STUDIA MATHEMATICA, T. LXIV. (1979)

Norm convergent expansion for L_Φ-valued Gaussian random elements

by

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Abstract. This paper is a continuation of [1]. We prove that every Gaussian random element with values in an Orlicz space L_Φ can be expanded into a norm convergent (a.s.) orthogonal series, where Φ is a concave Young function.

In [1] we have investigated Gaussian random elements with values in L_p spaces, $0 \leqslant p < \infty$. This paper is a continuation of [1]. We prove here that the Gaussian random element with values in an Orlicz space L_Φ can be expanded into a norm convergent (a.s.) orthogonal series (under the assumption that Φ is a concave Young function). This result has been proved by Jain and Kallianpur for Gaussian random elements taking values in an arbitrary Banach space [6]. In the proof of our theorem we use a theorem on the convergence of L_Φ-valued martingales (Theorem 1). This theorem is analogous to the theorem on the mean-convergence of Banach-valued martingales, due to Chaterji [2]. In our situation, however, there are some difficulties connected with the construction of conditional expectation because the classical notion of the Bochner integral cannot be used [11]. In recent years the integration theory for functions with values in non-locally convex linear spaces has been developed by several authors [3], [12], [14]. However, their constructions are restricted to the case of p-normed spaces and therefore are not suitable for our purposes. Our construction of integral and conditional expectation is suggested by the notions of an integrable space and an integral, recently introduced and investigated by Wilhelm for functions with values in a normed group [16], [17], [18]. This construction is based on the correspondence between random elements with values in an L_Φ space and measurable random processes with paths in an L_Φ space, given by Theorem 1.1 in [1] (see also Remark 4.4 in [1]).

Preliminaries. Let (T, F, m) be a measure space. Let Φ be an arbitrary Young function, i.e. a continuous non-decreasing function defined for $u \geqslant 0$ and such that $\Phi(u) = 0$ if and only if $u = 0$. Assume additionally that Φ is subadditive. Given such a function Φ, put

$$\|x\|_\Phi = \int_T \Phi(|x(t)|)\, m(dt)$$

for every F-measurable real-valued function x on T. Let L_Φ be the set of all real-valued F-measurable functions x on T such that $\|x\|_\Phi < \infty$. Clearly, L_Φ is a linear space under usual addition and scalar multiplication and $\|\ \|_\Phi$ is a (usually non-homogeneous) seminorm on L_Φ. Moreover, $(L_\Phi, \|\ \|_\Phi)$ is a complete linear metric space (we identify functions which are equal a.e.). $(L_\Phi, \|\ \|_\Phi)$ is an Orlicz space (usually on Φ less restrictive conditions are imposed) [8]–[10].

Now, let us assume that (T, F, m) is a σ-finite separable measure space. Then L_Φ is separable ([13], p. 30). Let X be a Gaussian random element with values in L_Φ. Then, by Theorem 1.1 in [1], it follows that there exists a measurable Gaussian random process ξ such that $\tilde{\xi} = X$ a.s. $[P]$, where $\tilde{\xi}(\omega)$ denotes the equivalence class of F-measurable functions corresponding to $\xi(\omega, \cdot)$. In particular, μ_ξ, the measure induced on L_Φ by ξ, coincides with the distribution of X. On the other hand, every measurable Gaussian random process with sample paths in L_Φ induces, in a natural way, a Gaussian random element with values in L_Φ (Theorem 4.1 in [1]).

Gaussian random elements with values in L_Φ spaces. Now, we generalize to separable Orlicz spaces some results proved in [1] for L_p spaces. First of all we prove a lemma which will be useful in the sequel.

LEMMA. *There exist real constants α, β, $0 < \alpha, \beta < 1$, such that for every symmetric Gaussian random variable η we have:*

(i) $$\Phi(E|\eta|) \leqslant \frac{1}{1-\alpha} E\Phi(|\eta|),$$

(ii) $$\Phi\big((E\eta^2)^{1/2}\big) \leqslant \frac{1}{1-\beta} E\Phi(|\eta|).$$

If we additionally assume that Φ is concave, then for every random variable ξ we have

(iii) $$E\Phi(|\xi|) \leqslant \Phi(E|\xi|).$$

Proof. Let $c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-x^2/2} dx$. Put

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{\{|y| \leqslant c\}} e^{-x^2/2}\, dx.$$

Let σ^2 be the variance of η. Then we have

$$\begin{aligned}(1-\alpha)\Phi(E|\eta|) &= (1-\alpha)\Phi\Big(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}|u|e^{-u^2/2\sigma^2}du\Big) \\ &= (1-\alpha)\Phi\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sigma|y|e^{-y^2/2}dy\Big) = (1-\alpha)\Phi(\sigma c) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\{|y|>c\}}\Phi(\sigma c)e^{-y^2/2}dy \leqslant \frac{1}{\sqrt{2\pi}}\int \Phi(\sigma|y|)e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\Phi(|u|)e^{-u^2/2\sigma^2}du = E\Phi(|\eta|),\end{aligned}$$

which proves (i). We obtain (ii) in the same way, taking into account that

$$(E\eta^2)^{1/2} = AE|\eta|,$$

where A is a positive constant independent of σ (see [15], Lemma 2.2.3).

Finally, inequality (iii) is a version of Jensen's inequality.

As it has been remarked in the Preliminaries, instead of L_Φ-valued random elements we can consider measurable Gaussian random processes with paths in L_Φ. Given a measurable Gaussian random process $\xi(\omega, t)$, write

$$\theta(t) = E\xi(t), \qquad m_\Phi(t) = E\Phi(|\xi(t)|),$$
$$M_\Phi(t) = E\Phi(|\xi(t)-\theta(t)|), \qquad K(s,t) = E\big(\xi(t)-\theta(t)\big)\big(\xi(s)-\theta(s)\big).$$

PROPOSITION 1. *The following statements are equivalent:*

(i) $\xi(\omega, \cdot) \in L_\Phi$ *a.s.* $[P]$,

(ii) $m_\Phi \in L_1$,

(iii) $\theta \in L_\Phi$ *and* $M_\Phi \in L_1$.

By these conditions it follows that

(iv) $\theta \in L_\Phi$ *and* $K^{1/2} \in L_\Phi$.

Moreover, if Φ is concave, then (iv) *implies* (i) (*and all these conditions are equivalent*).

Proof. We have

$$\begin{aligned}\int m_\Phi(t)m(dt) &= \int E\Phi(|\xi(t)|)m(dt) \\ &\leqslant \int E\Phi(|\xi(t)-\theta(t)|)m(dt) + \int E\Phi(|\theta(t)|)m(dt) \\ &= \int M_\Phi(t)m(dt) + \int \Phi(|\theta(t)|)m(dt).\end{aligned}$$

Thus, (iii) implies (ii). Since

$$\int m_\Phi dm = E\Big\{\int \Phi(|\xi(t)|)m(dt)\Big\} = E\|\tilde{\xi}\|_\Phi,$$

we infer that (ii) implies (i).

The proof that (i) implies (iii) is almost the same as the corresponding part of the proof of Theorem 4.2 in [1] and is omitted.

Next, let us observe that by inequality (ii) in the lemma we have

$$\Phi\left(K(t,t)^{1/2}\right) \leqslant \frac{1}{1-\beta} M_\Phi(t),$$

and so (i) always implies (iv). On the other hand, if we assume that Φ is concave, then, by inequality (iii) in the lemma, we obtain

$$M_\Phi(t) \leqslant \Phi\left(\frac{1}{A} K(t,t)^{1/2}\right).$$

Thus, if $K(t,t)^{1/2} \in L_\Phi$, then $M_\Phi \in L_1$, which proves that (iv) implies (iii), and completes the proof.

COROLLARY 1. *Let X be an L_Φ-valued Gaussian random element. Then*

$$X = \theta + Y$$

where $\theta \in L_\Phi$ and Y is a symmetric Gaussian random element with values in L_Φ.

Proof. Let ξ be a measurable Gaussian random process such that $\tilde{\xi} = X$ a.s. $[P]$. Then, by the above proposition $E\xi(t) = \theta(t) \in L_\Phi$ and $\eta(t) = \xi(t) - \theta(t)$ is a symmetric Gaussian random process with paths in L_Φ. Putting $Y = \tilde{\eta}$, we obtain the desired conclusion.

COROLLARY 2. *Let X be a Gaussian random element with values in L_Φ. The support of X is the algebraic sum of an element of L_Φ and a closed linear subspace.*

The proof follows immediately from Theorem 2.1 in [1] and Corollary 1.

COROLLARY 3. *Let $\{\xi(t): t \in T\}$ be a measurable Guassian process and let f be a real F-measurable function defined on T. Then either $f\xi(\omega,\cdot) \in L_\Phi$ a.s. $[P]$ — if $f\theta \in L_\Phi$ and $E\Phi(|f\xi - f\theta|) \in L_1$, or $f\xi(\omega,\cdot) \notin L_\Phi$ a.s. $[P]$ — if at least one of these conditions is not satisfied (when Φ is concave, we have either $f\xi(\omega,\cdot) \in L_\Phi$ a.s. $[P]$ — if $f\theta \in L_\Phi$ and $fK^{1/2} \in L_\Phi$, or $f\xi(\omega,\cdot) \notin L_\Phi$ a.s. $[P]$ — if at least one of these conditions is not satisfied).*

Proof ot this corollary follows immediately from Proposition 1 and the 0-1 law (Theorem 2.1 [1]).

Remark. We can also observe, in the same way as in [1], Remark 4.3, that if μ is a Gaussian measure on L_Φ and G is a complete measurable subgroup of L_Φ, then $\mu(G) = 0$ or $\mu(G) = 1$.

L_Φ-valued martingales. Let (Ω, Σ, P) be a probability space and let (T, F, m) be a σ-finite measure space. Let Φ be a Young function. Assume that Φ is concave (then it is, of course, subadditive).

Let $\mathscr{L}_\Phi$ be the set of all real-valued $\Sigma \times F$-measurable functions f defined on $\Omega \times T$ and such that $[f]_\Phi < \infty$, where

$$[f]_\Phi = \int_T \Phi(E|f|)\,dm(t).$$

It is easy to observe that $(\mathscr{L}_\Phi, [\ ]_\Phi)$ is a complete metric linear space (if we identify functions which are equal a.e.).

By $\mathscr{L}$ we shall denote the linear space of all functions of the form $\sum_{i=1}^{n} x_i(t)\chi_{A_i}(\omega)$, where $x_i \in L_\Phi$ and $A_i \in \Sigma$, $i = 1, \ldots, n$.

PROPOSITION 2. *$\mathscr{L}$ is a dense linear subspace of $(\mathscr{L}_\Phi, [\ ]_\Phi)$.*

The proof of this proposition depends on the standard approximation arguments and is omitted.

Now, let Σ_0 be a sub-σ-algebra of Σ. If $f(\omega,t) = \sum_{i=1}^{n} x_i(t)\chi_{A_i}(\omega)$, we define

$$I(f) = \sum_{i=1}^{n} x_i P\{A_i|\Sigma_0\},$$

where $P\{\cdot|\Sigma_a\}$ denotes the conditional probability with respect to Σ_0. Observe that

$$E(|I(f)|) \leqslant E(I(|f|)) = E|f|$$

whenever $f \in \mathscr{L}$. Therefore we have

$$[I(f)]_\Phi \leqslant [f]_\Phi.$$

By the above inequality we see that I is a continuous linear mapping from $\mathscr{L}$ into $\mathscr{L}_\Phi$. Since $\mathscr{L}$ is dense in $\mathscr{L}_\Phi$, we can extend I onto $\mathscr{L}_\Phi$. This extension will be denoted by the same symbol and will be called the *conditional expectation operator.*

Now, we shall establish some elementary properties of I.

PROPOSITION 3. *Let Σ_0 be a sub-σ-algebra of Σ and let I be the conditional expectation operator with respect to Σ_0. The following statements are valid:*

1. *$I(f)$ is a $\Sigma_0 \times F$-measurable function, for every $f \in \mathscr{L}_\Phi$.*
2. *$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s. $[P \times m]$, for $f, g \in \mathscr{L}_\Phi$, $\alpha, \beta \in \mathbf{R}$.*
3. *$[I(f)]_\Phi \leqslant [f]_\Phi$, for $f \in \mathscr{L}_\Phi$.*
4. *$\int_A I(f)\,dP = \int_A f\,dP$ a.s. $[m]$, for $f \in \mathscr{L}_\Phi$ and $A \in \Sigma_0$.*
5. *Let Σ_i, $i = 0, 1$, be sub-σ-algebras of Σ. Let I_i denote the conditional expectation operator with respect to Σ_i, $i = 0, 1$. If $\Sigma_0 \subset \Sigma_1$, then*

$$I_0(I_1(f)) = I_1(I_0(f)) = I_0(f) \text{ a.s. } [P \times m]$$

for $f \in \mathscr{L}_\Phi$.

6. *$I(f) = f$ a.s. $[P \times m]$ if f is $\Sigma_0 \times F$-measurable.*

Proof. Property 1 follows immediately from the fact that the topology in $(\mathscr{L}_\Phi, [\ ]_\Phi)$ is stronger than the topology of convergence in the measure $P\times m$ (see the lemma, (iii)).

Properties 2,3,5,6 follow immediately from the construction of I. We prove property 4. We have

$$\Big|\int_A I(f)\,dP - \int_A I(f_n)\,dP\Big| \leqslant E|f-f_n|$$

and

$$\Big|\int_A f\,dP - \int_A f_n\,dP\Big| \leqslant E|f-f_n|.$$

Thus we have

$$\Big\|\int_A I(f)\,dP - \int_A I(f_n)\,dP\Big\|_\Phi \leqslant [f-f_n]_\Phi$$

and

$$\Big\|\int_A f\,dP - \int_A f_n\,dP\Big\|_\Phi \leqslant [f-f_n]_\Phi.$$

Now, let $f_n \in \mathscr{L}$ and let $[f_n-f]_\Phi \to 0$. Since 4 obviously holds for f_n, we obtain the required conclusion from the following inequality:

$$\Big\|\int_A I(f)\,dP - \int_A f\,dP\Big\|_\Phi \leqslant \Big\|\int_A I(f)\,dP - \int_A I(f_n)\,dP\Big\|_\Phi + \Big\|\int_A f\,dP - \int_A f_n\,dP\Big\|_\Phi$$
$$\leqslant 2[f-f_n]_\Phi.$$

PROPOSITION 4. *Let $f\in\mathscr{L}_\Phi$. Then $I(f) = E\{f|\Sigma_0\}$ a.s. $[P\times m]$, where $E\{f(t)|\Sigma_0\}$ denotes the usual conditional expectation of the random variable $f(t) = f(\cdot, t)$.*

Proof. By the standard approximation arguments it easily follows that $E\{f|\Sigma_0\}$ is $\Sigma_0\times F$-measurable. The required equality is obvious if $f\in\mathscr{L}$. Now, let $f\in\mathscr{L}_\Phi$ and let $f_n\in\mathscr{L}$ be such that $[f_n-f]_\Phi\to 0$. By the property of I we have $[I(f_n)-I(f)]_\Phi\to 0$. On the other hand, we have

$$|E(E\{f_n(\cdot,t)-f(\cdot,t)|\Sigma_0\})| \leqslant E|f_n(\cdot,t)-f(\cdot,t)|.$$

Hence

$$[E\{f_n|\Sigma_0\} - E\{f|\Sigma_0\}]_\Phi \leqslant [f_n-f]_\Phi.$$

Therefore

$$I(f) = E\{f|\Sigma_0\} \quad \text{in} \quad (\mathscr{L}_\Phi, [\ ]_\Phi),$$

which means that

$$I(f) = E\{f|\Sigma_0\} \text{ a.s. } [P\times m].$$

Remarks. (i) Let X be a random element with values in L_Φ and let ξ be a measurable stochastic process such that $\tilde{\xi} = X$ a.s. $[P]$. Assume that $\xi\in\mathscr{L}_\Phi$. Let $\Sigma_0 = \{\Sigma, 0\}$ and put

$$EX = \widetilde{I\xi}.$$

It is easy to verify that the expectation E is well defined and that $EX\in L_\Phi$. Moreover, if we indentify $(\mathscr{L}_\Phi, [\ ]_\Phi)$ with the family of all L_Φ-valued random elements X having the property that the measurable process ξ representing X belongs to $\mathscr{L}_\Phi$, then the mapping $X\to EX$ is a continuous linear operator from $(\mathscr{L}_\Phi, [\ ]_\Phi)$ into $(L_\Phi, \|\ \|_\Phi)$.

(ii) It is easy to observe that $[f]'_\Phi = E\|f\|_\Phi$ is not suitable for our purposes. If $\Phi(t) = t^p$, $0<p<1$, and m is the Lebesgue measure on $[0,1]$, then it is easy to find a sequence $f_n\in\mathscr{L}$ such that $E\|f_n\|_p\to 0$ but $\|I(f_n)\|_p\to\infty$. More generally it can be shown that if the metric linear space is not locally convex, then there exists a sequence f_n of simple functions, defined on the unit interval with the Lebesgue measure, uniformly tending to zero and such that $\|Ef_n\|\nrightarrow 0$ [11].

(iii) In the construction of I the condition that Φ is concave was used only in order to ensure that the $\mathscr{L}_\Phi$-topology should be stronger than the topology of convergence in the measure $P\times m$. Observe that this condition is satisfied when m is finite or when $\limsup_{t\to 0+} t/\Phi(t) < \infty$.

Now, let Σ_i be an increasing sequence of sub-σ-algebras of Σ. A sequence f_i of $\Sigma_i\times F$-measurable mappings defined on $\Omega\times T$ is called an *L_Φ-valued martingale* if $[f_i]_\Phi<\infty$ and $i\leqslant j \Rightarrow E\{f_j|\Sigma_i\} = f_i$.

THEOREM 1. *Let $\{f_n, \Sigma_n\ n\geqslant 1\}$ be an L_Φ-valued martingale such that*

$$f_n = E\{f|\Sigma_n\}$$

where $f\in\mathscr{L}_\Phi$. Then

$$\lim [f_\infty - f_n]_\Phi = 0,$$

where $f_\infty = E\{f|\Sigma_\infty\}$ and Σ_∞ is the σ-algebra generated by $\bigcup_{n\geqslant 1}\Sigma_n$.

This theorem has been proved by Chaterji [2] for Banach space-valued martingales. It is easy to observe that the arguments used there remain valid also in our situation.

Expansion of Gaussian random elements. Assume that (T, F, m) is a σ-finite separable measure space and that Φ is a concave Young function.

We prove the main result of this paper.

THEOREM 2. *Let X be a Gaussian random element with values in L_Φ. There exists a function $\theta\in L_\Phi$, a sequence (ξ_j) of independent, normally distributed real random variables with mean zero and variance 1 and a sequence (φ_n) of elements of L_Φ such that*

$$X = \theta + \sum_{j=1}^{\infty} \xi_j \psi_j,$$

where the series $\sum \xi_j \psi_j$ *converges a.s.* $[P]$ *in the norm of* L_Φ.

Proof. By Corollary 1 it follows that we can assume without loss of generality that $\theta = 0$.

Now, let ξ be a $\Sigma \times F$-measurable Gaussian random process such that $\tilde{\xi} = X$ a.s. $[P]$. We can assume that $\xi(\cdot, t) \in L_2(\Omega)$, for every $t \in T$. Arguing as in the proof of Theorem 1 in [6], we obtain the equality

$$\xi(\cdot, t) = \sum_{j=1}^{\infty} \xi_j(\cdot)\psi_j(t),$$

where ξ_j's have the required properties and, for every $t \in T$, the series converges in $L_2(\Omega)$ and hence a.e. $[P]$. By the proof of Lemma 1 in [5] we have $K(t, t) = \sum_{n=1}^{n} \psi_n^2(t)$. By Proposition 1, (iv), we have $K(t, t)^{1/2} \in L_\Phi$. Thus we have $\psi_n \in L_\Phi$. By the lemma, we obtain

$$[\xi]_\Phi \leqslant \frac{1}{1-a} E\|X\|_\Phi.$$

Since $E\|X\|_\Phi < \infty$ [4], we have $\xi \in \mathscr{L}_\Phi$. Let $\mathscr{I}_n = E\{\xi | \Sigma_n\}$, where Σ_n is the σ-field generated by $\{\xi_1, \xi_2, \ldots, \xi_n\}$. By Theorem 1 it follows that $[\mathscr{I}_n - \mathscr{I}_\infty]_\Phi \to 0$, where $\mathscr{I}_\infty = E\{\xi | \Sigma_\infty\}$. By the lemma, (iii), we obtain $E\|\tilde{\mathscr{I}}_n - \tilde{\mathscr{I}}_\infty\|_\Phi \to 0$. Thus, $\tilde{\mathscr{I}}_n \to \tilde{\mathscr{I}}_\infty$ in probability. By Proposition 4 we obtain $\mathscr{I}_n = \sum_{j=1}^{n} \xi_j \psi_j$ and $\mathscr{I}_\infty = \xi$, a.s. $[P \times m]$. Since ξ_j are independent, we obtain $\tilde{\mathscr{I}}_n \to \tilde{\xi}$ with probability 1 (see Theorem 1 in [7]), which completes the proof.

Added in proof. The author has recently learned of a paper of З. Г. Горгадзе, *О мерах в банаховых пространствах измеримых функций*, Труды Тбилисского университета, 166 (1976), pp. 43–50. Using an inequality stated in this paper in the lemma, p. 48, one can observe that the conditions (i)–(iv) in our Proposition 1 are equivalent for arbitrary separable L_Φ space, under assumption that Φ is a continuous nondecreasing function satisfying (Δ_2) condition and such that $\Phi \geqslant 0$ and $\Phi(t) = 0$ iff $t = 0$. Also all the remaining results hold for all separable L_Φ spaces, with Φ as above.

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Received June 7, 1976 (1168)