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An axiomatic approach to joint spectra I

by

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Abstract. We introduce and study axioms for joint spectra and related concepts in Banach algebras. In particular, we prove the existence of the largest spectrum and the minimal spectra and give some of their properties.

1. Introduction. Let $\mathscr{A}$ be a commutative complex unital Banach algebra with the unit e. For any subset $\boldsymbol{x}_a \subset \mathscr{A}$, $\boldsymbol{x}_a = \{x_a\}_{a \in \alpha}$ its joint spectrum (or shortly spectrum) in $\mathscr{A}$ is defined as

$$\sigma_{\mathscr{A}}(\boldsymbol{x}_a) = \{(f(x_a))_{a\in\alpha} \in \boldsymbol{C}^{\alpha}\colon \ f \in \mathfrak{M}(\mathscr{A})\}, \tag{1}$$

where $\mathfrak{M}(\mathscr{A})$ is the set of all multiplicative-linear functionals of the algebra $\mathscr{A}$. The spectrum $\sigma_{\mathscr{A}}(\boldsymbol{x}_a)$ is always a non-void compact subset of $\boldsymbol{C}^{\alpha}$. The joint spectrum is a basic concept for one of the most important chapters in the theory of commutative Banach algebras, namely for the functional calculus in these algebras. However, the concept of a joint spectrum is not so clear in the non-commutative case, even if we reduce our attention to commuting families of elements of the algebra in question. Thus different writers have adopted different concepts of a joint spectrum. Bonsall and Duncan propose in [2] to define a spectrum as the union of the left and the right spectrum. The same concept is adopted in the papers of Harte [6], [7]. Dash in [5] considers the bicommutant spectrum while Taylor in [11], before introducing his very interesting concept, starts with the commutant spectrum. All these concepts concide in the case of a single element of the algebra in question. Some special subsets of joint spectra have also been studied, such as the left spectrum, the right spectrum, the defect spectrum, and, in particular, the approximate point spectrum (cf. [3], [4], [5], [6], [7], [10]). In Section 3 of this paper we propose axioms for the joint spectra and some of their subsets, extending our previous concept, called a spectral system ([10], Definition 2.1). As the main axiom we assume here the spectral mapping property for polynomial maps. Since not all spectra considered before obey this axiom, we propose here the term "spectroid". So, in particular, the commutant and bicommutant spectra are not spectra but spectroids in the sense of this paper. After the examples of Section 4 we give in Section 5 the functional representation of subspectra. We also obtain in this section the spectral mapping theorem for spectra under rational maps and give a description of some situations with a unique spectrum. In Section 6 we prove the existence of the largest spectrum and the minimal spectra and subspectra. In Section 7 we consider the maps of spectra induced by homomorphisms; in particular, we show that the largest spectrum is invariant under automorphisms of the algebra in question.

In this paper we do not consider questions connected with the functional calculi.

2. Notation. Let A be a complex unital Banach algebra with the unit e. The set of all invertible elements of A will be denoted by $G(A)$. The family of all non-void subsets of A consisting of pairwise commuting elements will be designated by $\mathfrak{c}(A)$. The elements of $\mathfrak{c}(A)$ will be denoted by boldfaced characters $\boldsymbol{x}_a$, where $\boldsymbol{a}$ is a (non-void) set of indexes. So $\boldsymbol{x}_a = \{x_a\}_{a\in\boldsymbol{a}}$. The set $\mathfrak{c}(A)$ is partially ordered by inclusion, its maximal elements coincide with the maximal commutative subalgebras of A. The set of all maximal commutative subalgebras of A will be denoted by $\mathfrak{m}(A)$, and so $\mathfrak{m}(A) \subset \mathfrak{c}(A)$. We shall write $\mathfrak{c}_0(A)$ for the family of all finite elements of $\mathfrak{c}(A)$. As usual, $\boldsymbol{C}^{\boldsymbol{a}} = \prod_{a\in\boldsymbol{a}} \boldsymbol{C}_a$, where all $\boldsymbol{C}_a$ are equal to the field $\boldsymbol{C}$ of all complex numbers. $\boldsymbol{C}^{\boldsymbol{a}}$ is then a topological linear space (with the usual product topology). If $\boldsymbol{a}$ and $\boldsymbol{\beta}$ are non-void sets of indexes, then $\boldsymbol{p}_\beta(\boldsymbol{t}_a)$ will stand for a family of polynomials $\{p_\beta(\boldsymbol{t}_a)\}_{\beta\in\boldsymbol{\beta}}$, with complex coefficients, in indeterminates $\boldsymbol{t}_a = \{t_a\}_{a\in\boldsymbol{a}}$ (each p_β depends only upon a finite number of indeterminates $t_{a_1}, t_{a_2}, \ldots, t_{a_n}$).

Each such system of polynomials induces a map, denoted by the same symbol $\boldsymbol{p}_\beta$: $\boldsymbol{C}^{\boldsymbol{a}} \to \boldsymbol{C}^{\boldsymbol{\beta}}$, given by $\boldsymbol{z}_a \to \big(p_\beta(\boldsymbol{z}_a)\big)_{\beta\in\boldsymbol{\beta}} \in \boldsymbol{C}^{\boldsymbol{\beta}}$, where $\boldsymbol{z}_a = (z_a)_{a\in\boldsymbol{a}} \in \boldsymbol{C}^{\boldsymbol{a}}$. Such a map will be called a *polynomial map*. Also, if $\boldsymbol{x}_a \in \mathfrak{c}(A)$, then taking it instead of $\boldsymbol{t}_a$ in $\boldsymbol{p}_\beta(\boldsymbol{t}_a)$, we again abtain an element $\boldsymbol{p}_\beta(\boldsymbol{x}_a) \in \mathfrak{c}(A)$.

Similarly, $\boldsymbol{r}_\beta(\boldsymbol{t}_a)$ will stand for a family of rational functions in variables $\{t_a\}_{a\in\boldsymbol{a}}$. So $\boldsymbol{r}_\beta(\boldsymbol{t}_a) = \{r_\beta(\boldsymbol{t}_a)\}_{\beta\in\boldsymbol{\beta}}$ and $r_\beta(\boldsymbol{t}_a) = \dfrac{p_\beta(\boldsymbol{t}_a)}{q_\beta(\boldsymbol{t}_a)}$, where p_β and q_β are complex polynomials. We shall assume that p_β and q_β have no common roots. As before, the system $\boldsymbol{r}_\beta$ defines a map from $\bigcap_{\beta\in\boldsymbol{\beta}} \{\boldsymbol{z}_a \in \boldsymbol{C}^{\boldsymbol{a}}\colon q_\beta(\boldsymbol{z}_a) \neq 0\}$ into $\boldsymbol{C}^{\boldsymbol{\beta}}$, which will be called a *rational map*. Also, if $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and $\boldsymbol{r}_\beta(\boldsymbol{t}_a) = \left\{\dfrac{p_\beta(\boldsymbol{t}_a)}{q_\beta(\boldsymbol{t}_a)}\right\}_{\beta\in\boldsymbol{\beta}}$ is a family of rational functions, then $\boldsymbol{r}_\beta(\boldsymbol{x}_a) \in \mathfrak{c}(A)$, provided $q_\beta(\boldsymbol{x}_a) \in G(A)$ for all $\beta \in \boldsymbol{\beta}$.

Occasionally, we shall omit the index set when it is clear from the context; e.g., we shall sometimes write $\boldsymbol{z}$ instead of $\boldsymbol{z}_a$, etc.

3. Axioms and definitions. Suppose that to each family $\boldsymbol{x}_a \in \mathfrak{c}(A)$ there corresponds a non-void compact subset of $\boldsymbol{C}^{\boldsymbol{a}}$

$$\boldsymbol{x}_a \to \tilde\sigma(\boldsymbol{x}_a) \subset \boldsymbol{C}^{\boldsymbol{a}}. \tag{2}$$

We shall formulate several conditions (axioms) for such a map

$$\tilde\sigma(\boldsymbol{x}_a) \subset \prod_{a\in\boldsymbol{a}} \sigma(x_a), \tag{I}$$

where $\boldsymbol{x}_a = \{x_a\}_{a\in\boldsymbol{a}} \in \mathfrak{c}(A)$, and $\sigma\{x_a\}$ is the usual spectrum of an element $x_a \in A$, defined as $\sigma(x_a) = \{\lambda \in C\colon x - \lambda e \notin G(A)\}$.

From this condition it follows, in particular, that

$$\tilde\sigma(x) \subset \sigma(x) \tag{3}$$

for every $x \in A$. Here, for simplicity, we write $\tilde\sigma(x)$ instead of $\tilde\sigma(\{x\})$.

Assume as the second axiom the following stronger version of formula (3)

$$\tilde\sigma(x) = \sigma(x) \tag{II}$$

for all $x \in A$.

The most essential will be the following axiom:

$$\tilde\sigma\big(\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big) = \boldsymbol{p}_\beta\tilde\sigma(\boldsymbol{x}_a); \tag{III}$$

here $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and $\boldsymbol{p}_\beta$ is a system of complex polynomials in indeterminates $\boldsymbol{t}_a$, or a polynomial map. The property of the map (2) given by axiom (III) will be called the *spectral mapping property of* $\tilde\sigma$.

3.1. Definition. A map (2) is called a *subspectrum* on A if axioms (I) and (III) are satisfied. It is called a *spectrum*, if, moreover, axiom (II) is satisfied (it is sufficient to assume only axiom (II) and (III) here since axiom (I) is their consequence). The set of all spectra on A will be denoted by $\mathrm{Sp}(A)$, and the set of all subspectra by $\mathrm{Sp}_1(A)$, so that $\mathrm{Sp}(A) \subset \mathrm{Sp}_1(A)$.

Let us formulate some consequences of the spectral mapping property. Suppose first that $\boldsymbol{\beta} \subset \boldsymbol{a}$ and put $p_\beta(\boldsymbol{t}_a) = t_\beta$ for all $\beta \in \boldsymbol{\beta}$. Axiom (III) then implies

$$\tilde\sigma(\boldsymbol{x}_\beta) = \pi\tilde\sigma(\boldsymbol{x}_a), \tag{IV}$$

where $\boldsymbol{x}_a \in \mathfrak{c}(A)$, $\boldsymbol{\beta}$ is a non-void subset of $\boldsymbol{a}$, and π is the projection of $\boldsymbol{C}^{\boldsymbol{a}}$ onto $\boldsymbol{C}^{\boldsymbol{\beta}}$ given by $\pi(\boldsymbol{z}_a) = \boldsymbol{z}_\beta$.

The property of $\tilde\sigma$ given by formula (IV) will be called the *projection property* of $\tilde\sigma$.

Now put $\boldsymbol{\beta} = \boldsymbol{a}$ and $p_a(\boldsymbol{t}_a) = t_a + \lambda_a$, where $\lambda_a \in \boldsymbol{C}$. Axiom (III) then implies

$$\tilde\sigma(\boldsymbol{x}_a + \boldsymbol{\lambda}_a\cdot e) = \tilde\sigma(\boldsymbol{x}_a) + \boldsymbol{\lambda}_a, \tag{V}$$

where $\boldsymbol{x}_a \in \mathfrak{c}(A)$, $\boldsymbol{\lambda}_a = (\lambda_a)_{a\in\boldsymbol{a}} \in \boldsymbol{C}^{\boldsymbol{a}}$, $\boldsymbol{x}_a + \boldsymbol{\lambda}_a e = \{x_a + \lambda_a e\}_{a\in\boldsymbol{a}}$.

3.2. Definition. A *spectroid* on A is a map (2) satisfying axioms (I) and (V). If, moreover, axiom (IV) is also satisfied, a spectroid will be called a *semispectrum*. The set of all spectroids on A will be denoted by $\mathrm{Sp}_3(A)$, and the set of all semispectra by $\mathrm{Sp}_2(A)$. Thus we have $\mathrm{Sp}(A) \subset \mathrm{Sp}_1(A) \subset \mathrm{Sp}_2(A) \subset \mathrm{Sp}_3(A)$.

Axiom (V) can be used for defining a spectroid by means of regularity. So we give the following

3.3. Definition. Let $\tilde{\sigma} \in \mathrm{Sp}_3(A)$. An element $\boldsymbol{x}_a \in \mathfrak{c}(A)$ will be called *$\tilde{\sigma}$-regular* if $0 \notin \tilde{\sigma}(\boldsymbol{x}_a)$, where 0 is the zero element of the linear space $\boldsymbol{C}^a$. The set of all $\tilde{\sigma}$-regular elements will be denoted by $\mathrm{Reg}(\tilde{\sigma})$. From axiom (V) it follows that each spectroid can be defined by means of $\tilde{\sigma}$-regular elements:

$$\tilde{\sigma}(\boldsymbol{x}_a) = \{\lambda_a \in \boldsymbol{C}^a\colon\ \boldsymbol{x}_a - \lambda_a \cdot e \notin \mathrm{Reg}(\tilde{\sigma})\}.$$

Let us remark that in order to define a semispectrum on $\mathfrak{c}(A)$ it is sufficient to define it only on maximal subalgebras of A. In fact, for every $\boldsymbol{x}_a \in \mathfrak{c}(A)$ there is an element $\mathscr{A} \in m(A)$ with $\boldsymbol{x}_a \subset \mathscr{A}$. If $\mathscr{A} = \boldsymbol{x}_\beta$ then $\boldsymbol{a} \subset \boldsymbol{\beta}$ and axiom (IV) implies that $\tilde{\sigma}(\boldsymbol{x}_a) = \pi\tilde{\sigma}(\boldsymbol{x}_\beta)$ if $\tilde{\sigma} \in \mathrm{Sp}_2(A)$. So all values of $\tilde{\sigma}$ are obtained by projecting the values $\tilde{\sigma}(\mathscr{A})$, $\mathscr{A} \in m(A)$.

Let us also remark that in many concrete situations we have a semispectrum defined only on elements of $\mathfrak{c}_0(A)$. In this case we can extend it to $\mathfrak{c}(A)$, using the following result of [10], which in our terminology reads as follows:

3.4. Proposition. *Let $\tilde{\sigma}$ be a semispectrum defined on elements of $\mathfrak{c}_0(A)$. There exists a unique semispectrum on $\mathfrak{c}(A)$ which, restricted to $\mathfrak{c}_0(A)$, equals $\tilde{\sigma}$. If $\tilde{\sigma}$ is a subspectrum or a spectrum, then its extension is also a subspectrum, or a spectrum, respectively.*

We have assumed here a Banach algebra convention, i.e. all concepts are related to a Banach algebra A. However, many important spectroids are defined in the so called *spatial convention*, i.e. they are defined on the algebra $L(X)$ of all continuous endomorphisms of a complex Banach space X, and in the definition elements of X are involved. If such a definition does not depend upon a particular choice of the space X, we can have this spectroid also in the Banach algebra convention. For a given Banach algebra A we simply put $X = A$ and interpret elements of A as endomorphisms of X. We can do it in two ways; interpreting elements of A as operators either of left multiplications or of right multiplication. Thus we embed A in $L(X)$ in two ways, and so a spectroid on $L(X)$ gives rise to two spectroids defined on A, namely the left one and the right one (we are not interested in the iteration of this procedure, since we can embed $L(X)$ in $L(L(X))$ again in two ways, etc).

In the next section, recalling the known and important examples

of spectroids, we shall exploit both conventions, though the general convention accepted in this paper is the Banach algebra one.

4. Examples of spectroids.

4.0. The usual joint spectrum (1) in a commutative Banach algebra A. It is an element of $\mathrm{Sp}(A)$.

4.1. The left spectrum σ_l and the right spectrum σ_r. An element $\boldsymbol{x}_a$ is σ_l-regular (σ_r-regular) if the left (right) ideal in A generated by the set $\boldsymbol{x}_a$ is improper and coincides with the whole of A. Both the left and the right spectra are in $\mathrm{Sp}_1(A)$, but not, in general, in $\mathrm{Sp}(A)$ (cf. [6], [10]).

4.2. The spectrum σ. It is defined by $\sigma(\boldsymbol{x}_a) = \sigma_l(\boldsymbol{x}_a) \cup \sigma_r(\boldsymbol{x}_a)$, $\boldsymbol{x}_a \in \mathfrak{c}(A)$. It belongs to $\mathrm{Sp}(A)$ (cf. [2], [7]).

4.3. The product $\boldsymbol{x}_a \to \prod_{a \in a} \sigma(x_a)$ is a semispectrum but not a spectrum (cf. [10]).

4.4. The bicommutant spectrum σ''. For any $\boldsymbol{x}_a \in \mathfrak{c}(A)$ its bicommutant $(\boldsymbol{x}_a)''$ is the intersection $\bigcap\{\mathscr{A} \in m(A)\colon\ \boldsymbol{x}_a \subset \mathscr{A}\}$. It is a commutative subalgebra of A and $\sigma''(\boldsymbol{x}_a)$ is defined as the usual joint spectrum 4.0 in this subalgebra. We have $\sigma'' \in \mathrm{Sp}_3(A)$, but, in general, it is not a semispectrum (cf. [10]).

4.5. The commutant spectrum σ'. For any $\boldsymbol{x}_a \in \mathfrak{c}(A)$ its commutant $(\boldsymbol{x}_a)'$ is the union $\bigcup\{\mathscr{A} \in m(A)\colon\ \boldsymbol{x}_a \subset \mathscr{A}\}$. An element $\boldsymbol{x}_a$ is σ'-regular if there are elements $x_1, x_2, \ldots, x_n \in \boldsymbol{x}_a$ and $y_1, y_2, \ldots, y_n \in (\boldsymbol{x}_a)'$ such that $\sum_{i=1}^{n} x_i y_j = e$. The commutant spectrum is a spectroid but not, in general, a semispectrum (cf. [10]).

4.6. The defect spectrum σ_d. We exploit here the spatial convention. An element $\boldsymbol{x}_a \in \mathfrak{c}(L(X))$ is σ_d-regular if there are elements $x_1, x_2, \ldots, x_n \in \boldsymbol{x}_a$ such that $\{\sum_{i=1}^{n} x_i \xi_i\colon\ \xi_i \in X\} = X$. In the Banach algebra convention we then have two spectra: the left defect spectrum σ_d^l and the right defect spectrum σ_d^r. The defect spectrum is a subspectrum (cf. [10], where it was denoted by σ_τ (Definition 2.9)).

4.7. The approximate point spectrum σ_π. Here we again apply the spatial convention. An element $\boldsymbol{x}_a \in \mathfrak{c}(L(X))$ is σ_π-regular if it is not σ_π-singular, and it is σ_π-singular if there is a net (ξ_τ) of elements of X, with $\|\xi_\tau\| = 1$, such that $\lim_\tau x_a \xi_\tau = 0$ for all $a \in \boldsymbol{a}$. Passing to the Banach algebra convention, we obtain two concepts of a left and right approximate point spectrum, both being subspectra (cf. [6], [10]).

4.8. The Taylor spectrum σ_T. Let X be a complex Banach space and consider the linear spaces $X_0, X_1, \ldots, X_n$ of homogeneous exterior forms of degrees, respectively, $0, 1, \ldots, n$, in indeterminates $e_1, e_2, \ldots, e_n$, with coefficients from X. Thus

$$X_k = \Big\{ \sum_{i_1<i_2\ldots<i_k} \xi_{i_1\ldots i_k} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \colon \ \xi_{i_1\ldots i_k} \in X \Big\}$$

and X_0 may be identified with X. Let $(x_1, x_2, \ldots, x_n)$ be a fixed n-tuple of pairwise commuting operators in $L(X)$. This n-tuple gives rise to linear maps

$$\partial_k\colon X_k \to X_{k1}^{+} \qquad (k = 0, 1, \ldots, n, \text{ with } X_{n+1} = \{0\})$$

defined by

$$\partial_k(\xi e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}) = \sum_{i=1}^{n} x_i(\xi) e_i \wedge e_{i_1} \wedge \ldots \wedge e_{i_k},$$

exploiting the usual convention $e_i \wedge e_j = -e_j \wedge e_i$. The n-tuple $(x_1, \ldots, x_n)$ is σ_{T}-regular if the sequence

$$0 \longrightarrow X_0 \xrightarrow{\partial_0} X_1 \longrightarrow \ldots \longrightarrow X_k \xrightarrow{\partial_k} X_{k+1} \longrightarrow \ldots \longrightarrow X_n \longrightarrow 0$$

is exact. This defines a spectrum on $\mathfrak{c}_0(L(X))$, which, by Proposition 3.4 can be extended to a spectrum on $\mathfrak{c}(L(X))$. If we want to have σ_{T} defined on $\mathfrak{c}(A)$, we proceed as at the end of Section 3, obtaining the left and right Taylor spectra (cf. [11], [12]).

For more examples cf. [9], where some families of subspectra contained in the Taylor spectrum are considered.

5. The functional representation of a subspectrum. If we have a semispectrum on A, then from the projection property (axiom (IV)) it follows that we know it on the whole of $\mathfrak{c}(A)$ if we know it on $m(A)$. For this reason we want to know how subspectra behave on the maximal commutative subalgebras of A. In this section we study the subspectra on A restricted to the elements of $m(A)$.

5.1. THEOREM. *Let $\tilde\sigma$ be a subspectrum on A. Then*

$$\tilde\sigma(\boldsymbol{x}_a) \subset \sigma_{\mathscr{A}}(\boldsymbol{x}_a) \tag{4}$$

for every $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and every $\mathscr{A} \in m(A)$ with $\boldsymbol{x}_a \subset \mathscr{A}$. (Here $\sigma_{\mathscr{A}}(\boldsymbol{x}_a)$ is the usual joint spectrum in the commutative Banach algebra $\mathscr{A}$.)

Proof. We first show that for any $\mathscr{A} \in m(A)$ we have

$$\tilde\sigma(\mathscr{A}) \subset \sigma_{\mathscr{A}}(\mathscr{A}). \tag{5}$$

In fact, let $\mathscr{A} = \boldsymbol{x}_\beta$ and suppose that there is in $\boldsymbol{C}^\beta$ an element λ such that $\lambda \in \tilde\sigma(\mathscr{A}) \setminus \sigma_{\mathscr{A}}(\mathscr{A})$. Since $\sigma_{\mathscr{A}}(\mathscr{A})$ is a polynomially convex subset of $\boldsymbol{C}^\beta$ (cf. [14]), it follows that there exists a polynomial $p(\boldsymbol{t}_\beta)$ such that

$$|p(\lambda)| > \sup\{|p(z)|\colon z \in \sigma_{\mathscr{A}}(\mathscr{A})\}. \tag{6}$$

Applying this polynomial to $\mathscr{A}$, we obtain an element $x = p(\mathscr{A}) \in \mathscr{A}$, and, by axiom (III), we have

$$p(\lambda) \in p\big(\tilde\sigma(\mathscr{A})\big) = \tilde\sigma\big(p(\mathscr{A})\big) = \tilde\sigma(x). \tag{7}$$

On the other hand, we also have

$$\sigma_{\mathscr{A}}(x) = \sigma_{\mathscr{A}}\big(p(\mathscr{A})\big) = p\big(\sigma_{\mathscr{A}}(\mathscr{A})\big),$$

which in view of (6), shows that $p(\lambda) \notin \sigma_{\mathscr{A}}(x) = \sigma(x)$. Together with relation (7) this shows that $\tilde\sigma(x) \not\subset \sigma(x)$, which contradicts formula (3). Thus we have established formula (5). Formula (4) now follows by projecting $\boldsymbol{C}^\beta$ onto $\boldsymbol{C}^a$ and using axiom (IV).

5.2. Remark. From Theorem 3 of [10] it immediately follows, that if $\tilde\sigma$ is a semispectrum on A and the relation (4) holds true for each $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and each $\mathscr{A} \in m(A)$ satisfying $\boldsymbol{x}_a \subset \mathscr{A}$, then $\tilde\sigma$ is a subspectrum.

We can now give the functional representation of a subspectrum.

5.3. THEOREM. *Let $\tilde\sigma$ be a subspectrum on A. For each $\mathscr{A} \in m(A)$ there exists a compact subset $\Delta(\tilde\sigma, \mathscr{A}) \subset \mathfrak{M}(\mathscr{A})$ such that*

$$\tilde\sigma(\boldsymbol{x}_a) = \big\{\big(f(x_a)\big)_{a\in a} \in \boldsymbol{C}^a\colon f \in \Delta(\tilde\sigma, \mathscr{A})\big\} \tag{8}$$

for all $\boldsymbol{x}_a \in \mathfrak{c}(A)$ with $\boldsymbol{x}_a \subset \mathscr{A}$.

Proof. Fix an element $\mathscr{A} \in m(A)$. By formula (5) for each $\lambda \in \tilde\sigma(\mathscr{A})$, $\lambda = (\lambda_x)_{x\in\mathscr{A}}$, there exists a unique functional $f \in \mathfrak{M}(\mathscr{A})$ such that $f(x) = \lambda_x$ for all $x \in \mathscr{A}$. The correspondence $\lambda \leftrightarrow f$ is a homeomorphism between $\tilde\sigma(\mathscr{A})$ treated, by (5), as a subset of $\sigma_{\mathscr{A}}(\mathscr{A})$ and a subset of $\mathfrak{M}(\mathscr{A})$ (which, in turn, is homeomorphic to $\sigma_{\mathscr{A}}(\mathscr{A})$). Denote this subset by $\Delta(\tilde\sigma, \mathscr{A})$. Since $\tilde\sigma(\mathscr{A})$ is compact $\Delta(\tilde\sigma, \mathscr{A})$ is also compact. We have

$$\tilde\sigma(\mathscr{A}) = \big\{\big(f(x)\big)_{x\in\mathscr{A}} \in \boldsymbol{C}^\beta\colon f \in \Delta(\tilde\sigma, \mathscr{A})\big\}, \tag{9}$$

where β is a suitable set of indexes. Formula (8) now follows from formula (9) and the projection property of $\tilde\sigma$.

From this result we obtain the following spectral mapping theorem for elements of $\operatorname{Sp} A$, which also gives a characterization of spectra among the subspectra:

5.4. THEOREM. *Let $\tilde\sigma$ be a subspectrum on A. Then $\tilde\sigma$ is a spectrum on A if and only for each $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and each system of rational functions $r_\beta(\boldsymbol{t}_a) = p_\beta(\boldsymbol{t}_a)/q_\beta(\boldsymbol{t}_a)$, $\beta \in \boldsymbol{\beta}$, such that*

$$\tilde\sigma(\boldsymbol{x}_a) \subset \{z_a \in \boldsymbol{C}^a\colon q_\beta(z_a) \neq 0 \text{ for all } \beta \in \boldsymbol{\beta}\} \tag{10}$$

we have

$$q_\beta(\boldsymbol{x}_a) \in G(A) \tag{11}$$

for all $\beta \in \boldsymbol{\beta}$ *and*

$$r_\beta \tilde{\sigma}(\boldsymbol{x}_a) = \tilde{\sigma}\big(r_\beta(\boldsymbol{x}_a)\big). \tag{12}$$

Proof. If $\tilde{\sigma} \in \mathrm{Sp}_1(A) \setminus \mathrm{Sp}(A)$, then there is an element $x \in A$ such that $\sigma(x) \setminus \tilde{\sigma}(x)$ is non-void. If $\lambda \in \sigma(x) \setminus \tilde{\sigma}(x)$, then $r(t) = (t-\lambda)^{-1}$ satisfies, together with $\boldsymbol{x}_a = \{x\}$, relation (10) but not relation (11).

On the other hand, suppose that $\tilde{\sigma}$ is a spectrum on A. Let $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and let $\boldsymbol{r}_\beta(\boldsymbol{t}_a)$ be a family of rational functions satisfying relation (10). For any $\beta \in \boldsymbol{\beta}$ it is $\sigma\big(q_\beta(\boldsymbol{x}_a)\big) = \tilde{\sigma}\big(q_\beta(\boldsymbol{x}_a)\big) = q_\beta\big(\tilde{\sigma}(\boldsymbol{x}_a)\big)$, and so, by (10), $0 \notin \sigma\big(q_\beta(\boldsymbol{x}_a)\big)$. This implies relation (11). Using formula (8), we have for any $\mathscr{A} \in m(A)$, with $\boldsymbol{x}_a \subset \mathscr{A}$,

$$\begin{aligned} \boldsymbol{r}_\beta \tilde{\sigma}(\boldsymbol{x}_a) &= \big\{\boldsymbol{r}_\beta\big((f(x_a))_{a\in a}\big) \in C^\beta\colon f \in \Delta(\tilde{\sigma}, \mathscr{A})\big\} \\ &= \Big\{\Big(r_\beta\big((f(x_a))_{a\in a}\big)\Big)_{\beta\in\beta} \in C^\beta\colon f \in \Delta(\tilde{\sigma}, \mathscr{A})\Big\} \\ &= \Big\{\Big(f\big(r_\beta(\boldsymbol{x}_a)\big)\Big)_{\beta\in\beta} \in C^\beta\colon f \in \Delta(\tilde{\sigma}, \mathscr{A})\Big\} = \tilde{\sigma}\big(\boldsymbol{r}_\beta(\boldsymbol{x}_a)\big), \end{aligned}$$

and formula (12) holds true.

5.5. Definition. Let $\mathscr{A}$ be a commutative unital Banach algebra. A compact subset $\Delta \in \mathfrak{M}(\mathscr{A})$ is called a *spectral set* for $\mathscr{A}$ if

$$\sigma(x) = \{f(x) \in C\colon f \in \Delta\} \tag{13}$$

for all $x \in \mathscr{A}$.

Every commutative Banach algebra possesses a minimal spectral set (with respect to inclusion), though usually such a set is not determined in a unique way (cf. Example 6.7). It is easy to see that a spectral set for $\mathscr{A}$ must contain its Shilov boundary $\Gamma(\mathscr{A})$. If $\tilde{\sigma}$ is a spectrum on A, then $\Delta(\tilde{\sigma}, \mathscr{A})$ must be a spectral set for each $\mathscr{A} \in m(A)$. This remark immediately implies the following corollary to Theorem 5.3.

5.6. Corollary. *If for each* $\mathscr{A} \in m(A)$ *we have* $\Gamma(\mathscr{A}) = \mathfrak{M}(\mathscr{A})$, *then* A *has a unique spectrum given by*

$$\sigma(\boldsymbol{x}_a) = \sigma_{\mathscr{A}}(\boldsymbol{x}_a) \tag{14}$$

for all $\boldsymbol{x}_a \in \mathfrak{c}(A)$ *and* $\mathscr{A} \in m(A)$ *with* $\boldsymbol{x}_a \subset \mathscr{A}$.

Such a situation holds e.g. for ES-algebras (for the definition cf. [13]; an ES-algebra is characterized by the property that all its elements have totally disconnected spectra), in particular for $A = L(C^n)$, or for a group algebra of a compact group (cf. [13]).

In the next theorem we shall use the theorem of Fuglede–Putnam (cf. [8]) which states that if x is a normal operator on a Hilbert space H such that $xy = yx$ for an operator y, then $x^*y = yx^*$.

5.7. Theorem. *Let* H *be a complex Hilbert space and let* $A = L(H)$. *Then all spectra on* A *coincide on elements of* $\mathfrak{c}(A)$ *consisitng of normal operators.*

Proof. Take any element $\boldsymbol{x}_a$ in $\mathfrak{c}(A)$ consisting of normal operators. Let $\mathscr{A}$ be a maximal subset of A consisting of pairwise commuting normal operators and containing the set $\boldsymbol{x}_a$. By the Fuglede–Putnam theorem we have $\mathscr{A}^* = \mathscr{A}$, and by the maximality $\mathscr{A}$ is a commutative C^*-algebra. We claim that $\mathscr{A} \in m(A)$. In fact, if x commutes with all elements of $\mathscr{A}$, then, by the Fuglede–Putnam theorem x^* also commutes and so do the elements $(x+x^*)/2$ and $(x-x^*)/2i$. The latter element, being hermitian, belongs to $\mathscr{A}$, and so $x \in \mathscr{A}$. Hence $\mathscr{A} \in m(A)$, and $\mathfrak{M}(\mathscr{A}) = \Gamma(\mathscr{A})$, since $\mathscr{A}$ is a C^*-algebra. Thus $\mathfrak{M}(\mathscr{A})$ is the unique spectral set for $\mathscr{A}$, and so $\Delta(\tilde{\sigma}, \mathscr{A}) = \mathfrak{M}(\mathscr{A})$ for all $\tilde{\sigma} \in \mathrm{Sp}(A)$. This means that $\tilde{\sigma}(\mathscr{A}) = \sigma_{\mathscr{A}}(\mathscr{A})$ and so, by the projection property, $\tilde{\sigma}(\boldsymbol{x}_a) = \sigma_{\mathscr{A}}(\boldsymbol{x}_a)$ for each $\tilde{\sigma} \in \mathrm{Sp}(A)$, which gives the desired result.

6. The largest spectrum and the minimal spectra and subspectra.

6.1. Definition. For two subspectra $\tilde{\sigma}_1, \tilde{\sigma}_2$ defined on a Banach algebra A we shall write $\tilde{\sigma}_1 \leqslant \tilde{\sigma}_2$ if $\sigma_1(\boldsymbol{x}_a) \subset \sigma_2(\boldsymbol{x}_a)$ for all $\boldsymbol{x}_a \in \mathfrak{c}(A)$.

In this section we shall show that every Banach algebra possesses a largest spectrum σ_m, i.e. a spectrum such that $\tilde{\sigma} \leqslant \sigma_m$ for all $\tilde{\sigma} \in \mathrm{Sp}_1(A)$. We shall show also that there exist minimal spectra and subspectra.

The proof of the following lemma is an easy exercise on compact sets and continuous maps.

6.2. Lemma. *Let* p *be a continuous map from* C^a *into* C^β, *and let* Ω *be a subset of* C^a *with compact closure* $\overline{\Omega}$. *Then* $p(\overline{\Omega}) = \overline{p(\Omega)}$.

This lemma will be used in the proof of the following

6.3. Theorem. *There exists a largest spectrum* σ_m *on* A.

Proof. For a fixed $\boldsymbol{x}_a \in \mathfrak{c}(A)$ define

$$\sigma_m(\boldsymbol{x}_a) = \overline{\bigcup\{\tilde{\sigma}(\boldsymbol{x}_a)\colon \tilde{\sigma} \in \mathrm{Sp}_1(A)\}} \tag{15}$$

it is a compact subset of C^a satisfying axiom (I).

Take any family of complex polynomials $\boldsymbol{p}_\beta(\boldsymbol{t}_a)$. Since it defines a continuous map from C^a into C^β, we can apply Lemma 6.2, taking as Ω the union $\bigcup\{\tilde{\sigma}(\boldsymbol{x}_a)\colon \tilde{\sigma} \in \mathrm{Sp}_1(A)\}$ and as p the map defined by the family $\boldsymbol{p}_\beta(\boldsymbol{t}_a)$. Thus

$$\begin{aligned} \boldsymbol{p}_\beta\big(\sigma_m(\boldsymbol{x}_a)\big) &= \boldsymbol{p}_\beta\Big(\overline{\bigcup\{\sigma(\boldsymbol{x}_a)\colon \tilde{\sigma} \in \mathrm{Sp}_1(A)\}}\Big) \\ &= \overline{\boldsymbol{p}_\beta\Big(\bigcup\{\tilde{\sigma}(\boldsymbol{x}_a)\colon \tilde{\sigma} \in \mathrm{Sp}_1(A)\}\Big)} = \overline{\bigcup\{\boldsymbol{p}_\beta\big(\tilde{\sigma}(\boldsymbol{x}_a)\big)\colon \tilde{\sigma} \in \mathrm{Sp}_1(A)\}} \\ &= \overline{\bigcup\{\tilde{\sigma}\big(\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big)\colon \tilde{\sigma} \in \mathrm{Sp}_1(A)\}} = \sigma_m\big(\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big), \end{aligned}$$

and so σ_m satisfies axiom (III). Axiom (II) is satisfied as well, since σ_m is larger than some spectrum, e.g. the spectrum of Example 4.2.

By Theorem 5.1 we obtain the following corollary:

6.4. COROLLARY. *For all* $\boldsymbol{x}_a \in \mathfrak{c}(A)$ *we have*

$$\sigma_m(\boldsymbol{x}_a) \subset \bigcap \{\sigma_{\mathscr{A}}(\boldsymbol{x}_a) \colon \mathscr{A} \in m(A), \boldsymbol{x}_a \subset \mathscr{A}\} \subset \sigma''(\boldsymbol{x}_a).$$

Remark. For the algebra A in which $\sigma_{\mathscr{A}}(\boldsymbol{x}_a)$ does not depend upon the subalgebra $\mathscr{A} \in m(A)$ containing the set $\boldsymbol{x}_a \in \mathfrak{c}(A)$, the largest spectrum is given by $\sigma_m(\boldsymbol{x}_a) = \sigma_{\mathscr{A}}(\boldsymbol{x}_a)$. Such a situation holds e.g. for the situation described in Corollary 5.6. In the general case, however, the value of $\sigma_{\mathscr{A}}(\boldsymbol{x}_a)$ depends upon $\mathscr{A}$, as shown in paper [1]. Thus the problem arises of a more efective description of the largest spectrum. In the commutative case the largest spectrum equals $\sigma_{\mathscr{A}}$, but in the non-commutative we know it only in situations described by Corollary 5.6.

We now prove a theorem concerning the existence of minimal spectra and subspectra.

6.5. THEOREM. $\mathrm{Sp}(A)$ *and* $\mathrm{Sp}_1(A)$ *contain the minimal elements.*

Proof. Suppose that we have a linearly ordered subset $\{\sigma_\iota\} \subset \mathrm{Sp}_1(A)$. We shall show that there exists an element $\sigma_0 \in \mathrm{Sp}_1(A)$ such that $\sigma_0 \leqslant \sigma$ for all ι. To this end put $\sigma_0(\boldsymbol{x}_a) = \bigcap_\iota \sigma_\iota(\boldsymbol{x}_a)$ for each $\boldsymbol{x}_a \in \mathfrak{c}(A)$. By the finite intersection property $\sigma_0(\boldsymbol{x}_a)$ is a non-void compact subset of $\boldsymbol{C}^a$, which clearly satisfies axiom (I). If $\{\sigma_\iota\} \subset \mathrm{Sp}(A)$, then σ_0 satisfies also axiom (II). In fact, let us fix an index $a \in a$. For each $\lambda \in \sigma(\boldsymbol{x}_a)$ and each ι there is a point $\lambda_\iota \in \sigma_\iota(\boldsymbol{x}_a)$ such that $\pi(\lambda_\iota) = \lambda$, where π is the projection of $\boldsymbol{C}^a$ onto the ath coordinate plane. The net $\{\lambda_\iota\}$ contains a subset convergent to a point $\lambda \in \boldsymbol{C}^a$. Since $\lambda_\iota \in \sigma_\varkappa(\boldsymbol{x}_a)$ for all $\varkappa \geqslant \iota$ (the ordering of indexes ι is the same as the ordering of the corresponding spectra and we write $\varkappa \geqslant \iota$ if $\sigma_\varkappa \geqslant \sigma_\iota$), it follows that λ is in the intersection $\sigma_0(\boldsymbol{x}_a)$. We have $\pi(\lambda) = \lim_\iota(\lambda_\iota) = \lambda$, and so $\pi\sigma_0(\boldsymbol{x}_a)$ contains the spectrum $\sigma(x_a)$. Together with the relation given by (I), this implies that σ_0 satisfies axiom (II).

Axiom (III) is also satisfied since for any element $\boldsymbol{x}_a \in \mathfrak{c}(A)$ and any system of complex polynomials $\boldsymbol{p}_\beta(\boldsymbol{t}_a)$ we have

$$\boldsymbol{p}_\beta\big(\sigma_0(\boldsymbol{x}_a)\big) = \boldsymbol{p}_\beta\big(\bigcap_\iota \sigma_\iota(\boldsymbol{x}_a)\big) = \bigcap_\iota \boldsymbol{p}_\beta\big(\sigma_\iota(\boldsymbol{x}_a)\big) = \bigcap_\iota \sigma_\iota\big(\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big) = \sigma_0\big(\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big).$$

Applying the Kuratowski–Zorn lemma, we see that there are minimal elements both in $\mathrm{Sp}(A)$ and $\mathrm{Sp}_1(A)$.

If $\mathscr{A}$ is a commutative Banach algebra, then the largest spectrum coincides with the usual joint spectrum, while the minimal spectra are of the form $\sigma(\boldsymbol{x}_a) = \{(f(x_a))_{a\in a} \in \boldsymbol{C}^a \colon f \in \Delta\}$, where Δ is a minimal spectral set. Of course, each minimal subspectrum of a commutative algebra consists of a single point for each $\boldsymbol{x}_a \in \mathfrak{c}(\mathscr{A})$. (1)

(1) In paper [15] there is described the situation when there exist single point subspectra on a Banach algebra A.

We now give two examples of minimal spectra for a commutative algebra.

6.6. EXAMPLE. Let $\Omega = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1|^2 + |z_2|^2 < 1\}$, and let $\mathscr{A}$ be the algebra of all functions holomorphic in Ω and continuous on $\overline{\Omega}$ provided with the supremum norm. The set $\partial\Omega = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\}$ is a spectral set for $\mathscr{A}$, which coincides with its Shilov boundary. Thus $\mathscr{A}$ possesses a unique minimal spectral set, and so a unique minimal spectrum.

6.7. EXAMPLE. Let $\Omega = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1| < 1,\ |z_2| < 1\}$, and let $\mathscr{A}$ be the algebra of all functions holomorphic on Ω and continuous on $\overline{\Omega}$, provided with the supremum norm. The algebra $\mathscr{A}$ possesses two minimal spectral sets,

$$\Delta_1 = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon z_1 = 0,\ |z_2| \leqslant 1\} \cup \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1| \leqslant 1,\ |z_2| = 1\}.$$

and

$$\Delta_2 = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1| \leqslant 1, z_2 = 0\} \cup \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1| = 1,\ |z_2| \leqslant 1\};$$

consequently there are two different minimal spectra.

7. Maps of subspectra defined by homomorphisms.

7.1. PROPOSITION. *Let A_1 and A_2 be unital Banach algebras. Let σ_2 be a spectroid (semispectrum, subspectrum) on A_2. Let h be a unital homomorphism of A_1 into A_2. Define $\sigma_1 = h^*\sigma_2$ by setting*

$$\sigma_1(\boldsymbol{x}_a) = \sigma_2(h\boldsymbol{x}_a), \tag{16}$$

where $\boldsymbol{x}_a \in \mathfrak{c}(A_1)$ and $h\boldsymbol{x}_a = (hx_a)_{a\in a}$. Then σ_1 is a spectroid (semispectrum, subspectrum) on A_1.

Proof. Clearly, we have $h\boldsymbol{x}_a \in \mathfrak{c}(A_2)$, whenever $\boldsymbol{x}_a \in \mathfrak{c}(A_1)$, and so σ_1 is well defined, and for any $\boldsymbol{x}_a \in \mathfrak{c}(A_1)$ the set $\sigma_1(\boldsymbol{x}_a)$ is a non-void compact subset of $\boldsymbol{C}^a$. Moreover,

$$\sigma_1(\boldsymbol{x}_a) = a_2(h\boldsymbol{x}_a) \subset \prod_a \sigma(hx_a) \subset \prod_a \sigma(x_a),$$

since, clearly, $\sigma(hx) \subset \sigma(x)$ for each $x \in A_1$. Thus σ_1 satisfies axiom (I). Suppose now that σ_2 is a subspectrum. Let $\boldsymbol{x}_a \in \mathfrak{c}(A_1)$ and let $\boldsymbol{p}_\beta(\boldsymbol{t}_a)$ be a family of complex polynomials in indeterminates $\boldsymbol{t}_a$. Clearly, we have $h\boldsymbol{p}_\beta(\boldsymbol{x}_a) = \boldsymbol{p}_\beta(h\boldsymbol{x}_a)$ and so

$$\sigma_1\big(\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big) = \sigma_2\big(h\boldsymbol{p}_\beta(\boldsymbol{x}_a)\big) = \sigma_2\big(\boldsymbol{p}_\beta(h\boldsymbol{x}_a)\big) = \boldsymbol{p}_\beta\big(\sigma_2(h\boldsymbol{x}_a)\big) = \boldsymbol{p}_\beta\big(\sigma_1(\boldsymbol{x}_a)\big),$$

which means that σ_1 is a subspectrum too. The same reasoning shows also that σ_1 is a spectroid (semispectrum), provided σ_2 is a spectroid (semispectrum) too.

Remarks. It can occur that σ_1 fails to be a spectrum on A_1 while σ_2 is a spectrum on A_2 and one can easily create a situation is which σ_2 is the largest spectrum and σ_1 is a minimal subspectrum. Even if h is a homomorphism onto, or an isomorphism into, $h^*\sigma$ may fail to be a spectrum for a spectrum σ. Only when h is an isomorphism onto and σ is a spectrum, $h^*\sigma$ is also a spectrum. This is because $\sigma(x) = \sigma(hx)$ for any $x \in A_1$ and so axiom (II) is satisfied for $h^*\sigma$. The last remark we formulate as

7.2. Corollary. *Let h be an automorphism of a Banach algebra A and let $\tilde{\sigma}$ be a spectrum on A. Then $h^*\sigma$ is also a spectrum on A.*

From Proposition 7.1 we shall obtain the following

7.3. Theorem. *The maximal spectrum σ_m on a Banach algebra A is invariant under automorphisms of A, i.e.*

$$\sigma_m = h^*\sigma_m \tag{17}$$

for each automorphism h of A.

Proof. By Proposition 7.1 $h^*\sigma_m$ is a subspectrum (a spectrum by Corollary 7.2), and so from the definition of σ_m we see that

$$(h^*\sigma_m)(\boldsymbol{x}_a) \subset \sigma_m(\boldsymbol{x}_a) \tag{18}$$

for all $\boldsymbol{x}_a \in \mathfrak{c}(A)$. Taking h^{-1} instead of h, we have, by (18),

$$\big((h^{-1})^*\sigma_m\big)(\boldsymbol{x}_a) \subset \sigma_m(\boldsymbol{x}_a) \tag{19}$$

for all $\boldsymbol{x}_a \in \mathfrak{c}(A)$. Relation (19) implies

$$\sigma_m(\boldsymbol{x}_a) = [h^*(h^{-1})^*\sigma_m](\boldsymbol{x}_a) \subset (h^*\sigma_m)(\boldsymbol{x}_a). \tag{20}$$

Relations (18) and (20) imply (17).

7.4. Definition. A spectroid $\tilde{\sigma}$ on the Banach algebra A is called *regular* if it is invariant under automorphisms of A, i.e. if $h^*\tilde{\sigma} = \tilde{\sigma}$ for each automorphism h of A.

The spectroids σ', σ'', σ_l, σ_r, σ and σ_m are regular (we do not know whether spatially defined spectroids σ_π, σ_δ, σ_{T} are regular; we only know that they are invariant under inner automorphism, or inner-regular). However, we have not added the requirement of regularity, or inner regularity, because we do not see any consequences of this property. Moreover, the minimal spectra, in general, fail to be regular.

7.5. Example. Let A be the bicylinder algebra of Example 6.7 and let σ_i be the minimal spectrum defined by the spectral set Δ_i, $i = 1, 2$. Let h be the automorphism of A given by $(hx)(z_1, z_2) = x(z_2, z_1)$. Then $h^*\sigma_1 = \sigma_2 \neq \sigma_1$.

7.6. Proposition. *For each $\tilde{\sigma} \in \mathrm{Sp}_1(A)$ there exists a smallest regular subspectrum $\tilde{\sigma}_1$ larger than $\tilde{\sigma}$.*

Proof. We define

$$\tilde{\sigma}_1(x_a) = \overline{\bigcup \{(h^*\tilde{\sigma})(\boldsymbol{x}_a) : h \in \mathrm{Aut}(A)\}},$$

for all $\boldsymbol{x}_a \in \mathfrak{c}(A)$, where $\mathrm{Aut}(A)$ is the set of all automorphisms of A. Then we proceed as in proofs of Theorem 6.3 and Proposition 7.1.

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