \bigcup_{i \in J} V_i \times [i, i+1]$. Since $f_{\lambda_0}|U_{\delta_0}$ is proper there exists a $\delta_1 \in \Delta$, $\delta_1 \geqslant \delta_0$, such that

$$f_{\lambda_0}\big(U_{\delta_0} \cap M \times [\delta_1(i), \infty)\big) \subset V_i \times [i, \infty) \quad \text{for} \quad i \in J.$$

By applying Borsuk's homotopy extension theorem repeatedly we can extend $qf_{\lambda_0}|U_{\delta_0}$ to a fundamental map $G\colon X \to Y$ in M, N such that

$$G\big(U_{\delta_1(i)} \times [\delta_1(i), \infty)\big) \subset V_i \quad \text{for} \quad i \in J,$$

where q is the projection: $N \times R_+ \to N$. It remains to prove that $\psi(G) \underset{p}{\simeq} f$. Remind that

(4.19) $\psi(G) = \{g_\delta\colon \delta \in \Delta\}, \quad g_\delta(x, t) = \big(G(x, \varrho_{\delta_G \delta}(t)), t\big) \quad \text{for} \quad (x, t) \in M \times R_+.$

Let W be any neighborhood of $Y \times R_+$ in $N \times R_+$. We may assume that $W = V_{\bar\delta}$, $\bar\delta \in \Delta$. There exist indices $\lambda_1 \in \Lambda$ and $\delta_2, \delta_3 \in \Delta$ such that $\lambda_1 \geqslant \lambda_0$, $\delta_3 \geqslant \delta_0$ and

$$f_\lambda|U_{\delta_3} \underset{p}{\simeq} f_{\lambda_1}|U_{\delta_3} \quad \text{in } V_{\bar\delta}, \quad \lambda \geqslant \lambda_1,$$

$$g_\delta|U_{\delta_3} \underset{p}{\simeq} g_{\delta_2}|U_{\delta_3} \quad \text{in } V_{\bar\delta}, \quad \delta \geqslant \delta_2.$$

By (4.18) there is a proper homotopy $H\colon U_{\delta_0} \times I \to W_0$ connecting $f_{\lambda_1}|U_{\delta_0}$ and $f_{\lambda_0}|U_{\delta_0}$. Since H is proper and $(U_{\delta_0} \cap M \times [0, i]) \times I$ is compact, we can find $\delta_M, \delta_H \in \Delta$ such that

(4.20) $q'H\big((U_{\delta_0} \cap M \times [0, i]) \times I\big) \subset [0, \delta_M(i)] \quad \text{for} \quad i \in J,$

(4.21) $q'H\big((U_{\delta_0} \cap M \times [\delta_H(i), \infty)) \times I\big) \cap [0, i] = \emptyset \quad \text{for} \quad i \in J,$

where q' is the projection: $N \times R_+ \to R_+$. From (4.21)

(4.22) $qH\big((U_{\delta_0} \cap M \times [\delta_H(i), \infty)) \times I\big) \subset V_i \quad \text{for} \quad i \in J.$

By (4.20), (4.21) and (4.22)

(4.23) $qH(U_{\delta_0 \delta_H \bar\delta(i+1)} \times [\delta_H \bar\delta(i), \delta_H \bar\delta(i+1)] \times I) \subset V_{\bar\delta(i)} \quad \text{for} \quad i \in J,$

$$q'H(U_{\delta_0 \delta_H(i+1)} \times [\delta_H(i), \delta_H(i+1)] \times I) \subset [i, \delta_M \delta_H(i+1)] \quad \text{for} \quad i \in J.$$

Let us define indices $\delta_4, \delta_5 \in \Delta$ as follows.

$$\delta_4(i) = \mathrm{Max}\big(\delta_0 \delta_H \bar\delta \delta_M \delta_H(i+1), \delta_3 \delta_H \bar\delta \delta_M \delta_H(i+1)\big) \quad \text{for} \quad i \in J,$$
$$\delta_5(i) = \delta_H \bar\delta \delta_M \delta_H(i) \quad \text{for} \quad i \in J.$$

Define $L\colon U_{\delta_4} \times I \to V_{\bar\delta}$ by

$$\begin{aligned} qL(x, t, s) &= qf_{\lambda_1}\big(x, (1-3s)t + 3s\varrho_{\delta_5}(t)\big), \quad 0 \leqslant s \leqslant \tfrac{1}{3}, \\ &= qH\big(x, (3s-1)\varrho_{\delta_5}(t)\big), \quad \tfrac{1}{3} \leqslant s \leqslant \tfrac{2}{3}, \\ &= qf_{\lambda_0}\big(x, (3s-2)\varrho_{\delta_G \bar\delta}(t) + (3-3s)\varrho_{\delta_5}(t)\big), \quad \tfrac{2}{3} \leqslant s \leqslant 1, \ (x, t) \in U_{\delta_4} \times I. \end{aligned}$$

$$\begin{aligned} q'L(x, t, s) &= q'f_{\lambda_1}\big(x, (1-3s)t + 3s\varrho_{\delta_H}(t)\big), \quad 0 \leqslant s \leqslant \tfrac{1}{3}, \\ &= q'H\big(x, (3s-1)\varrho_{\delta_H}(t)\big), \quad \tfrac{1}{3} \leqslant s \leqslant \tfrac{2}{3}, \ (x, t) \in U_{\delta_4} \times I, \\ &= (3s-2)t + (3-3s)q'f_{\lambda_0}\big(x, \varrho_{\delta_H}(t)\big), \quad \tfrac{2}{3} \leqslant s \leqslant 1. \end{aligned}$$

It is easy to see that L is a proper homotopy connecting $f_{\lambda_1}|U_{\delta_4}$ and $g_{\delta_3}|U_{\delta_4}$ (cf. (4.19)). This implies $\psi(G) \underset{p}{\simeq} f$ and completes the proof of (4.17).

Now, to complete the proof of Theorem 2, for a morphism ξ from X to Y in $\mathscr{C}_f$, take a fundamental map $F\colon X \to Y$ in M, N representing ξ. Define $\Psi(\xi)$ as the pn-class determined by the proper fundamental net $\psi(F)\colon X \times R_+ \to Y \times R_+$ in $M \times R_+$, $N \times R_+$. Obviously Ψ is a functor from $\mathscr{C}_f$ to $\tilde{\mathscr{C}}_p$. (4.14) and (4.17) show that Ψ is a category isomorphism. This completes the proof.

Finally, we have the following theorem.

THEOREM 3. *Let X, X' be compacta and let Y, Y' be metrizable spaces. If $\mathrm{Sh}_f(X) \geqslant \mathrm{Sh}_f(X')$ and $\mathrm{Sh}(Y) \geqslant \mathrm{Sh}(Y')$, then $\mathrm{Sh}(X \times Y) \geqslant \mathrm{Sh}(X' \times Y')$, where $\mathrm{Sh}(Z)$ is the shape of a metrizable space Z in the sense of Fox.*

This theorem can be proved similarly as [9], Theorem 1, so we shall omit the proof. We do not know whether Sh_f can be replaced by Sh in Theorem 3.

PROBLEM. For compacta X and Y, does $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ imply $\mathrm{Sh}_f(X) \geqslant \mathrm{Sh}_f(Y)$?

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