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Fat equicontinuous groups of homeomorphisms of linear topological spaces and their application to the problem of isometries in linear metric spaces

by

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Abstract. It is proved that if X is a locally convex linear topological space with the strong Krein–Milman property, then every equicontinuous group of homeomorphism of X which includes translations and "minus identity", consists of affine mappings only. Also, it is proved that for every metrizable t.v.s. X the following two statements are equivalent: (i) every equicontinuous group of homeomorphisms of X which includes translations and "minus identity" consists of affine mappings only, (ii) for every translation invariant metric d on X inducing the topology of X, every isometry from (X, d) onto another linear metric space is affine.

The question whether every surjective isometry between two linear metric spaces is affine has not been solved yet (the metrics are assumed to be translation invariant). The first result in this direction has been obtained by S. Mazur and S. Ulam in [6]. Namely, they proved that the question has a positive answer if the metrics are given by norms. Their proof is based on the argument that, if the metric is sufficiently regular, then one can define, in terms of the metric, the midpoint of a given interval. This argument has been used by several authors to prove the corresponding results for other classes of metrics (a survey of these results is to be found in [8]). In our opinion, it is worthwhile to change the approach and to ask whether the corresponding version of the Mazur–Ulam theorem holds for arbitrary metrics, provided, may be, that the spaces involved are sufficiently "nice". The first attempt along this line was made by Z. Charzyński [1], who proved that the theorem in question holds when both spaces are finite-dimensional. In [5], the author has generalized Charzyński's result to the case of metrizable locally convex Montel spaces.

In the present note, we study equicontinuous subgroups of groups of homeomorphisms of linear topological spaces, which include translations and "minus identity". We show that, if X is a locally convex linear space with the strong Krein–Milman property, then

(∗) *every equicontinuous group of homeomorphisms of X including translations and "minus identity" must consist of affine mappings only.*

Also, we prove that for a metrizable t.v.s., property (*) is equivalent to the fact that the corresponding version of the Mazur–Ulam theorem holds for arbitrary metrics inducing the topology of X. The last result shows that the equicontinuous groups of homeomorphisms are a useful tool in studing the problem of isometries.

For the sake of completeness, we include in this note several results and arguments previously presented in [5].

1. Preliminaries. X, Y will denote real topological Hausdorff vector spaces; explicit mention of the topology will usually be suppressed. $H(X)$ will denote the group (under composition) of all homeomorphisms of X onto itself. A subgroup of $H(X)$ is distinguished, given by the algebraic structure of X, $T(X)$, generated by translations and "minus identity":

$$T(X) = \{\varphi \in H(X)\colon \varphi(x) = \varepsilon x + y,\ \varepsilon = \pm 1,\ y \in X\}.$$

A subgroup of $H(X)$ is called *fat* if it includes $T(X)$. If $f \in H(X)$, $G(f)$ is the subgroup of $H(X)$ generated by f and $T(X)$. For any subgroup G of $H(X)$, we define

$$G_0 = \{f \in G\colon f(0) = 0\}.$$

In particular, $(G(f))_0$ will be written as $G_0(f)$.

A fat subgroup G of $H(X)$ is said to be *equicontinuous* iff its elements are equiuniformly continuous, i.e. for any neighbourhood U of the origin in X, there is a neighbourhood V of the origin X such that $f \in G$ and $x-y \in V$ imply $f(x)-f(y) \in U$ for all $f \in G$.

Remark 1. If G is a fat subgroup of $H(X)$, then

$$G_0 = \{g \in G\colon g(x) = f(x)-f(0) \text{ for some } f \in G \text{ and for all } x \in X\},$$

$$G = \{f \in H(X)\colon f(x) = g(x)+y \text{ for some } g \in G_0,\ y \in X \text{ and all } x \in X\}.$$

Remark 2. It can easily be deduced from the previous remark that a fat subgroup G is equicontinuous iff G_0 is equicontinuous at the origin.

It is clear that $T(X)$ is equicontinuous.

Aff (X) and Lin (X) will denote the group of all affine homeomorphisms and the group of all linear homeomorphisms of X, respectively.

If the space X is metrizable, we shall denote by Metr (X) the set of all metrics d inducing the topology of X which are translation invariant, i.e. such that $d(x, y) = d(x-y, 0)$, for all $x, y \in X$.

By $I(X, d)$ we shall denote the subgroup of $H(X)$ consisting of all isometries with respect to the metric $d \in$ Metr (X).

2. Isometries in linear metric spaces. The following fact is due to Z. Charzyński [1] (cf. [5]).

PROPOSITION 1. *For a metrizable topological vector space X, the following conditions are equivalent*:

(i) *for every $d \in$ Metr(X), every isometry from (X, d) onto another linear metric space is affine,*

(ii) $I(X, d) \subset$ Aff (X), *for every $d \in$ Metr(X).*

Charzyński's argument will be presented in Section 6 in the proof of Theorem 3, which generalizes Proposition 1.

The next proposition shows a connection between fat equicontinuous groups and groups of isometries.

PROPOSITION 2. *For a metrizable topological vector space X, the following conditions are equivalent*:

(i) $I(X, d) \subset$ Aff (X), *for every $d \in$ Metr(X),*

(ii) *every equicontinuous fat subgroup of $H(X)$ is contained in* Aff (X).

Proof. (i) $\Rightarrow$ (ii). Let G be an arbitrary fat equicontinuous subgroup of $H(X)$, and let h be an arbitrary bounded metric in Metr(X). Define $d(x, y) = \sup\{h(g(x), g(y))\colon g \in G\}$. Obviously, d is a metric and

(1) $$d(x, y) = d(f(x), f(y)) \quad \text{for every } x, y \in X \text{ and } f \in G.$$

We shall prove that $d \in$ Metr (X). Since all translations belong to G, we deduce, by (1), that $d(x, y) = d(x-y, 0)$ for all $x, y \in X$. Now, observe that

(2) $$d(x, y) \geqslant h(x, y) \quad \text{for all } x, y \in X.$$

On the other hand, take $\varepsilon > 0$. Since G is equicontinuous and $h \in$ Metr (X), there is $\eta > 0$ such that $h(x, y) < \eta$ implies $h(g(x), g(y)) < \varepsilon$ for every $g \in G$. But this means that, for every $\varepsilon > 0$, there is an $\eta > 0$ such that $h(x, y) < \eta$ implies $d(x, y) < \varepsilon$. This, with (2), shows that d induces the same topology as h. Thus, $d \in$ Metr (X).

Finally, by (1), we have $G \subset I(X, d)$. Hence, by (i), $G \in$ Aff (X).

(ii) $\Rightarrow$ (i). It suffices to observe that, for every metric $d \in$ Metr (X), the group $I(X, d)$ is a fat equicontinuous subgroup of $H(X)$.

3. Affine groups property. In view of the previous section, one sees that investigations of fat equicontinuous groups of homeomorphisms may be useful in studying the question whether every surjective isometry between two linear metric spaces must be affine. In what follows, we shall say that a linear topological space X has the *affine groups property* if and only if every fat equicontinuous subgroup of $H(X)$ is contained in Aff (X). In the theorem below, we give several necessary and sufficient conditions for a space X to have the affine groups property.

THEOREM 1. *Let X be a linear topological space. Then the following conditions are equivalent*:

(i) *X has the affine groups property,*

(ii) *for every fat equicontinuous subgroup G of $H(X)$, G_0 is contained in* Lin (X),

(iii) *for every equicontinuous fat subgroup G of $H(X)$, we have* $g(x) = -g(-x)$, *for every* $g \in G_0$ *and* $x \in X$,

(iv) *for every* $f \in H(X)$ *such that* $G(f)$ *is equicontinuous,* $G(f) \subset \mathrm{Aff}(X)$,

(v) *for every* $f \in H(X)$ *such that* $G(f)$ *is equicontinuous,* $G_0(f) \subset \mathrm{Lin}(X)$,

(vi) *for every* $f \in H(X)$ *such that* $G(f)$ *is equicontinuous, the mapping* $g(x) = f(x) - f(0)$, for $x \in X$, *is linear and*

$$G_0(f) = G_0(g) = \{\varepsilon g^n\colon \varepsilon = \pm 1,\ n = 0, \pm 1, \pm 2, \ldots\},$$

(vii) *for every* $f \in H(X)$ *such that* $G(f)$ *is equicontinuous,* $G_0(f)$ *is abelian,*

(viii) *for every* $f \in H(X)$ *such that* $G(f)$ *is equicontinuous, we have* $g(x) = -g(-x)$ *for every* $g \in G_0(f)$ *and* $x \in X$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) are trivial.

(iii) $\Rightarrow$ (i). (iii) means that $g\big(\frac{1}{2}(x+(-x))\big) = \frac{1}{2}\big(g(x)+g(-x)\big)$, for every $g \in G_0$ and every $x \in X$. Using Remark 1, one can easily show that this implies that $f\big(\frac{1}{2}(x+y)\big) = \frac{1}{2}\big(f(x)+f(y)\big)$ for every $f \in G$ and every $x, y \in X$. Therefore, G consists of affine mappins only.

The implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) are trivial.

(v) $\Rightarrow$ (vi). We have $g \in G_0(f)$ and therefore the mapping g is linear. Obviously, $G(f) = G(g)$. Hence $G_0(f) = G_0(g)$. It is easy to check, by using the linearity of g, that $G(g)$ consists of all mappings of the form

$$h(x) = \varepsilon g^n(x) + y \quad \text{for} \quad x \in X,$$

where $\varepsilon = \pm 1$, $n = 0, \pm 1, \pm 2, \ldots$ and $y \in X$. Since $G_0(f) = G_0(g)$, (vi) follows from the definition of $G_0(g)$.

(vi) $\Rightarrow$ (vii). Obvious.

(vii) $\Rightarrow$ (viii). $-\mathrm{Id} \in G_0(f)$ and therefore g commutes with it.

(viii) $\Rightarrow$ (iii). Let G be an arbitrary fat subgroup of $H(X)$. Fix any $g \in G$. Trivially, $G(g) \subset G$. Hence $G(g)$ is equicontinuous and $G_0(g) \subset G_0$. By (viii), we infer that $g(x) = -g(-x)$ for every $x \in X$.

4. Invariant subsets. If G is a subgroup of $H(X)$ and A a subset of X, we write GA for $\bigcup_{g \in G} g(A)$, which is the smallest G-invariant subset of X including A. We define

$$\mathrm{Inv}(G) = \{A \subset X\colon A = GA\},$$

the class of all G-invariant subsets of X. The lemma below presents the properties of G-invariant subsets needed in the sequel.

Lemma 1. *Let X be a linear topological space and let G be a fat subgroup of $H(X)$. Then*

(i) *for every* $A \in \mathrm{Inv}(G_0)$, $x \in X$ *and* $g \in G_0$, *we have*

$$g(x+A) = g(x) + A = g(x) + g(A),$$

(ii) *every G_0-invariant subset is symmetric,*

(iii) *if* $A_t \in \mathrm{Inv}(G_0)$, *for* $t \in T$, *where T is an arbitrary set of indices, then* $\bigcap_{t \in T} A_t \in \mathrm{Inv}(G_0)$,

(iv) *if A is G_0-invariant, so is its closure,*

(v) *if* $A_1, A_2, \ldots, A_n \in \mathrm{Inv}(G_0)$, *then* $A_1 + A_2 + \ldots + A_n \in \mathrm{Inv}(G_0)$,

(vi) *if G is equicontinuous, then the topology of X admits a base of G_0-invariant neighbourhoods of the origin.*

Proof. (i). Fix $x \in X$, $A \in \mathrm{Inv}(G_0)$, $g \in G_0$. The statement follows directly from the definition of $\mathrm{Inv}(G_0)$ applied to the mapping $f \in G_0$, defined by

$$f(u) = g(u+x) - g(x) \quad \text{for} \quad u \in X.$$

(ii), (iii), (iv) are obvious.

(v). It suffices to show that $A + B \in \mathrm{Inv}(G_0)$ provided that $A, B \in \mathrm{Inv}(G_0)$. To this end fix $A, B \in \mathrm{Inv}(G_0)$. By (i), we have

$$g(A+B) = \bigcup_{x \in A} g(x+B) = \bigcup_{x \in A} g(x) + B = g(A) + B = A + B,$$

for every $g \in G_0$.

(vi). Let $\mathscr{U}$ be an arbitrary base of neighbourhoods of the origin in X. It can easily be checked (in the same way as in the proof of Proposition 2) that $\{G_0 U\colon U \in \mathscr{U}\}$ forms a G_0-invariant base of neighbourhoods of the origin equivalent to $\mathscr{U}$.

Lemma 2. *Let X be a locally convex linear topological space and let G be an equicontinuous fat subgroup of $H(X)$. Then, for every* $A \subset X$, we *infer that $G_0 A$ is bounded if and only if A is bounded.*

Proof. Assume that A is a bounded subset of X. Let $\mathscr{U}$ be a base of G_0-invariant neighbourhoods of the origin in X. For every $U \in \mathscr{U}$, let U' be a convex neighbourhood of the origin contained in U. For any positive integer k and for every $V \subset X$, denote

$$k \oplus V = V + \ldots + V, \text{ the sum of } k \text{ copies of } V.$$

Since A is bounded, for every $U \in \mathscr{U}$ there is a positive integer $k(U)$ such that $A \subset k(U)U'$. Since U' is convex and $U' \subset U$, we have

$$A \subset k(U)U' = k(U) \oplus U' \subset k(U) \oplus U. \tag{3}$$

By (3) and by Lemma 1 (v), each set $k(U) \oplus U$ is G_0-invariant and contains A. By Lemma 1 (iii), the set $B = \bigcap_{U \in \mathscr{U}} k(U) \oplus U$ is G_0-invariant.

Since $A \subset B$, we have $G_0 A \subset B$. Given any convex neighbourhood V of the origin, take $U \in \mathscr{U}$ with $U \subset V$. We have

$$G_0 A \subset B \subset k(U) \oplus U \subset k(U) \oplus V = k(U) V.$$

Thus $G_0 A$ is bounded.

The reverse implication of the lemma is an obvious consequence of the fact that $A \subset G_0 A$.

QUESTION. Can the assumption that X is locally convex be ommited in the lemma above?

5. Extreme points and spaces with the strong Krein–Milman property. Main result. We recall that, if A is a subset of a linear space X, then $x_0 \in A$ is said to be an *extreme point* of A iff x_0 does not belong to the convex hull of $A \setminus \{x_0\}$. To prove the main theorem of this note, we shall need the following

LEMMA 3. *Let X be a linear topological space and let G be a fat subgroup of $H(X)$. Assume that $A \subset X$ is a G_0-invariant subset of X and x_0 is an extreme point of A. Then, for every $g \in G_0$, we have*

$$g(x_0) = -g(-x_0).$$

Proof. By Lemma 1 (ii), A is symmetric and $-x_0 \in A$, therefore $-x_0$ is also an extreme point of A. Observe that the set $B = (x_0 + A) \cap \cap (-x_0 + A)$ is symmetric. Now, if $0 \neq x \in B$, then $0 = \frac{1}{2}(x + (-x))$ and 0 is not an extreme point of $-x_0 + A$. This would imply that x_0 is not an extreme point of A—a contradiction. Since $0 \in B$, we obtain $B = \{0\}$.

Fix an arbitrary $g \in G_0$. By Lemma 1 (i), we have

$$\{0\} = g(\{0\}) = g\big((x_0 + A) \cap (-x_0 + A)\big) = \big(g(x_0) + g(A)\big) \cap \big(g(-x_0) + g(A)\big).$$

Since $g(x_0) + g(-x_0) = \big(g(x_0) + g(A)\big) \cap \big(g(-x_0) + g(A)\big)$, we deduce that $0 = g(x_0) + g(-x_0)$. Thus $g(x_0) = -g(-x_0)$.

Linear topological spaces with the property that every closed bounded subset possesses at least one extreme point are said to have the *strong Krein–Milman property*. In such a case, we shall write for short: X has the SKMP.

MAIN THEOREM. *Let X be a locally convex linear topological space with the SKMP. Then X has the affine groups property.*

Proof. Fix $f \in H(X)$ such that $G(f)$ is equicontinuous and set

$$B = \{x \in X \colon g(x) = -g(-x) \text{ for } g \in G_0(f)\}.$$

Observe that for every $y \in B$ and every $h \in G_0(f)$, setting $y' = h(y)$, we have

$$g(y') = g\big(h(y)\big) = (gh)(y) = -(gh)(-y) = -g\big(h(-y)\big)$$
$$= -g\big(-h(y)\big) = -g(-y'),$$

for every $g \in G_0(f)$. This means that B is $G_0(f)$-invariant.

By Theorem 1 (viii), it suffices to show that $B = X$. Assume the contrary, i.e. that $B \neq X$, and fix $x \in X \setminus B$. It is easy to see that B is closed. Hence, by Lemma 1 (vi), there exists a $G_0(f)$-invariant neighbourhood of the origin, say V, such that $x + V$ is disjoint with B. Thus $x \notin B + V$. Note that, by Lemma 1 (v), $g(x) \notin g(B + V) = B + V$, for every $g \in G_0(f)$ This means that the set $A = \{g(x) \colon g \in G_0(f)\}$ is disjoint with $B + V$. Therefore, the closure of A is disjoint with B. On the other hand, A is the smallest $G_0(f)$-invariant subset containing x. By Lemma 2, A is bounded. Hence the closure of A contains at least one extreme point, say x_0. By Lemma 1 (iv), the closure of A is $G_0(f)$-invariant and by Lemma 3 we obtain $x_0 \in B$—a contradiction.

The theorem above and the results presented in Section 2, together can be summarized in

COROLLARY. *If X is a metrizable locally convex linear space with the* SKMP *and d is a metric on X, then every isometry from (X, d) onto another linear metric space is affine.*

QUESTION. *Does the theorem above remain valid without the assumption that X is locally convex?*

QUESTION. *Does every linear topological space possess the affine groups property?* It would be interesting even to know whether the answer is affirmative for locally convex or metrizable spaces.

Remark 3. In view of the Main Theorem it is worthwhile to mention what spaces possess the SKMP. It is known ([7], [4]) that for Banach spaces the SKMP is equivalent to the Radon–Nikodym property. This implies that every reflexive and every separable conjugate Banach space has the SKMP (cf. [3]). Also, it can easily be proved that every locally convex Montel space has the SKMP.

LEMMA 4. *Let X be a finite-dimensional space and let $T \in H(X)$ be a linear mapping such that $G(T)$ is equicontinuous. Then each $x \in X$ is an extreme point of its $G_0(T)$-orbit (i.e. of the set $G_0(T)\{x\} = \{g(x) \colon g \in \in G_0(T)\}$).*

Proof. Without any loss of generality we may assume that X is an n-dimensional Euclidian space. Put $G = G(T)$. Observe that

$$G_0\{x\} = \{\varepsilon T^n(x) \colon \varepsilon = \pm 1,\ n = 0, \pm 1, \pm 2, \ldots\}, \tag{4}$$

for each $x \in X$. Take any bounded convex neighbourhood U of the origin in X. Let V be the closed convex hull of $G_0 U$. Using the linearity of T, it is easy to check that V is a bounded G_0-invariant convex symmetric body in X. Therefore, there is a unique symmetric ellipsoid $E \in X$ with the minimal volume, among all the ellipsoids in X containing V [9]. Since T leaves V invariant, we infer that T does not change the volume. On the other hand, since $T(V) = V$, we deduce that $T(E) = E$. Hence T is an isometry with respect to the Euclidean norm $\|\ \|_E$ induced on X by E. This and (4) imply that $G_0\{x\} \subset \{y \in X\colon \|y\|_E = \|x\|_E\}$ for each $x \in X$. The lemma follows from the fact that each point on the boundary of E is an extreme point of E.

THEOREM 2. *Let X be a linear topological space. Then the following conditions are equivalent*:

(i) *X has the affine groups property,*

(ii) *for every $f \in H(X)$ such that $G(f)$ is equicontinuous and for every $x \in X$, x is an extreme point of its $G_0(f)$-orbit (i.e. of the set $G_0(f)\{x\}$).*

Proof. (i) $\Rightarrow$ (ii). Take $f \in H(X)$ such that $G(f)$ is equicontinuous and fix $x \in X$. By (i), f is affine. Without any loss of generality we may assume that f is linear. Put $G = G(f)$. By Lemma 1 (vi), we have $G_0 = \{\varepsilon f^n\colon \varepsilon = \pm 1,\ n = 0, \pm 1, \pm 2, \ldots\}$. Therefore $G_0\{x\} = \{\varepsilon f^n(x)\colon \varepsilon = \pm 1,\ n = 0, \pm 1, \pm 2, \ldots\}$. Let $z_n = f^n(x)$ for $n = 0, \pm 1, \pm 2, \ldots$ Assume that x is not an extreme point of $G_0\{x\}$. Then

$$x = z_0 = \sum_{i=1}^{m} \varepsilon_i t_{n_i} z_{n_i}, \tag{5}$$

where $\varepsilon_i = \pm 1$, $t_{n_i} > 0$ and $\sum_{i=1}^{m} t_{n_i} = 1$. Put $k = \min\{0, n_1, n_2, \ldots, n_m\}$ and $l = \max\{0, n_1, n_2, \ldots, n_m\}$ and define $Y = \operatorname{span}\{z_i\colon k \leqslant i \leqslant l\}$. It follows from (5) that z_l can be written in the form

$$z_l = \sum_{i=k}^{l-1} a_i z_i, \quad \text{where} \quad a_i \in \mathbf{R}.$$

Therefore

$$f(z_l) = \sum_{i=k}^{l-1} a_i f(z_i) = \sum_{i=k}^{l-1} a_i z_{i+1} \in Y. \tag{6}$$

Since $f(z_i) = z_{i+1}$ and because of (6), we deduce that $f(Y) = Y$. Thus $G_0\{x\} \subset Y$. Let $T = f|Y$. Then Y is finite dimensional. Hence, by Lemma 4, we conclude that x is an extreme point of $G_0(T)\{x\}$. Since $G_0\{x\} = G_0(T)\{x\}$, we infer that x is an extreme point of $G_0\{x\}$ — a contradiction.

The reverse implication is an easy consequence of Lemma 3 and Theorem 1 (viii).

6. Pseudoisometries and unimorphies. A mapping f from a linear topological space X onto another linear topological space Y will be called a *pseudoisometry* iff there exist such systems $\{d_t\colon t \in T\}$ and $\{h_t\colon t \in T\}$ of translation-invariant pseudometrics inducing the topology of X and Y, respectively, that, for every $t \in T$,

$$h_t\big(f(x), f(y)\big) = d_t(x, y) \quad \text{for every } x, y \in X.$$

If for $t \in T$, d_t and h_t are given by pseudonorms, then f is said to be a *unimorphy*. It is known (cf. [2], p. 111) that every unimorphy is affine.

THEOREM 3. *Let X be a linear topological space. Then the following conditions are equivalent*:

(i) *X has the affine groups property,*

(ii) *every pseudoisometry from X onto another linear topological space is affine.*

Proof. (i) $\Rightarrow$ (ii). Fix any pseudoisometry f from X onto another linear topological space Y and let $\{d_t\colon t \in T\}$, $\{h_t\colon t \in T\}$ be such systems of translation-invariant pseudometrics inducing the topologies of X and Y, respectively, that, for every $t \in T$,

$$h_t\big(f(x), f(y)\big) = d_t(x, y) \quad \text{for every } x, y \in X.$$

Without any loss of generality we may assume that $f(0) = 0$.

Let G be the subgroup of $H(X)$ consisting of all $g \in H(X)$ such that, for every $t \in T$,

$$d_t\big(g(x), g(y)\big) = d_t(x, y) \quad \text{for every } x, y \in X.$$

Obviously, G is a fat equicontinuous group, and therefore consists of affine mappings. For every $y \in X$, define

$$u_y(x) = f^{-1}\big(f(x) + f(y)\big) - y \quad \text{for} \quad x \in X.$$

It is easy to see that, for every $y \in X$, the mapping u_y is a pseudoisometry of X onto itself with respect to the pseudometrics $\{d_t\colon t \in T\}$, and $u_y(0) = 0$. Thus $u_y \in G$ for every $y \in X$. Hence u_y is linear for every $y \in X$. Set

$$v(x, y) = u_y(x) - x \quad \text{for every } x, y \in X.$$

We have

$$v(x, y) = f^{-1}\big(f(x) + f(y)\big) - (x + y) = v(y, x), \tag{7}$$

for every $x, y \in X$. Obviously, $v(x, y)$ is linear at x and, by (7), we infer that $v(x, y)$ is linear at y as well. We shall prove that $v(x, y) = 0$ for every $x, y \in X$. To this end, take an arbitrary real a. Then we have, for every

$t \in T$,

$$\begin{aligned} d_t(ax, 0) &= d_t\big(u_{a^{-1}y}(ax), 0\big) \\ &\geqslant d_t\big(u_{a^{-1}y}(ax) - ax, 0\big) - d_t(ax, 0) = d_t\big(v(ax, a^{-1}y), 0\big) - d_t(ax, 0) \\ &= d_t\big(v(x, y), 0\big) - d_t(ax, 0). \end{aligned}$$

Finally, we infer that, for every $t \in T$,

$$2d_t(ax, 0) \geqslant d_t\big(v(x, y), 0\big) \quad \text{for every } x, y \in X.$$

Letting a tend to zero, we infer that, for every $t \in T$,

$$d_t\big(v(x, y), 0\big) = 0 \quad \text{for every } x, y \in X.$$

Thus $v(x, y) = 0$ for every $x, y \in X$. By (7), this means that

$$f^{-1}\big(f(x) + f(y)\big) = x + y \quad \text{for every } x, y \in X.$$

Hence $f(x+y) = f(x) + f(y)$ for every $x, y \in X$. Since f is continuous, we deduce that f is linear.

(ii) $\Rightarrow$ (i). Let G be an arbitrary fat equicontinuous subgroup of $H(X)$ and let $\{d_t: t \in T\}$ be an arbitrary system of translation-invariant pseudometrics inducing the topology of X. Using a similar argument to that used in the proof od Proposition 2, one can show that G consists of pseudoisometries with respect to the system of pseudometrics $\{d'_t: t \in T\}$, where

$$d'_t(x, y) = \sup\big\{\min\{d_t(g(x), g(y)), 1\}: g \in G\big\}$$

for $t \in T$. Thus G consists of affine mappings.

Remark 4. If X is a locally convex space with the affine groups property, then every pseudoisometry f from X onto another space is a unimorphy. Indeed, f is an affine homeomorphism and if $\{p_t: t \in T\}$ is any system of pseudonorms inducing the topology of X, then f is a unimorphy with respect to $\{p_t: t \in T\}$ and $\{q_t: t \in T\}$, where

$$q_t(y) = p_t\big(f^{-1}(y) - f^{-1}(0)\big) \quad \text{for every } y \in Y \text{ and } t \in T.$$

A mapping $f \in H(X)$ is said to be a *pseudoisometry of X onto itself* iff there is such a system of translation-invariant pseudometrics $\{d_t: t \in T\}$ inducing the topology of X that, for every $t \in T$,

$$d_t\big(f(x), f(y)\big) = d_t(x, y) \quad \text{for every } x, y \in X.$$

Remark 5. If X is a locally convex space with the affine groups property, then, for every pseudoisometry f of X onto itself, there exists such a system of pseudonorms $\{p_s: s \in S\}$ inducing the topology of X that for every $s \in S$,

$$p_s\big(f(x) - f(y)\big) = p_s(x - y) \quad \text{for} \quad x, y \in X.$$

To show this, it is enough to observe that if G is a group of all pseudoisometries of X onto itself with respect to pseudometrics $\{d_t: t \in T\}$ and $\{q_s: s \in S\}$ is an arbitrary system of pseudonorms inducing the topology of X, then it suffices to define, for $s \in S$,

$$p_s(x) = \sup\big\{q_s\big(g(x) - g(0)\big): g \in G\big\} \quad \text{for} \quad x \in X.$$

(p_s is a pseudonorm because of the fact that G consists of affine mappings.)

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