as determined by x in the manner described in Section 1. See Figure 7 for a picture of $R(S_x)$ where R is a figure eight, and Figure 8 for another example.

We can map $R(S_x)$ onto S_x by mapping S^1 identically onto itself, all of the k spirals homeomorphically onto one of themselves, and A onto the endpoint of that spiral. It may be checked by means of nerves of ε-covers that $R(S_x)$ is R-like. But there can be no model for $\mathscr{R}$ since, if there were, it would also be a model for $\mathscr{S}$. ■

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Decompositions in the product of a measure space and a Polish space

by

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Abstract. Let X, $\mathcal{M}$, μ be a complete probability space and Y a Polish space with Borel field $\mathcal{B}_Y$. It is shown that if $A \in \mathcal{M}\otimes\mathcal{B}_Y$, then $\{x \in X;\ A(x) \text{ is } F_\sigma\}$ and $\{x \in X;\ A(x) \text{ is } F_{\sigma\delta}\}$ are both measurable. Furthermore, we prove the existence of "measurable decompositions". From those results, we deduce a theorem on the stability of the class of the Baire-2 functions under integration.

Introduction. Assume X, μ a probability space and let $\mathscr{M}$ be the σ-algebra of σ-measurable subsets of X. Let Y be a Polish space with Borel field $\mathscr{B}_Y$. By well known arguments, we obtain that if $A \in \mathscr{M}\otimes\mathscr{B}_Y$, then the sections $A(x)$, where x is taken in X, are of bounded Baire class. Hence $\mathscr{M}\otimes\mathscr{B}_Y$ is the union of the classes $\mathscr{S}_\alpha$ $(\alpha<\omega_1)$, consisting of the sets $A \in \mathscr{M}\otimes\mathscr{B}_Y$, such that $A(x)$ is of Baire class at most α, for each $x \in X$, where the Baire class is defined with respect to the closed sets. Let $\mathscr{F}_0 = \mathscr{S}_0$, which is stable under countable intersections. Starting from $\mathscr{F}_0$, we obtain a Baire system $(\mathscr{F}_\alpha)_{\alpha<\omega_1}$. It is a natural question if $\mathscr{S}_\alpha$ and $\mathscr{F}_\alpha$ coincide for all $\alpha<\omega_1$. We will answer if affirmatively for $\alpha = 1$ and $\alpha = 2$.

Let $\mathscr{P} = \{M\times F;\ M \in \mathscr{M},\ F \text{ closed in } Y\}$. The class of the $\mathscr{P}$-analytic subsets of $X\times Y$ will be denoted by $\mathscr{A}(X, Y)$, or simply $\mathscr{A}$, if no confusion is possible. Let $A \in \mathscr{A}$ and assume $A = = \bigcup_\nu \bigcap_k (M_{\nu|k}\times F_{\nu|k})$, where ν runs over $\mathscr{N} = N^N$. In such a representation, it will be always assumed that

$$M_{\nu|k}\times F_{\nu|k} \neq \emptyset, \quad M_{\nu|k+1}\times F_{\nu|k+1} \subset M_{\nu|k}\times F_{\nu|k} \quad \text{and} \quad \operatorname{diam} F_{\nu|k} \leqslant 1/k,$$

for each $\nu \in \mathscr{N}$ and $k \in N$. It is easily seen that $\mathscr{A}$ contains $\mathscr{M}\otimes\mathscr{B}_Y$.

Definition 1. If $A \subset X\times Y$, then $\bar{A}^s \subset X\times Y$ is defined by $\bar{A}^s(x) = \overline{A(x)}$.

The following description of the set $\bar{A}^s$ will be useful. If $y \in Y$ and $\varepsilon>0$, then $B(y, \varepsilon)$ is the open ball with midpoint y and radius ε. Let $(y_n)_n$ be a dense sequence in Y. For every $n \in N$ and $k \in N$ we take $M_{nk} = \pi_1\big(A \cap (X\times B(y_n, 1/k))\big)$, where π_1 is the projection on X. Then $\bar{A}^s = \bigcap_k \bigcup_n \big(M_{nk}\times B(y_n, 1/k)\big)$. From this observation, we obtain immediately:

PROPOSITION 2. *Let* $A\subset X\times Y$ *and suppose* $\mu(\pi_1(A))=0$. *If* $A(x)$ *is closed in* Y *for each* $x\in X$, *then* $A\in\mathcal{M}\otimes\mathcal{B}_Y$.

PROPOSITION 3. *If* $A\in\mathcal{A}$, *then* $\pi_1(A)\in\mathcal{M}$.

Proof. Assume $A=\bigcup_\nu\bigcap_k(M_{\nu|k}\times F_{\nu|k})$, then $\pi_1(A)=\bigcup_\nu\bigcap_k M_{\nu|k}$ and therefore an $\mathcal{M}$-analytic set. Thus $\pi_1(A)\in\mathcal{M}$.

Clearly (3) implies the following:

PROPOSITION 4. *If* $A\in\mathcal{A}$, *then* $\bar{A}^s\in\mathcal{M}\otimes\mathcal{B}_Y$.

PROPOSITION 5. *Let* Y, Z *be Polish spaces and assume* $f\colon X\times Y\to X\times Z$ *an* $\mathcal{M}\otimes\mathcal{B}_Y-\mathcal{M}\otimes\mathcal{B}_Z$ *measurable map satisfying* $\pi_1\circ f=\pi_1$. *Then* $f(A)\in\mathcal{A}(X,Z)$, *for each* $A\in\mathcal{A}(X,Y)$.

Proof. Let $g=\pi_2\circ f$, which is $\mathcal{M}\otimes\mathcal{B}_Y-\mathcal{B}_Z$ measurable. Let $h\colon X\times Y\times Z\to Z^2$ be given by $h(x,y,z)=\big(g(x,y),z\big)$. Clearly h is $\mathcal{M}\otimes\mathcal{B}_Y\otimes\mathcal{B}_Z-\mathcal{B}_Z\otimes\mathcal{B}_Z$ measurable. Hence $h^{-1}(\Delta)\in\mathcal{M}\otimes\mathcal{B}_Y\otimes\mathcal{B}_Z$, where Δ is the diagonal of Z^2. Thus $h^{-1}(\Delta)\cap(A\times Z)$ belongs to $\mathcal{A}(X,Y\times Z)$. Therefore $h^{-1}(\Delta)\cap(A\times Z)$ has a representation $\bigcup_\nu\bigcap_k(M_{\nu|k}\times F'_{\nu|k}\times F''_{\nu|k})$, implying

$$f(A)=\pi_{X\times Z}(h^{-1}(\Delta)\cap(A\times Z))=\bigcup_\nu\bigcap_k(M_{\nu|k}\times F''_{\nu|k})\in\mathcal{A}(X,Z).$$

PROPOSITION 6. *If* $A\in\mathcal{A}(X,Y)$, *then there exist a subset* D *of* $X\times\mathcal{N}$ *and a map* $\varphi\colon X\times\mathcal{N}\to X\times Y$, *verifying the following properties*:

1. $D\in\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}$,
2. φ *is* $\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}-\mathcal{M}\otimes\mathcal{B}_Y$ *measurable*,
3. $D(x)$ *is closed in* $\mathcal{N}$ *for each* $x\in X$,
4. $\pi_1\circ\varphi=\pi_1$ *and* $\pi_2\circ\varphi_x$ *is continuous for each* $x\in X$,
5. $\varphi(D)=A$.

Proof. Assume $A=\bigcup_\nu\bigcap_k(M_{\nu|k}\times F_{\nu|k})$. For every $k\in N$ and $(n_1,\dots,n_k)\in N^k$, let

$$\mathcal{N}_{n_1\dots n_k}=\{\nu\in\mathcal{N};\ \nu_1=n_1,\dots,\nu_k=n_k\}.$$

Obviously

$$D=\bigcup_\nu\bigcap_k(M_{\nu|k}\times\mathcal{N}_{\nu|k})=\bigcap_k\bigcup_{n_1,\dots,n_k}(M_{n_1\dots n_k}\times\mathcal{N}_{n_1\dots n_k})$$

verifies (1) and (3). The map φ is given by $\varphi(x,\nu)=(x,\bigcap_k F_{\nu|k})$, where $\bigcap_k F_{\nu|k}$ is a unique point of Y. Clearly $\pi_1\circ\varphi=\pi_1$ and using standard arguments we obtain that $\pi_2\circ\varphi_x$ is continuous for each $x\in X$. It follows immediately that φ is $\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}-\mathcal{M}\otimes\mathcal{B}_Y$ measurable. Further

$$\varphi(D)=\bigcup_\nu\varphi(\bigcap_k M_{\nu|k}\times\{\nu\})=\bigcup_\nu(\bigcap_k M_{\nu|k}\times\bigcap_k F_{\nu|k})=A,$$

completing the proof.

Decomposition results. Let $X,\mathcal{M},\mu$ be a complete probability space and Y a Polish space. π_1 will denote the projection on X and π_2 the projection on Y. The bulk of this paper is the following proposition, which is essentially based on the Baire category theorem.

PROPOSITION 7. *For each* $x\in X$, *let* $\mathcal{C}_x$ *be a set of closed subsets of* Y, *verifying the following condition*:

(S) *A closed subset of a set in* $\mathcal{C}_x$ *also belongs to* $\mathcal{C}_x$ *and* $\emptyset\in\mathcal{C}_x$.

Suppose $A\in\mathcal{A}(X,Y)$ *and take* $C=\{x\in X;\ A(x)$ *can be covered by countably many elements of* $\mathcal{C}_x\}$. *Then there are sequences* $(A_n)_n$, $(B_k)_k$ *in* $\mathcal{M}\otimes\mathcal{B}_Y$ *and a sequence* $(X_k)_k$ *of subsets of* X, *such that*:

1. $A_n(x)\in\mathcal{C}_x$ *for each* $n\in N$ *and each* $x\in X$.

For each $k\in N$:

2. $\mu_*(X_k)=0$,
3. $\{x\in\pi_1(B_k);\ B_k(x)\in\mathcal{C}_x\}\subset X_k$,
4. $X_k\setminus\{x\in\pi_1(B_k);\ B_k(x)\in\mathcal{C}_x\}$ *is negligible*,
5. $C\cap\pi_1(A\setminus\bigcup_n A_n)\subset\bigcup_k X_k$.

Proof. Using Proposition 6, there is a subset D of $X\times\mathcal{N}$ and a map $\varphi\colon X\times\mathcal{N}\to X\times Y$ satisfying the following properties:

1. $D\in\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}$,
2. φ is $\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}-\mathcal{M}\otimes\mathcal{B}_Y$ measurable,
3. $D(x)$ is closed in $\mathcal{N}$ for each $x\in X$,
4. $\pi_1\circ\varphi=\pi_1$ and $\pi_2\circ\varphi_x$ is continuous for each $x\in X$,
5. $\varphi(D)=A$.

Let $(V_k)_k$ be a countable base for the topology of $\mathcal{N}$. By transfinite induction, we define for each $\alpha<\omega_1$ a sequence $(M_{k\alpha})_k$ in $\mathcal{M}$ and a sequence $(\psi_{k\alpha})_k$ in $\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}$ as following:

1. Let $M_{k0}=\psi_{k0}=\emptyset$ for each $k\in N$.
2. Now assume $(M_{k\alpha})_k$ and $(\psi_{k\alpha})_k$ obtained for each $\alpha<\beta$. Let $k\in N$ be fixed. Take

$$\Phi_{k\beta}=(X\times V_k)\setminus\bigcup_{\alpha<\beta}\bigcup_l\psi_{l\alpha}\in\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}.$$

Let $M_{k\beta}$ be a measurable subset of

$$\{x\in X;\ \overline{\varphi(\Phi_{k\beta}\cap D)^s(x)}\in\mathcal{C}_x\},$$

with measure the inner measure of this set. Finally, take

$$\psi_{k\beta}=\Phi_{k\beta}\cap(M_{k\beta}\times\mathcal{N})\in\mathcal{M}\otimes\mathcal{B}_{\mathcal{N}}.$$

This completes the construction.

It is clear that when $k \in N$ is fixed, $\mu(M_{k\alpha})$ increases when α increases. Hence there exists some ordinal $\eta < \omega_1$ so that $\mu(M_{k\eta}) = \mu(M_{k,\eta+1})$ for each $k \in N$. The sequence $(A_n)_n$ will be the countable family $\{\overline{\varphi(\psi_{k\alpha} \cap D)}^s;\ \alpha \leqslant \eta,\ k \in N\}$. For each $k \in N$, define

$$B_k = \overline{\varphi(\Phi_{k,\eta+1} \cap (M_{k\eta}^c \times \mathcal{N}) \cap D)}^s \quad \text{and} \quad X_k = \{x \in M_{k\eta}^c;\ \overline{\varphi(\Phi_{k,\eta+1} \cap D)}^s(x) \in \mathscr{C}_x\}.$$

Using successively Proposition 5 and Proposition 4, we see that $(A_n)_n$ and $(B_k)_k$ are sequences in $\mathscr{M} \otimes \mathscr{B}_Y$. We prove that the required conditions are verified.

1. Let $\alpha \leqslant \eta$, $k \in N$ and $x \in X$. We remark that

$$\varphi(\psi_{k\alpha} \cap D)(x) = (\pi_2 \circ \varphi_x)(\psi_{k\alpha}(x) \cap D(x)).$$

If $x \notin M_{k\alpha}$, then

$$\overline{\varphi(\psi_{k\alpha} \cap D)}^s(x) = \emptyset \in \mathscr{C}_x.$$

If $x \in M_{k\alpha}$, we have

$$\overline{\varphi(\psi_{k\alpha} \cap D)}^s(x) = \overline{\varphi(\Phi_{k\alpha} \cap D)}^s(x) \in \mathscr{C}_x.$$

2. This follows from the property

$$\mu(M_{k\eta}) = \mu(M_{k,\eta+1}) = \mu_*\{x \in X;\ \overline{\varphi(\Phi_{k,\eta+1} \cap D)}^s(x) \in \mathscr{C}_x\}.$$

3. If $x \in \pi_1(B_k)$, then $x \in M_{k\eta}^c$ and $B_k(x) = \overline{\varphi(\Phi_{k,\eta+1} \cap D)}^s(x)$.

4. First, we have that $M_{k\eta}^c \setminus \pi_1(B_k) \subset X_k$. Indeed, if $x \in M_{k\eta}^c \setminus \pi_1(B_k)$, then

$$\emptyset = B_k(x) = \overline{\varphi(\Phi_{k,\eta+1} \cap D)}^s(x) \in \mathscr{C}_x.$$

Since $M_{k\eta}^c \setminus \pi_1(B_k) \in \mathscr{M}$, $M_{k\eta}^c \setminus \pi_1(B_k)$ is negligible. Now

$$X_k \setminus \{x \in \pi_1(B_k);\ B_k(x) \in \mathscr{C}_x\} \subset (M_{k\eta}^c \setminus \pi_1(B_k)) \cup \{x \in X_k;\ B_k(x) \notin \mathscr{C}_x\}.$$

Because $\{x \in X_k;\ B_k(x) \notin \mathscr{C}_x\} = X_k \cap \{x \in M_{k\eta}^c;\ B_k(x) \notin \mathscr{C}_x\} = \emptyset$, we obtain (4).

5. Let $x \in X$ be fixed and define $I_x = \{k \in N;\ x \in M_{k\eta}\}$. Obviously

$$V_k \subset \bigcup_{\alpha < \eta} \bigcup_{l} \psi_{l\alpha}(x) \cup \Phi_{k\eta}(x) \quad \text{for each } k \in N.$$

Therefore $\bigcup_{l \in I_x} V_l \subset \bigcup_{\alpha \leqslant \eta} \bigcup_{l} \psi_{l\alpha}(x)$.

Now suppose $x \in C$ and $D(x) \not\subset \bigcup_{\alpha \leqslant \eta} \bigcup_{l} \psi_{l\alpha}(x)$. Clearly $D(x) \setminus \bigcup_{l \in I_x} V_l \neq \emptyset$.

Let $(F_r)_r$ be a sequence in $\mathscr{C}_x$ with $A(x) \subset \bigcup_r F_r$. Because $D(x) \subset \bigcup_r (\pi_2 \circ \varphi_x)^{-1}(F_r)$ and each set $(\pi_2 \circ \varphi_x)^{-1}(F_r)$ is closed in $\mathcal{N}$, we obtain by the Baire category theorem some $r \in N$ such that $(\pi_2 \circ \varphi_x)^{-1}(F_r) \cap (D(x) \setminus \bigcup_{l \in I_x} V_l)$ has nonempty interior in $D(x) \setminus \bigcup_{l \in I_x} V_l$.

Take $k \in N$ satisfying

$$\emptyset \neq V_k \cap (D(x) \setminus \bigcup_{l \in I_x} V_l) \subset (\pi_2 \circ \varphi_x)^{-1}(F_r).$$

It follows that $(\Phi_{k,\eta+1} \cap D)(x) \subset (\pi_2 \circ \varphi_x)^{-1}(F_r)$ and thus $\varphi(\Phi_{k,\eta+1} \cap D)(x) \subset F_r$. Since (S) holds, we find $\overline{\varphi(\Phi_{k,\eta+1} \cap D)}^s(x) \in \mathscr{C}_x$. On the other side, $k \notin I_x$, implying that $x \in X_k$. Hence, if $x \in C \setminus \bigcup_k X_k$, then $D(x) \subset \bigcup_{\alpha \leqslant \eta} \bigcup_{l} \psi_{l\alpha}(x)$ and thus $A(x) \subset \bigcup_n A_n(x)$. This completes the proof.

COROLLARY 8. *Assume that the class $\mathscr{C}_x$ $(x \in X)$ satisfy the following additional hypothesis*:

(M) *If $B \in \mathscr{M} \otimes \mathscr{B}_Y$, then $\{x \in X;\ B(x) \in \mathscr{C}_x\} \in \mathscr{M}$.*

Suppose $A \in \mathscr{A}(X, Y)$ and let C be as in Proposition 7. Then:

1. *$C \in \mathscr{M}$.*

2. *If $C = X$, then exists a sequence $(F_n)_n$ in $\mathscr{M} \otimes \mathscr{B}_Y$ such that $F_n(x) \in \mathscr{C}_x$ for each n and each $x \in X$ and $A \subset \bigcup_n F_n$.*

Proof. We consider the sequences $(A_n)_n$, $(B_k)_k$ and $(X_k)_k$ obtained in Proposition 7.

Let $k \in N$ be fixed. Since (M) holds, $\{x \in \pi_1(B_k);\ B_k(x) \in \mathscr{C}_x\} \in \mathscr{M}$ and thus also $X_k \in \mathscr{M}$, by (3) and (4). It follows that X_k is negligible. It is clear that

$$X \setminus \pi_1(A \setminus \bigcup_n A_n) \subset C \subset (X \setminus \pi_1(A \setminus \bigcup_n A_n)) \cup \bigcup_k X_k,$$

showing that C is measurable.

Assume now $C = X$. For each $x \in \pi_1(A \setminus \bigcup_n A_n)$, we consider a sequence $(F_n^x)_n$ in $\mathscr{C}_x$ so that $A(x) \subset \bigcup_n F_n^x$. For each $n \in N$, take $F_n(x) = A_n(x)$ if $x \notin \pi_1(A \setminus \bigcup_n A_n)$ and $F_n(x) = F_n^x$ otherwise. Applying Proposition 2, we see that $F_n \in \mathscr{M} \otimes \mathscr{B}_Y$. Furthermore $F_n(x) \in \mathscr{C}_x$ for each $x \in X$. By construction $A \subset \bigcup_n F_n$.

THEOREM 9. *If $A \in \mathscr{M} \otimes \mathscr{B}_Y$, then $\{x \in X;\ A(x) \text{ is } F_\sigma\} \in \mathscr{M}$. If $A(x)$ is an F_σ-set for each $x \in X$, then there exists a sequence $(F_n)_n$ in $\mathscr{F}_0$ so that $A = \bigcup_n F_n$. We conclude that $\mathscr{S}_1 = \mathscr{F}_1$.*

Proof. For each $x \in X$, let $\mathscr{C}_x = \{F \subset A(x);\ F \text{ closed}\}$. Clearly condition (S) is satisfied. To show (M), take $B \in \mathscr{M} \otimes \mathscr{B}_Y$. Then

$$\{x \in X;\ B(x) \in \mathscr{C}_x\} = X \setminus (\pi_1(\overline{B}^s \setminus B) \cup \pi_1(B \setminus A)) \in \mathscr{M}.$$

The proof is completed by Corollary 8.

Our next objective is to show that $\mathscr{S}_2 = \mathscr{F}_2$. To do this, we need more material.

DEFINITION 10. Let $\mathscr{G} = \bigcup_{k=0}^{\infty} N^k$, where $N^0 = \{\emptyset\}$. Let $S \in \mathscr{A}(X, Y)$ be fixed and let $\bigcup_{\nu}\bigcap_{k}(M_{\nu|k}\times F_{\nu|k})$ be a representation of S. If $c\in N^k$, we take $S_c=S$ if $k=0$ and $S_c=\bigcup_{\nu|k=c}\bigcap_{l}(M_{\nu|l}\times F_{\nu|l})$ otherwise. We introduce for each $x\in X$ and $c\in\mathscr{G}$ a *transfinite system* $(\mathscr{D}_x^{\alpha}(c))_{\alpha<\omega_1}$ of subsets of 2^Y as following:

1. $\mathscr{D}_x^0(c)=\{P\subset Y;\ \bar{P}\cap S_c(x)=\emptyset\}$,

2. $\mathscr{D}_x^{\beta}(c)=\{P\subset Y$; for each $r\in N$, P has a countable closed covering $(F_n)_n$, such that $P\cap F_n\in\bigcup_{\alpha<\beta}\mathscr{D}_x^{\alpha}(c,r)$ for each $n\in N\}$.

If $A\subset X\times Y$, $x\in X$ and $c\in\mathscr{G}$, let $\mathrm{cl}_c(A,x)=\omega_1$ if $A(x)\notin\bigcup_{\alpha<\omega_1}\mathscr{D}_x^{\alpha}(c)$ and otherwise the smallest $\alpha<\omega_1$ satisfying $A(x)\in\mathscr{D}_x^{\alpha}(c)$. Propositions 11 and 12 are related to that definition.

PROPOSITION 11. *For each $x\in X$ and $c\in\mathscr{G}$*:

1. *$\mathscr{D}_x^{\alpha}(c)$ increases when α increases.*

2. *For each $\alpha<\omega_1$, $P\in\mathscr{D}_x^{\alpha}(c)$, $Q\subset P\Rightarrow Q\in\mathscr{D}_x^{\alpha}(c)$.*

3. *If $\alpha<\omega_1$ and $P\in\mathscr{D}_x^{\alpha}(c)$, then there exists an $F_{\sigma\delta}$-subset of Y, which contains P and is disjoint from $S_c(x)$.*

4. *Conversely, if P is contained in an $F_{\sigma\delta}$-set, which is disjoint from $S_c(x)$, then $P\in\bigcup_{\alpha<\omega_1}\mathscr{D}_x^{\alpha}(c)$.*

Suppose $A\in\mathscr{A}(X,Y)$, then:

5. *For each $\alpha<\omega_1$, $\{x\in X;\ \mathrm{cl}_c(A,x)\leqslant\alpha\}\in\mathscr{M}$, for each $c\in\mathscr{G}$.*

6. *If $\mathrm{cl}_c(A,x)\leqslant\beta$ for each $x\in X$, then for every $r\in N$ there is a sequence $(F_n)_n$ in $\mathscr{F}_0$ so that $\mathrm{cl}_{(c,r)}(A\cap F_n,x)<\beta$ for each $n\in N$ and each $x\in X$ and $A\subset\bigcup_n F_n$.*

Proof. 1. This follows immediately from the definition.

2. The property is true if $\alpha=0$. Using induction on $\alpha<\omega_1$, the proof is easily completed.

3. The proof is given by induction on $\alpha<\omega_1$. The case $\alpha=0$ is obvious. Assume now the property true for each $\alpha<\beta$ and let $P\in\mathscr{D}_x^{\beta}(c)$. Since $S_c=\bigcup_r S_{(c,r)}$, we only have to show that for each $r\in N$ the set P is contained in an $F_{\sigma\delta}$-set, which is disjoint from $S_{(c,r)}(x)$. Let $(F_n)_n$ be a countable closed covering of P with $P\cap F_n\in\bigcup_{\alpha<\beta}\mathscr{D}_x^{\alpha}(c,r)$ for each n. By induction hypothesis we have for each $n\in N$ an $F_{\sigma\delta}$-subset Q_n of Y satisfying $P\cap F_n\subset Q_n$ and $Q_n\cap S_{(c,r)}(x)=\emptyset$. The set $Q=\bigcup_n F_n\setminus\bigcup_n(F_n\setminus Q_n)$ is an $F_{\sigma\delta}$-set containing P and disjoint from $S_{(c,r)}(x)$.

4. Assume $P=\bigcap_m\bigcup_n F_{mn}$ disjoint from $S_c(x)$ and $P\notin\bigcup_{\alpha<\omega_1}\mathscr{D}_x^{\alpha}(c)$. We claim that there are 2 sequences $(r_m)_m$ and $(n_m)_m$ of integers so that

$$P\cap F_{1n_1}\cap\ldots\cap F_{mn_m}\notin\bigcup_{\alpha<\omega_1}\mathscr{D}_x^{\alpha}(c,r_1,\ldots,r_m)\quad\text{for each } m\in N.$$

Indeed, suppose $r_1,\ldots,r_m$ and $n_1,\ldots,n_m$ obtained. Since $P\cap F_{1n_1}\cap\ldots\cap F_{mn_m}$ is covered by the closed sets $F_{m+1,n}$ $(n\in N)$, there must exist some $r_{m+1}\in N$ and some $n_{m+1}\in N$ satisfying

$$P\cap F_{1n_1}\cap\ldots\cap F_{mn_m}\cap F_{m+1,n_{m+1}}\notin\bigcup_{\alpha<\omega_1}\mathscr{D}_x^{\alpha}(c,r_1,\ldots,r_m,r_{m+1}),$$

proving the claim. It follows that

$$F_{1n_1}\cap\ldots\cap F_{mn_m}\cap S_{(c,r_1,\ldots,r_m)}(x)\neq\emptyset\quad\text{for each } m\in N.$$

Clearly we can take $\operatorname{diam}F_{mn}<1/m$. Since Y is Polish, we obtain that $\bigcap_m F_{mn_m}$ and $\bigcap_m(M_{(c,r_1,\ldots,r_m)}\times F_{(c,r_1,\ldots,r_m)})(x)$ have nonempty intersection, contradicting the hypothesis $P\cap S_c(x)=\emptyset$. By (2), this completes the proof.

5. We proceed by introduction on $\alpha<\omega_1$. We have that

$$\{x\in X;\ \mathrm{cl}_c(A,x)=0\}=X\setminus\pi_1(\bar{A}^s\cap S_c)$$

is measurable. Let $\beta<\omega_1$ and assume the property true for every $\alpha<\beta$. Let $r\in N$ be fixed. For each $x\in X$, take

$$\mathscr{C}_x^r=\{F\subset Y;\ F\text{ closed and } A(x)\cap F\in\bigcup_{\alpha<\beta}\mathscr{D}_x^{\alpha}(c,r)\}.$$

By (2), the sets $\mathscr{C}_x^r$ verify condition (S) of Proposition 7. We show that condition (M) of Corollary 8 is also satisfied. Let thus $B\in\mathscr{M}\otimes\mathscr{B}_Y$. Then

$$\{x\in X,\ B(x)\in C_x\}=(X\setminus\pi_1(\bar{B}^s\setminus B))\cap\bigcup_{\alpha<\beta}\{x\in X;\ \mathrm{cl}_{(c,r)}(A\cap B,x)\leqslant\alpha\},$$

which is measurable by induction hypothesis.

Let C^r be as Proposition 7. Applying the first part of Corollary 8, we have

$$\{x\in X;\ \mathrm{cl}_c(A,x)\leqslant\beta\}=\bigcap_r C^r\in\mathscr{M}.$$

6. This follows from the second part of Corollary 8.

PROPOSITION 12. *If $A\in\mathscr{A}(X,Y)$, then the following "stabilisation property" holds*:

There exists $\alpha<\omega_1$ such that $\{x\in X;\ \alpha<\mathrm{cl}_{\varphi}(A,x)<\omega_1\}$ is negligible. Hence $\{x\in X;\ \mathrm{cl}_{\varphi}(A,x)<\omega_1\}\in\mathscr{M}$.

We first have to show the following lemma:

LEMMA 13. *Let $T\in\mathscr{A}(X,Y)$, $c\in\mathscr{G}$ and $R\subset\{x\in X;\ \mathrm{cl}_c(T,x)<\omega_1\}$ such that $\{x\in R,\ \mathrm{cl}_c(T,x)\leqslant\alpha\}$ is negligible for each $\alpha<\omega_1$. Then there exists a sequence $(T_p,R_p,r_p)_p$ in $\mathscr{A}(X,Y)\times 2^X\times N$ with following properties*:

1. *$R_p\subset\{x\in X;\ \mathrm{cl}_{(c,r_p)}(T_p,x)<\omega_1\}$ $(p\in N)$.*

2. *$\{x\in R_p,\ \mathrm{cl}_{(c,r_p)}(T_p,x)\leqslant\alpha\}$ is negligible for each $\alpha<\omega_1$ $(p\in N)$.*

3. *If $x\in R_p$, then $\mathrm{cl}_{(c,r_p)}(T_p,x)<\mathrm{cl}_c(T,x)$ $(p\in N)$.*

4. *$R\setminus\bigcup_p R_p$ is negligible.*

Proof. Let $X_0 = \{x \in X;\ \mathrm{cl}_c(T, x) = 0\}$. Let $r \in N$ be fixed. For each $x \in X$ we define a family $\mathscr{C}_x^r$ of closed subsets of Y by taking $\mathscr{C}_x^r = \{F \subset Y;\ F$ closed and there is $\alpha < \mathrm{cl}_c(T, x)$ such that $T(x) \cap F \in \mathscr{D}_x^\alpha(c, r)\}$ if $x \notin X_0$ and $\mathscr{C}_x^r = \{\emptyset\}$ otherwise. Clearly the classes $\mathscr{C}_x^r$ verify condition (S) of Proposition 7. Let C^r be as in this proposition and let $(A_n^r)_n$, $(B_k^r)_k$ in $\mathscr{M} \otimes \mathscr{B}_Y$ and $(X_k^r)_k$ in 2^X be the sequences provided by the proposition. For each $k \in N$, take $V_k^r = \{x \in \pi_1(B_k^r);\ B_k^r(x) \in \mathscr{C}_x^r\}$. If $n \in N$ is fixed, then by (5) of the preceding proposition there is some $\alpha_{nr} < \omega_1$ so that

$$M_{nr} = \{x \in X;\ \mathrm{cl}_{(c,r)}(T \cap A_n^r, x) \leqslant \alpha_{nr}\} \quad \text{and} \quad \{x \in X;\ \mathrm{cl}_{(c,r)}(T \cap A_n^r, x) \leqslant \beta\}$$

have the same measure for all $\beta > \alpha_{nr}$. Let $\alpha > \alpha_{nr}$ for all $n, r \in N$. Obviously

$$\bigcap_{n,r} M_{nr} \setminus \bigcup_r \pi_1(T \setminus \bigcup_n A_n^r) \subset \{x \in X;\ \mathrm{cl}_c(T, x) \leqslant \alpha\}.$$

On the other side $R \setminus X_0 \subset \{x \in X;\ \mathrm{cl}_c(T, x) < \omega_1\} \setminus X_0 \subset \bigcap_r C^r$. Since

$$C^r \cap \pi_1(T \setminus \bigcup_n A_n^r) \subset \bigcup_k X_k^r \quad \text{for each } r \in N,$$

we obtain that

$$R \subset X_0 \cup (X_0^c \setminus \bigcup_r \pi_1(T \setminus \bigcup_n A_n^r)) \cup \bigcup_{k,r} (X_0^c \cap X_k^r)$$

$$\subset X_0 \cup (\bigcap_{n,r} M_{nr} \setminus \bigcup_r \pi_1(T \setminus \bigcup_n A_n^r)) \cup \bigcup_{n,r} (X_0^c \cap M_{nr}^c) \cup \bigcup_{k,r} (X_0^c \cap X_k^r).$$

By hypothesis, $X_0 \cap R$ and $(\bigcap_{n,r} M_{nr} \setminus \bigcup_r \pi_1(T \setminus \bigcup_n A_n^r)) \cap R$ are both negligible. The sequence $(T_p, R_p, r_p)_p$ will be the following countable family

$$\{(T \cap A_n^r, M_{nr}^c \cap X_0^c, r);\ n \in N, r \in N\} \cup \{(T \cap B_k^r, V_k^r \cap X_0^c, r);\ k \in N, r \in N\}.$$

It follows from (4) of Proposition 7 that $R \setminus \bigcup_p R_p$ is negligible. We verify the properties (1), (2), (3).

1. This follows from the fact that $A_n^r(x) \in \mathscr{C}_x^r$ for each $x \in X$ and $B_k^r(x) \in \mathscr{C}_x^r$ if $x \in V_k^r$.

2′. $\{x \in M_{nr}^c \cap X_0^c;\ \mathrm{cl}_{(c,r)}(T \cap A_n^r, x) \leqslant \alpha\} \subseteq \{x \in X;\ \mathrm{cl}_{(c,r)}(T \cap A_n^r, x) \leqslant \alpha\} \setminus M_{nr}$ is empty if $\alpha \leqslant \alpha_{nr}$ and negligible otherwise, by the choice of α_{nr}.

2″. $\{x \in V_k^r \cap X_0^c;\ \mathrm{cl}_{(c,r)}(T \cap B_k^r, x) \leqslant \alpha\} = \pi_1(B_k^r) \cap X_0^c \cap \{x \in X;\ \mathrm{cl}_{(c,r)}(T \cap B_k^r, x) \leqslant \alpha$ and $\mathrm{cl}_{(c,r)}(T \cap B_k^r, x) < \mathrm{cl}_c(T, x)\} = \pi_1(B_k^r) \cap X_0^c \cap \bigcup_{\beta \leqslant \alpha} \{x \in X;\ \mathrm{cl}_{(c,r)}(T \cap B_k^r, x) \leqslant \beta$ and $\mathrm{cl}_c(T, x) > \beta\}$, measurable by (5) of the preceding proposition. Since $\mu_*(V_k^r) = 0$, our set is negligible.

3. Follows immediately from the definition of the classes $\mathscr{C}_x^r$.

This completes the proof of the lemma.

Proof of Proposition 12. Clearly, the only thing to show is that if R satisfies the hypothesis of Lemma 13, then R is negligible. By successive applications of that lemma, we obtain for each $k \in N$ and $(p_1, \ldots, p_k) \in N^k$ a set $T_{p_1 \ldots p_k} \in \mathscr{A}(X, Y)$, a subset $R_{p_1 \ldots p_k}$ of X and an integer $r_{p_1 \ldots p_k}$ so that the following properties are verified:

1. $R_{p_1 \ldots p_k} \subset \{x \in X;\ \mathrm{cl}_{(c, r_{p_1}, \ldots, r_{p_1 \ldots p_k})}(T_{p_1 \ldots p_k}, x) < \omega_1\}$.
2. $\{x \in R_{p_1 \ldots p_k};\ \mathrm{cl}_{(c, r_{p_1}, \ldots, r_{p_1 \ldots p_k})}(T_{p_1 \ldots p_k}, x) \leqslant \alpha\}$ is negligible for each $\alpha < \omega_1$.
3. If $x \in R_{p_1 \ldots p_k}$, then $\mathrm{cl}_{(c, r_{p_1}, \ldots, r_{p_1 \ldots p_k})}(T_{p_1 \ldots p_k}, x) < \mathrm{cl}_{(c, r_{p_1}, \ldots, r_{p_1 \ldots p_{k-1}})}(T_{p_1 \ldots p_{k-1}}, x)$.
4. $R_{p_1 \ldots p_{k-1}} \setminus \bigcup_p R_{p_1 \ldots p_{k-1} p}$ is negligible.

We deduce from (4) that $R \setminus \bigcup_{\pi \in \mathscr{N}} \bigcap_k R_{\pi|k}$ is negligible. Furthermore, we claim that $\bigcap_k R_{\pi|k}$ is empty for each $\pi \in \mathscr{N}$. Indeed, for $x \in \bigcap_k R_{\pi|k}$, we would obtain:

$$\mathrm{cl}_c(T, x) > \mathrm{cl}_{(c, r_{\pi|1})}(T_{\pi|1}, x) > \ldots > \mathrm{cl}_{(c, r_{\pi|1}, \ldots, r_{\pi|k})}(T_{\pi|k}, x)$$

which is a strictly decreasing sequence of ordinals. This is impossible and the proof is complete.

THEOREM 13. *Let $A \in \mathscr{M} \otimes \mathscr{B}_Y$. Then $\{x \in X;\ A(x)$ is $F_{\sigma\delta}\} \in \mathscr{M}$. If $A(x)$ is an $F_{\sigma\delta}$-set for each $\alpha \in X$, then there exists a double sequence $(F_{mn})_{m,n}$ in $\mathscr{F}_0$ so that $A = \bigcap_m \bigcup_n F_{mn}$. Hence $\mathscr{S}_2 = \mathscr{F}_2$.*

Proof. The set S of Definition 10 will be the set A^c. From (3) and (4) of Proposition 11, we obtain that

$$\{x \in X;\ A(x) \text{ is } F_{\sigma\delta}\} = \{x \in X;\ \mathrm{cl}_\varphi(A, x) < \omega_1\}.$$

Thus the first statement follows from Proposition 12. Assume now $A(x)$ is an $F_{\sigma\delta}$-set for each $x \in X$. Again using Proposition 12, there exists $\alpha < \omega_1$ such that $\{x \in X;\ \mathrm{cl}_\varphi(A, x) > \alpha\}$ is negligible. By Proposition 2, it is therefore not a restriction to assume in the proof of the second statement that $\mathrm{cl}_\varphi(A, x)$ is bounded by some $\alpha < \omega_1$. We will show by induction on $\alpha < \omega_1$ that if $c \in \mathscr{G}$, $A \in \mathscr{A}(X, Y)$ and $\mathrm{cl}_c(A, x) \leqslant \alpha$ for each $x \in X$, then there is an element Q in $\mathscr{F}_2$ so that $A \subset Q \subset S_c^c$. Clearly, this will finish the proof.

The case $\alpha = 0$ is trivial, since then $\bar{A}^s \cap S_c = \emptyset$. Assume now the property true for every $\alpha < \beta$ and let $\mathrm{cl}_c(A, x) \leqslant \beta$ for every $x \in X$. We remark that it is enough to prove that for each $r \in N$ there is an element Q in $\mathscr{F}_2$ so that $A \subset Q \subset S_{(c,r)}^c$. Let thus $r \in N$ be fixed. By (6) of Proposition 11, there exists a sequence $(F_n)_n$ in $\mathscr{F}_0$ so that $\mathrm{cl}_{(c,r)}(A \cap F_n, x) < \beta$ for each $n \in N$ and each $x \in X$ and $A \subset \bigcup_n F_n$. For each $n \in N$ and $\alpha < \beta$, let

$$M_{n\alpha} = \{x \in X,\ \mathrm{cl}_{(c,r)}(A \cap F_n, x) = \alpha\}.$$

Applying the induction hypothesis on $(A \cap F_n) \cap (M_{n\alpha} \times Y)$ we find an element $Q_{n\alpha}$ in $\mathscr{F}_2$ so that $(A \cap F_n) \cap (M_{n\alpha} \times Y) \subset Q_{n\alpha} \subset S_{(c,r)}^c$. Let $n \in N$ be fixed. Since

$(M_{n\alpha})_{\alpha<\beta}$ is a measurable partition of X, the set $Q_n = \bigcup_{\alpha<\beta} (Q_{n\alpha} \cap (M_{n\alpha} \times Y))$ is still in $\mathscr{F}_2$ and $A \cap F_n \subset Q_n \subset S^c_{(c,r)}$. It is easily seen that

$$A \subset \bigcup_n F_n \setminus \bigcup_n (F_n \setminus Q_n) \subset S^c_{(c,r)},$$

where

$$Q = \bigcup_n F_n \setminus \bigcup_n (F_n \setminus Q_n) = \bigcup_n F_n \cap \bigcap_n (F_n^c \cup Q_n) \in \mathscr{F}_2 .$$

This completes the proof.

Application to the Baire functions. We refer to [2] for the following result on the integration of functions of the first class:

THEOREM 14. *Let X, $\mathscr{M}$, μ be a complete probability space and Y a Polish space. Let Φ: $X \times Y \to \boldsymbol{R}$ be a uniformly bounded function which is measurable in the first variable and of first Baire class in the second. Then the function φ on Y given by $\varphi(y) = \int_X \Phi(x, y)\mu(dx)$ is also of the first class.*

Under the continuum hypothesis it is easy to define a function Φ on $[0, 1]^2$ only taking the values $0, 1$, which is of the first Baire class in the first variable, of second Baire class in the second one and such that the function φ on $[0, 1]$ given by $\varphi(y) = \int_0^1 \Phi(x, y)m(dx)$ is not even measurable. However, the following is true:

THEOREM 15. *Let Φ: $X \times Y \to \boldsymbol{R}$ be a uniformly bounded function which is $\mathscr{M} \otimes \mathscr{B}_Y$-measurable and of the second Baire class in the second variable. Then the function φ introduced in Theorem 14 is still of the second class.*

Using Theorem 14 and the Lebesgue-theorem, Theorem 15 will be a consequence of the following result:

THEOREM 16. *Let Φ: $X \times Y \to \boldsymbol{R}$ be $\mathscr{M} \otimes \mathscr{B}_Y$-measurable and of second Baire class in the second variable. Then there exists a sequence $(\Phi_n)_n$ of $\mathscr{M} \otimes \mathscr{B}_Y$-measurable functions so that each Φ_n is of the first class in the second variable and Φ is the pointwise limit of $(\Phi_n)_n$.*

Proof. We use the terminology and results of [4], IX. Let $\mathcal{O} = \{f \in \boldsymbol{R}^{X \times Y};$ f is $\mathscr{M} \otimes \mathscr{B}_Y$-measurable and f is of first Baire class in the second variable$\}$. Clearly $\mathcal{O}$ is a complete ordinary function system. The sets M, N are the sets

$$\{(x, y) \in X \times Y;\ f(x, y) > a\} \quad \text{and} \quad \{(x, y) \in X \times Y;\ f(x, y) \geqslant a\},$$

respectively, where $f \in \mathcal{O}$ and $a \in \boldsymbol{R}$.

Hence the sets M are the members of $\mathscr{S}_1 = \mathscr{F}_1$. Indeed, each set M is clearly in $\mathscr{S}_1$. Conversely, if $A = \bigcup_n F_n$ with $F_n \in \mathscr{F}_0$ for each n, then

$$A = \{(x, y) \in X \times Y;\ f(x, y) > 0\}, \quad \text{where} \quad f\colon \sum_n \frac{1}{2^n} 1_{F_n} \in \mathcal{O} .$$

To show that Φ is an f^*, i.e. a pointwise limit of a sequence in $\mathcal{O}$, we have to prove that if $a \in \boldsymbol{R}$, then the sets $\{(x, y) \in X \times Y;\ \Phi(x, y) \leqslant a\}$ and

$$\{(x, y) \in X \times Y;\ \Phi(x, y) \geqslant a\}$$

are both $N^* = M_\delta = \mathscr{F}_2$. Now, those two sets are both in $\mathscr{S}_2$ and Theorem 13 completes the proof.

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