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# Weak-strong convolution operators on certain disconnected groups

by

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**Abstract.** In this paper we produce estimates for certain types of convolution operators on $L^p(G)$, where $G$ is a type of totally disconnected LCA group. The operators under consideration are analogues of the weak-strong convolution operators on $\boldsymbol{R}^n$ of Fefferman and Stein [2]. As in [2], our estimates are produced by observing the action of the operators on the space $\mathrm{BMO}(G)$ of functions of bounded mean oscillation on $G$. However, there is one significant difference in that our results are obtained without the intervention of an analogue of the space $H^1$.

As a consequence of the operator estimates, we provide sufficient conditions for a function to be an $L^p$ Fourier multiplier, for indices $p$ lying in a closed sub-interval $[r, r']$ of $(1, \infty)$. We also show that the range $[r, r']$ is best possible, by giving an example of a function $\psi$ which is an $L^p$ Fourier multiplier for all $p$ in $[r, r']$, but not for any $p$ outside $[r, r']$.

## 1. Definitions and notation.

1.1. Throughout this paper, we let $G$ denote an LCA group having the following properties:

(i) there exists a strictly decreasing sequence $\{G_n\}_{n\in Z}$ of open compact subgroups of $G$ such that the index $G_{n+1}:G_n$ of $G_{n+1}$ in $G_n$ is bounded by some integer $b \geqslant 2$;

(ii) $\bigcup G_n = G$ and $\bigcap G_n = \{0\}$;

(iii) $\mu(G_0) = 1$, $\mu$ denoting the Haar measure on $G$.

Let $\Gamma$ denote the dual group of $G$ and, for each $n$, $\Gamma_n$ the annihilator of $G_n$ in $\Gamma$. Then $\{\Gamma_n\}$ is a strictly increasing sequence of open, compact subgroups of $\Gamma$ such that $\Gamma_n:\Gamma_{n+1} \leqslant b$, and

(iv) $\bigcup \Gamma_n = \Gamma$ and $\bigcap \Gamma_n = \{0\}$.

1.2. If $\varphi \in L^\infty(\Gamma)$, denote by $T_\varphi$ the continuous linear operator on $L^2(G)$, for which

$$(T_\varphi f)^\wedge = \varphi \hat{f}.$$

If $p \in [1, \infty]$ and $\varphi \in L^\infty(\Gamma)$, we say that $\varphi$ is a *Fourier multiplier* of $L^p(G)$ if there exists a number $B$ such that

$$\|T_\varphi f\|_p \leqslant B \|f\|_p \tag{1}$$

for all $f$ in $L^2 \cap L^p(G)$. Denote the space of such functions by $M_p(\Gamma)$. The norm of an element $\varphi$ of $M_p(\Gamma)$ is the smallest number $B$ for which (1) holds.

For the basic facts about multipliers, the reader is referred to Edwards [1], Chapter 16.

1.3. For an element $f$ of $L^2_{\text{loc}}(G)$, define

$$f^{\#}(x) = \sup_{n\in Z}\left\{\frac{1}{\mu(G_n)}\int_{G_n+x}|f(y)-(f)_{G_{n-1}+x}|^2\,dy\right\}^{1/2},$$

where, for a measurable set $S$ of positive measure,

$$(f)_S = \frac{1}{\mu(S)}\int_S f(x)\,dx.$$

We say that $f\in \mathrm{BMO}(G)$ if and only if $f^{\#}\in L^\infty(G)$, and we set $\|f\|_{\mathrm{BMO}} = \|f^{\#}\|_\infty$. The space $\mathrm{BMO}(G)$ is a Banach space modulo the space of constant functions. Clearly, if $f\in L^\infty$, then $f\in \mathrm{BMO}$ and

$$\|f\|_{\mathrm{BMO}} \leqslant 2\|f\|_\infty.$$

1.4. A *radial* function on $G$ (resp., $\Gamma$) is one which is constant on each set of the form $G_n\setminus G_{n+1}$ (resp., $\Gamma_{n+1}\setminus\Gamma_n$).

1.5. If $S$ is a coset modulo $G_n$ in $G$, that is, $S = G_n+x$ for some $x$, then $[S]_{-1}$, called the *predecessor* of $S$, is defined by the formula

$$[S]_{-1} = G_{n-1}+x.$$

1.6. If $0<\theta<1$, and $n\in\mathscr{Z}$, we define the subgroup $G_n^{1-\theta}$ of $G$ by the formula

$$G_n^{1-\theta} = \begin{cases} G_n & \text{if } n\leqslant 0,\\ G_j & \text{if } n>0,\end{cases}$$

where $j$ is chosen so that

$$\mu(G_j)\leqslant \mu(G_n)^{1-\theta} < \mu(G_{j-1}).$$

1.7. For each integer $N$, let $D_N$ denote the $N$th "Dirichlet kernel" on $G$ defined by the formula

$$D_N(x) = \xi_{G_N}/\mu(G_N).$$

(For a set $S$, $\xi_S$ is the indicator function of $S$.)

**2. Estimates on BMO for convolution operators.**

2.1. THEOREM. *Suppose that $k\in L^1(G)$ and that*

(i) $|\hat{k}(\gamma)|\leqslant B$ *for all $\gamma$ in $\Gamma$,*

*and*



(ii) *for all $n$, and all $y$ in $G_{n+1}$,*

$$\int_{G\setminus G_n}|k(x-y)-k(x)|\,dx\leqslant B.$$

*Then*

$$\|k*f\|_{\mathrm{BMO}}\leqslant A\|f\|_{\mathrm{BMO}},$$

*for all $f$ in $L^\infty(G)$, where $A$ is a number independent of $\|k\|_1$ and of $f$.*

Proof. The proof may be obtained as a simplification of that of Theorem 2.2. ■

2.2. THEOREM. *Suppose that $0<\theta<1$, and that $\Theta$ is the radial function on $\Gamma$ such that*

(1) $$\Theta(\gamma) = \mu(G_{n+1})^{\theta/2}\quad on\quad \Gamma_{n+1}\setminus\Gamma_n,$$

*for each $n$. Let $k$ be an integrable function on $G$ such that*

(i) *for all $\gamma$,*

(2) $$|\hat{k}(\gamma)|\leqslant B\quad and\quad |\hat{k}(\gamma)|\leqslant B\Theta(\gamma)$$

*and*

(ii) *for all $n$, and all $y$ in $G_n$,*

(3) $$\int_{G\setminus[G_n^{1-\theta}]_{-1}}|k(x-y)-k(x)|\,dx\leqslant B.$$

*Then there is a number $C$ such that*

$$\|k*f\|_{\mathrm{BMO}}\leqslant C\|f\|_{\mathrm{BMO}}$$

*for all $f$ in $L^\infty(G)$.*

Proof. Fix $f$ in $L^\infty(G)$. In estimating $\|k*f\|_{\mathrm{BMO}}$ it suffices, by translation invariance, to consider the quantity

(4) $$\left(\frac{1}{\mu(G_N)}\int_{G_N}|k*f(x)-(k*f)_{G_{N-1}}|^2\,dx\right)^{1/2}.$$

Write $f = f_1+f_2$, where

(5) $$f_1 = f\xi_{[G_{N-1}^{1-\theta}]_{-1}}.$$

We first estimate

(6) $$\left(\frac{1}{\mu(G_N)}\int_{G_N}|k*f_2(x)-(k*f_2)_{G_{N-1}}|^2\,dx\right)^{1/2}.$$

In doing so, we shall suppose without loss of generality that

(7) $$D_{N-1}*f_2 = 0.$$

Now, if

(8) $$k' = k\xi_{G\setminus[G_{N-1}^{1-\theta}]_{-1}},$$

and $k'_z(x) = k'(x-z)$, then

(9) $$\left(\frac{1}{\mu(G_N)}\int\limits_{G_N} |k*f_2(x)-(k*f_2)_{G_{N-1}}|^2\,dx\right)^{1/2}$$
$$=\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}\left|\frac{1}{\mu(G_{N-1})}\int\limits_{G_{N-1}}(k'-k'_z)*f_2(x)\,dz\right|^2dx\right)^{1/2}$$
$$\leqslant \frac{1}{\mu(G_{N-1})}\int\limits_{G_{N-1}}\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|(k'-k'_z)*f_2(x)|^2\,dx\right)^{1/2}dz,$$

the equality following from (5) and (8), the estimate from Minkowski's inequality.

Now, if $\psi\in L^2(G_N)$, and

(10) $$\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|\psi|^2\right)^{1/2}\leqslant 1,$$

then

(11) $$\left|\frac{1}{\mu(G_N)}\int\limits_{G_N}(k'-k'_z)*f_2(s)\,\psi(s)\,ds\right|$$
$$=\frac{1}{\mu(G_N)}\left|\int\limits_{G_N}\int\limits_{G}(k'-k'_z)(t)\,f_2(s-t)\,\psi(s)\,dt\,ds\right|$$
$$\leqslant\int\limits_{G}|(k'-k'_z)(t)|\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|f_2(s-t)|^2\,ds\right)^{1/2}dt,$$

by Fubini's theorem and (10). But

(12) $$\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|f_2(s-t)|^2\,dt\right)^{1/2}$$
$$=\left(\frac{1}{\mu(G_N)}\int\limits_{G_N+t}|f_2(s)-D_{N-1}*f_2(s)|^2\,ds\right)^{1/2}\leqslant\|f\|_{\mathrm{BMO}},$$

thanks to assumption (7). Hence, by (9), (11), and (12),

(13) $$\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|k*f_2(x)-(k*f_2)_{G_{N-1}}|^2\,dx\right)^{1/2}$$
$$\leqslant\|f\|_{\mathrm{BMO}}\frac{1}{\mu(G_{N-1})}\int\limits_{G_{N-1}}dz\int\limits_{G}|k'-k'_z|(t)\,dt$$
$$\leqslant\|f\|_{\mathrm{BMO}}\sup_{z\in G_{N-1}}\int\limits_{G\setminus[G_{N-1}^{1-\theta}]_{-1}}|k(t)-k(t-z)|\,dt\leqslant B\|f\|_{\mathrm{BMO}},$$

by (8) and (3).



We next estimate the component of (4), due to $f_1$. Write $h=f_1-D_{N-1}*f_1$. Then

(14) $$\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|k*f_1(x)-(k*f_1)_{G_{N-1}}|^2\,dx\right)^{1/2}$$
$$=\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|k*h|^2\,dx\right)^{1/2}\leqslant\mu(G_N)^{-1/2}\|k*h\|_2$$
$$=\mu(G_N)^{-1/2}\left(\int\limits_{\Gamma\setminus\Gamma_{N-1}}|\hat{k}(\gamma)\hat{h}(\gamma)|^2\,d\gamma\right)^{1/2}\leqslant\mu(G_N)^{-1/2}B\,\mu(G_N)^{\theta/2}\left(\int\limits_{\Gamma}|\hat{h}(\gamma)|^2\,d\gamma\right)^{1/2}$$
$$=B\,\mu(G_N)^{(\theta-1)/2}\|f_1-D_{N-1}*f_1\|_2$$

by (1) and (2). But, by (5) and the fact that $G_{N-1}\subseteq[G_{N-1}^{1-\theta}]_{-1}$,

$$\|f_1-D_{N-1}*f_1\|_2=\left(\int\limits_{[G_{N-1}^{1-\theta}]_{-1}}|f-D_{N-1}*f|^2\,dx\right)^{1/2}.$$

So, by (14),

(15) $$\left(\frac{1}{\mu(G_N)}\int\limits_{G_N}|k*f_1(x)-(k*f_1)_{G_{N-1}}|^2\,dx\right)^{1/2}$$
$$\leqslant B\left(\mu(G_N)^{\theta-1}\int\limits_{[G_{N-1}^{1-\theta}]_{-1}}\left|f-f*\frac{\xi_{[G_{N-1}^{1-\theta}]_{-2}}}{\mu([G_{N-1}^{1-\theta}]_{-2})}\right|^2dx\right)^{1/2}$$
$$+B\left(\mu(G_N)^{\theta-1}\int\limits_{[G_{N-1}^{1-\theta}]_{-1}}\left|D_{N-1}*f-D_{N-1}*f*\frac{\xi_{[G_{N-1}^{1-\theta}]_{-2}}}{\mu([G_{N-1}^{1-\theta}]_{-2})}\right|^2dx\right)^{1/2}$$
$$\leqslant 2\,Bb^{(2-\theta)/2}\|f\|_{\mathrm{BMO}}$$

by 1.6, assumption 1.1, and the definition of BMO.

Combining (13) and (15) completes the proof. ■

2.3. Corollary. *If $k$ is as in 2.2, then*

$$\|k*f\|_{\mathrm{BMO}}\leqslant 2C\|f\|_\infty$$

*for all $f$ in $L^\infty(G)$, $C$ being the constant in the conclusion of Theorem 2.2.* ■

**3. Interpolation.** A key step in the production of our estimates for $L^p$ Fourier multipliers, is an interpolation theorem for analytic families of operators simultaneously mapping $L^\infty$ to BMO and $L^2$ to $L^2$. We shall omit the proof, which is a generalisation of the corresponding result on $R^n$ of Fefferman and Stein [loc. cit.].

3.1. THEOREM. *Suppose that $z \to T_z$ is a mapping from the strip $S = \{z \in \mathscr{C}\colon 0 \leqslant \operatorname{Re} z \leqslant 1\}$ to the space of bounded linear operators on $L^2(G)$. We assume this mapping to be strongly continuous and uniformly bounded on $S$, and analytic in the interior of $S$.*

*Suppose that there are constants $M_0$ and $M_1$ such that*

$$\sup_{-\infty<y<\infty} \|T_{iy}f\|_{\mathrm{BMO}} \leqslant M_0\|f\|_\infty \quad \text{if} \quad f \in L^2 \cap L^\infty(G)$$

*and*

$$\sup_{-\infty<y<\infty} \|T_{1+iy}f\|_2 \leqslant M_1\|f\|_2 \quad \text{if} \quad f \in L^2(G).$$

*If $0 < t < 1$, $p = 2/t$, and $f$ is a simple integrable function, then*

$$\|T_tf\|_p \leqslant M_t\|f\|_p,$$

*where $M_t$ is independent of $f$.*

**4. Multiplier theorems.** In proving our multiplier theorems, we shall make use of the following observation.

4.1. LEMMA. *Let $\Phi$ be a bounded radial function on $\Gamma$. For each integer $N$, let $\Phi_N = \Phi\xi_{\Gamma_N\setminus\Gamma_{-N}}$, and $F_N$ be the function on $G$ such that $\hat{F}_N = \Phi_N$. Then*

(1) $$\|F_N * f\|_{\mathrm{BMO}} \leqslant C\|f\|_{\mathrm{BMO}},$$

*for all $f$ in $L^\infty(G)$, where $C$ is a number independent of $N$ and $f$.*

Proof. One way is to observe that $F_N$ is also radial, and so

$$\int_{G\setminus G_n} |F_N(x-y) - F_N(x)|\, dx = 0$$

for all $y$ in $G_{n+1}$ and all $n$. Hence (1) follows from Theorem 2.1.

Alternatively, one may observe that the BMO norm is equivalent to the norm

$$\Big\|\sup_{\substack{N,n\\ N\geqslant n}} \Big\{D_n * \sum_{k=n}^{N} |(D_k - D_{k-1}) * f|^2\Big\}^{1/2}\Big\|_\infty. \ \blacksquare$$

We wish to deal with kernels $k$ on $G$ which are not necessarily integrable, but satisfy conditions like those in 2.2(i) and 2.2(ii). So we shall need the following regularisation result.

4.2. LEMMA. *Let $k$ be a pseudomeasure with compact support on $G$, integrable on each compact subset of $G\setminus\{0\}$. Assume that*

(i) *if $\Theta$ is as in 2.2(1), then*

(2) $$\|\hat{k}\|_\infty \leqslant B \quad \text{and} \quad |\hat{k}(\gamma)| \leqslant B\Theta(\gamma) \text{ a.e.}$$

*and*

(ii) *for all $n$, and all $y$ in $G_n$,*

(3) $$\int_{G\setminus[G_n^{1-\theta}]_{-1}} |k(x-y) - k(x)|\, dx \leqslant B.$$



*Let $k_i = D_i * k$. Then the functions $k_i$ are integrable and satisfy the conditions of Theorem 2.2 uniformly in $i$ (with a different value for $B$).*

Proof. The only nontrivial assertion to check is 2.2(ii). Suppose $n$ given, and that $y \in G_n$. Then

(4) $$\begin{aligned}&\int_{G\setminus[G_n^{1-\theta}]_{-1}} |D_i*k(x-y) - D_i*k(x)|\, dx \\ &\leqslant \int_G |D_i(z)| \Big\{\int_{G\setminus[G_n^{1-\theta}]_{-1}} |k(x-z) - k(x-y-z)|\, dx\Big\}\, dz \\ &\leqslant \int_{G_i} |D_i(z)| \Big\{\int_{G\setminus[G_n^{1-\theta}]_{-1}} \big(|k(x) - k(x-z)| + |k(x) - k(x-y)| + \\ &\qquad + |k(x-y) - k(x-y-z)|\big)\, dx\Big\}\, dz.\end{aligned}$$

Two cases now arise: (a) $G_i \subseteq G_n$; and (b) $G_n \subseteq G_i$.

*Case* (a). Observe that if $z \in G_i \subseteq G_n$, then by (3),

(5) $$\int_{G\setminus[G_n^{1-\theta}]_{-1}} |k(x) - k(x-z)|\, dx \leqslant B.$$

Again, by hypothesis (3),

(6) $$\int_{G\setminus[G_n^{1-\theta}]_{-1}} |k(x) - k(x-y)|\, dx \leqslant B,$$

since $y \in G_n$. Finally,

(7) $$\begin{aligned}&\int_{G\setminus[G_n^{1-\theta}]_{-1}} |k(x-y) - k(x-y-z)|\, dx \\ &= \int_{(G\setminus[G_n^{1-\theta}]_{-1})+y} |k(x) - k(x-z)|\, dx = \int_{G\setminus[G_n^{1-\theta}]_{-1}} |k(x) - k(x-z)|\, dx \leqslant B,\end{aligned}$$

by (5) and the fact that $y \in G_n \subseteq [G_n^{1-\theta}]_{-1}$. From (4)–(7), we deduce that

$$\int_{G\setminus[G_n^{1-\theta}]_{-1}} |D_i*k(x-y) - D_i*k(x)|\, dx \leqslant 3B,$$

when $y \in G_n$.

*Case* (b). In this case,

$$\int_{G\setminus[G_n^{1-\theta}]_{-1}} |D_i*k(x-y) - D_i*k(x)|\, dx = 0,$$

since $y \in G_n \subseteq G_i$, and $D_i(s-t) = D_i(s)$ for all $t$ in $G_i$ and all $s$. $\blacksquare$

Here now is the main multiplier theorem.

4.3. THEOREM. *Suppose that $k$ is a pseudomeasure satisfying all the conditions in 4.2. If $0<\alpha<1$, then $\hat{k}\Theta^{-\alpha}$ is a Fourier multiplier of $L^p$ whenever $p \in [2/(2-\alpha), 2/\alpha]$.*

Proof. It suffices to show that, when $p = 2/\alpha$, $\Theta^{-\alpha}\hat{k}\xi_{\Gamma_j\setminus\Gamma_{-j}} \in M_p(\Gamma)$ for every $j$, with norm bounded independently of $j$. To this end, fix $j$, and let $\{T_z\}_{z\in S}$ be the family of operators on $L^2(G)$, one for each $z$ in $S = \{z \in \mathscr{C}\colon 0 \leqslant \operatorname{Re} z \leqslant 1\}$, defined by the formula

$$(T_z f)^\wedge = \hat{k}_{\Gamma_j\setminus\Gamma_{-j}}\Theta^{-z}\hat{f}. \tag{8}$$

It is quite simple to check that, thanks to the conditions in 4.2(i), the mapping $z \to T_z$ is strongly continuous and uniformly bounded on $S$, and analytic in the interior of $S$.

Clearly, if $f \in L^2(G)$,

$$\sup_{-\infty<y<\infty} \|T_{1+iy}f\|_2 \leqslant B\|f\|_2 \tag{9}$$

because of (8) and 4.2(i).

Suppose next that $f \in L^2\cap L^\infty(G)$, and write

$$(T_{iy}f)^\wedge = \Theta^{-iy}\xi_{\Gamma_j\setminus\Gamma_{-j}}(\hat{k}\xi_{\Gamma_j\setminus\Gamma_{-j}}\hat{f}). \tag{10}$$

Let $S_j$ be the operator (on $L^2\cap L^\infty(G)$) defined by the formula

$$(S_j g)^\wedge = \hat{k}\xi_{\Gamma_j\setminus\Gamma_{-j}}\hat{g}.$$

By Corollary 2.3 and Lemma 4.2,

$$\|S_j f\|_{\mathrm{BMO}} \leqslant A\|f\|_\infty \tag{11}$$

where $A$ is independent of $f$ and $j$. Lemma 4.1 shows that

$$\|I_{iy}(S_j f)\|_{\mathrm{BMO}} \leqslant C\|S_j f\|_{\mathrm{BMO}} \tag{12}$$

where $C$ is independent of $f$, $j$ and $y$, and $I_{iy}$ is the operator defined by the formula

$$(I_{iy}g)^\wedge = \Theta^{-iy}\xi_{\Gamma_j\setminus\Gamma_{-j}}\hat{g}.$$

It follows from (11) and (12) that

$$\sup_{-\infty<y<\infty} \|T_{iy}f\|_{\mathrm{BMO}} \leqslant AC\|f\|_\infty. \tag{13}$$

The result now follows from (9), (13) and Theorem 3.1, (with $t = \alpha$). ■

A particular case of Theorem 3.1 is where we have a single linear mapping which is simultaneously continuous from $L^2$ to $L^2$ and from $L^\infty$ to BMO. Such an operator will then be continuous from $L^p$ to $L^p$ wherever $2 \leqslant p < \infty$. We are thus led to a second, simpler, multiplier theorem.

4.4. THEOREM. *Let $k$ be a pseudomeasure having all the properties laid down in 4.2. Then $\hat{k} \in M_p(\Gamma)$ for all $p$ in $(1, \infty)$.*



**5. Applications and examples.** For the remainder of this paper we assume, solely for simplicity of presentation, that $G_{n+1}\colon G_n = 2$ for all $n$, and take $\theta = 1/2$.

5.1. THEOREM. *Let $\varphi$ be a bounded function on $\Gamma$ such that $\varphi = 0$ on $\Gamma_0$, and, for each $n \geqslant 0$, $\varphi$ is constant on the cosets of $\Gamma_n$ in $\Gamma_{2n+2}\setminus\Gamma_{2n}$. Then*

(i) *$\varphi\Theta \in M_p(\Gamma)$ when $1<p<\infty$;*

*and*

(ii) *if $0<\alpha<1$,*

$$\varphi\Theta^{1-\alpha} \in M_p(\Gamma) \quad \text{when} \quad 2/(2-\alpha) \leqslant p \leqslant 2/\alpha.$$

*(Note $\Theta(\gamma) = 2^{-n/4}$ when $\gamma \in \Gamma_n\setminus\Gamma_{n+1}$.)*

Proof. Since the function $\varphi\Theta$ is constant on cosets of $\Gamma_0$, and bounded, there is a pseudomeasure $k$, supported in $G_0$, such that $\hat{k} = \varphi\Theta$. At the same time, for each $n \geqslant 0$, the function $\varphi\Theta\xi_{\Gamma\setminus\Gamma_{2n}}$ is constant on cosets of $\Gamma_n$, and is therefore the Fourier transform of a pseudomeasure supported in $G_n$. In other words, the behaviour of $k$ on the set $G\setminus G_n$ depends only on the part of $\varphi\Theta$ supported in $\Gamma_{2n}$. Clearly, therefore, $k$ is integrable on each compact subset of $G\setminus\{0\}$. If we verify condition (3) in Lemma 4.2 (with $\theta = 1/2$) for $k$, statements (i) and (ii) above will then follow directly from Theorems 4.4 and 4.3, respectively. In other words, it remains to prove that

$$\int_{G\setminus[G_n^{1/2}]_{-1}} |k(x-y)-k(x)|\,dx \leqslant B \tag{1}$$

for all integers $n$, and all $y$ in $G_n$.

Since $k$ is supported in $G_0$, we may restrict our attention to integers $n>0$. However, by Definition 1.6, $G_n^{1/2} = G_{n/2}$ or $G_{(n+1)/2}$ according as $n$ is even or odd. It is easily seen therefore that (1) will be established if we prove that

$$\int_{G\setminus G_m} |k(x-y)-k(x)|\,dx \leqslant B \tag{2}$$

for all $y$ in $G_{2m}$ and all integers $m \geqslant 1$.

Fix an integer $m \geqslant 1$, and write $k = k_1+k_2$, where

$$\hat{k}_1 = \xi_{\Gamma_{2m}}\hat{k} \quad \text{and} \quad \hat{k}_2 = \xi_{\Gamma\setminus\Gamma_{2m}}\hat{k}.$$

By hypothesis, $\hat{k}_2$ is constant on cosets of $\Gamma_m$; hence $k_2$ is supported in $G_m$, and makes no contribution to the left side of (2). On the other hand, $\hat{k}_1$ is supported in $\Gamma_{2m}$, and hence $k_1$ is constant on cosets of $G_{2m}$. Clearly, then $k_1(x-y)-k_1(x) = 0$ for all $y$ in $G_{2m}$, and we have established (2) with $B = 0$. ■

5.2. Example. We now show that the range $[2/(2-\alpha),\ 2/\alpha]$ in Theorem 5.1(ii) is best possible by exhibiting a function $\varphi$ satisfying the hypotheses of Theorem 5.1, but such that

$$\psi = \varphi\Theta^{1-\alpha} \notin M_q(\Gamma)$$

whenever $q \notin [2/(2-\alpha),\ 2/\alpha]$. We achieve this by showing more, viz. that if $p = 2/(2-\alpha)$, then $\psi$ does not belong to the closure of $M_q$ in $M_p$-norm for any $q < p$. We also show that $\psi \notin M_p^2(\Gamma)$, that is, the operator $T_\psi$ is not bounded from $L^p(G)$ to $L^2(G)$. This makes an interesting contrast with the "singular" multiplier example of Figà-Talamanca and Gaudry [3]. (At this stage, the reader is urged to consult the paper [4] of the authors.)

We construct Rudin–Shapiro-like polynomials on $G$ as follows. For each $n \geqslant 0$, fix $\gamma_0^n$ in $\Gamma_{2n+2}\setminus\Gamma_{2n+1}$ and set

$$\begin{aligned} \varrho_0^n &= \sigma_0^n = \xi_{G_n}\gamma_0^n, \\ \varrho_k^n &= \varrho_{k-1}^n + \gamma_k^n\sigma_{k-1}^n, \qquad (k = 1, \ldots, n+1), \\ \sigma_k^n &= \varrho_{k-1}^n - \gamma_k^n\sigma_{k-1}^n, \end{aligned}$$

where the $\gamma_k^n$ are chosen in $\Gamma_{2n+2}\setminus\Gamma_{2n+1}$ in such a way that $(\varrho_k^n)^\wedge$ and $(\sigma_k^n)^\wedge$ are both constant and nonzero on precisely $2^k$ cosets of $\Gamma_n$ in $\Gamma_{2n+2}\setminus\Gamma_{2n+1}$. (In fact, $(\varrho_k^n)^\wedge$ and $(\sigma_k^n)^\wedge$ will both take the values $\pm 2^{-n}$ on $2^k$ cosets of $\Gamma_n$ in $\Gamma_{2n+2}\setminus\Gamma_{2n+1}$.) It is easily seen that

$$|\varrho_k^n|^2 + |\sigma_k^n|^2 = 2\{|\varrho_{k-1}^n|^2 + |\sigma_{k-1}^n|^2\} = \ldots = 2^{k+1}\xi_{G_n},$$

so

$$\|\varrho_k^n\|_\infty \leqslant 2^{(k+1)/2}$$

and

$$\|\varrho_k^n\|_p \leqslant 2^{(k+1)/2-n/p}$$

for $k = 0, \ldots, n+1$.

We define $\varphi$ to be 0 on $\Gamma_0$ and on all sets of the form $\Gamma_{2n+1}\setminus\Gamma_{2n}$ $(n \geqslant 1)$. For $n \geqslant 0$, we set $\varphi$ equal to $\operatorname{sgn}(\varrho_{n+1}^n)^\wedge$ on $\Gamma_{2n+2}\setminus\Gamma_{2n+1}$. By Theorem 5.1, $\psi = \varphi\Theta^{1-\alpha} \in M_p$ where $p = 2/(2-\alpha)$.

To show that $\psi \notin \overline{M_q(\Gamma)}$, it suffices, by Proposition 1 of [4], to produce a sequence $\{h_n\}$ in $A_p(G)$ such that $\|h_n\|_{A_p}$ is bounded, $\|h_n\|_{A_q} \to 0$, and

$$\langle\psi, h_n\rangle = \int_\Gamma \psi(\gamma)\hat{h}_n(\gamma)\,d\gamma \not\to 0.$$

Take, for $n \geqslant 1$,

$$h_n = \varrho_{n+1}^n / 2^{(n+2)/2+(n+1)/p'}.$$

Since

$$(\varrho_{n+1}^n)^\wedge = (\varrho_{n+1}^n)^\wedge\,\xi_{\Gamma_{2n+2}\setminus\Gamma_{2n+1}},$$



we see that

$$\begin{aligned} \|\varrho_{n+1}^n\|_{A_p} &\leqslant \|\varrho_{n+1}^n\|_{p'}\|(\xi_{\Gamma_{2n+2}\setminus\Gamma_{2n+1}})^\wedge\|_p \\ &\leqslant 2^{(n+2)/2-n/p'}\cdot 2^{(2n+1)/p'} = 2^{(n+2)/2+(n+1)/p'}. \end{aligned}$$

Hence $\|h_n\|_{A_p} \leqslant 1$ for all $n$. However.

$$\begin{aligned} \|h_n\|_{A_q} &\leqslant \|\varrho_{n+1}^n\|_{q'}\|(\xi_{\Gamma_{2n+2}\setminus\Gamma_{2n+1}})^\wedge\|_q / 2^{(n+1)/2-(n+1)/p'} \\ &= 2^{(n+1)(1/q'-1/p')} \to 0 \end{aligned}$$

as $n \to \infty$ since $q < p$. Finally

$$\begin{aligned} \langle\psi, h_n\rangle &= \int_{\Gamma_{2n+2}\setminus\Gamma_{2n+1}} 2^{-n}2^{-((n+2)/2+(n+1)/p')}2^{-(1-\alpha)((2n+2)/4)}d\gamma \\ &= 2^{-1/2}; \end{aligned}$$

and so $\psi \notin \overline{M_q(\Gamma)}$ whenever $q < p$.

Finally we show that $T_\psi \notin M_p^2(\Gamma)$ by exhibiting a sequence of functions $\{f_n\}$ in $L^2(G)$ such that

$$\frac{\|T_\psi f_n\|_{p'}}{\|f_n\|_2} \to \infty \quad \text{as} \quad n \to \infty.$$

We take

$$f_n = \varrho_{n+1}^n.$$

Clearly,

$$(T_\psi f_n)^\wedge = 2^{-n}2^{-(1-\alpha)((2n+2)/4)}\xi_{\Gamma_{2n+2}\setminus\Gamma_{2n+1}},$$

and so

$$|T_\psi f_n| = 2^{n+1-(1-\alpha)((n+1)/2)}\xi_{G_{2n+1}}.$$

Hence

$$\|T_\psi f_n\|_{p'} = 2^{n+1-(1-\alpha)((n+1)/2)-(2n+1)/p'},$$

while

$$\|f_n\|_2 = \|\hat{f}_n\|_2 = 2^{1/2}.$$

Therefore

$$\frac{\|T_\psi f_n\|_{p'}}{\|f_n\|_2} = 2^{n(1-\alpha)/2} \to \infty \quad \text{as} \quad n \to \infty,$$

since $0 < \alpha < 1$. ■

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# Fat equicontinuous groups of homeomorphisms of linear topological spaces and their application to the problem of isometries in linear metric spaces


by

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**Abstract.** It is proved that if $X$ is a locally convex linear topological space with the strong Krein–Milman property, then every equicontinuous group of homeomorphism of $X$ which includes translations and "minus identity", consists of affine mappings only. Also, it is proved that for every metrizable t.v.s. $X$ the following two statements are equivalent: (i) every equicontinuous group of homeomorphisms of $X$ which includes translations and "minus identity" consists of affine mappings only, (ii) for every translation invariant metric $d$ on $X$ inducing the topology of $X$, every isometry from $(X, d)$ onto another linear metric space is affine.


The question whether every surjective isometry between two linear metric spaces is affine has not been solved yet (the metrics are assumed to be translation invariant). The first result in this direction has been obtained by S. Mazur and S. Ulam in [6]. Namely, they proved that the question has a positive answer if the metrics are given by norms. Their proof is based on the argument that, if the metric is sufficiently regular, then one can define, in terms of the metric, the midpoint of a given interval. This argument has been used by several authors to prove the corresponding results for other classes of metrics (a survey of these results is to be found in [8]). In our opinion, it is worthwhile to change the approach and to ask whether the corresponding version of the Mazur–Ulam theorem holds for arbitrary metrics, provided, may be, that the spaces involved are sufficiently "nice". The first attempt along this line was made by Z. Charzyński [1], who proved that the theorem in question holds when both spaces are finite-dimensional. In [5], the author has generalized Charzyński's result to the case of metrizable locally convex Montel spaces.

In the present note, we study equicontinuous subgroups of groups of homeomorphisms of linear topological spaces, which include translations and "minus identity". We show that, if $X$ is a locally convex linear space with the strong Krein–Milman property, then

(*) *every equicontinuous group of homeomorphisms of $X$ including translations and "minus identity" must consist of affine mappings only.*