$\to Y_{2n}$; $Q_m^{2n+1}\colon Y_{2n+1}\to \tilde{T}_m(Y_{2n})$ for all $m\in N$. Another application of the above Lemma by replacing $T_m$ by $\tilde{T}_m^{-1}$, $E_m$ by $\tilde{T}_m(Y_{2n})$, $P_m^{2n}$ by $Q_m^{2n+1}$, $T_m(E_m)$ by $Y_{2n}$ and $Q_m^{2n}$ by $P$ for all $m$ yields $Y_{2n+2}\supset Y_{2n+1}$, isometric extensions $S_m$ of $\tilde{T}_m$, $S_m\colon F_m\to Y_{2n+1}$, where $F_m$ are subspaces of $Y_{2n+2}$ with $Y_{2n}\subset F_m$ and $S_m(F_m) = Y_{2n+1}$ for all $m\in N$. We obtain in addition contractive projections $P_m^{2n+2}\colon Y_{2n+2}\to F_m$; $Q\colon Y_{2n+2}\to Y_{2n+1} = S_m(F_m)$. We set $R_{2n+2} = R_{2n}\cdot P\cdot Q$.

Consider now a countable dense subset $\Gamma$ of the unit sphere of $Y_{2n+2}$ and define

$$\Omega = \{T\colon \langle x\rangle \to \langle y\rangle \mid T \text{ linear isometric; } x, y\in\Gamma; T(x) = y\},$$

where $\langle x\rangle$ denotes the linear span of $x$.

Then certainly $\Omega$ is countable. The theorem of Hahn–Banach provides us with contractive projections from $Y_{2n+2}$ onto $\langle x\rangle$ for all $x\in\Gamma$. Set $\Omega_{2n+2} = \Omega_{2n}\cup\Omega$ and continue the induction by defining $Y_{2n+3}$.

Finally, set $Z = \overline{\bigcup_{n\in N} Y_n}$ and define a contractive projection from $Z$ onto $X$ by the $R_{2n}$, $n\in N$. ■

### References


[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.
[2] A. M. Davie, *The approximation problem for Banach spaces*, Bull. London Math. Soc. 5 (1973), pp. 261–266.
[3] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. 130 (1973), pp. 309–317.
[4] F. Forelli, *The isometries of $H^p$*, Canad. J. Math. 18 (1964), pp. 721–728.
[5] V. I. Gurarij, *Space of universal disposition, isotopic spaces and the Mazur problem on rotation of Banach spaces*, Sibirsk Mat. Ž. 7 (1966), pp. 1002–1013.
[6] J. Lamperti, *On the isometries of some function spaces*, Pacific J. Math. 8 (1958), pp. 459–466.
[7] W. Lusky, *The Gurarij spaces are unique*, Arch. Math. 27 (1976), pp. 627–635.
[8] A. Pełczyński and H. P. Rosenthal, *Localization techniques in $L^p$-spaces*, Studia Math. 52 (1975), pp. 263–289.
[9] R. R. Phelps, *Lectures on Choquet's theorem*, Princeton 1966.


*Received February 1, 1977* (1256)

STUDIA MATHEMATICA, T. LXV. (1979)

# A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers

by

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*Dedicated to Ralph Boas on the occasion of his sixty-fifth birthday*

**Abstract.** Localized Bessel potential spaces $S(q,\gamma)$, $\gamma>0$, were recently introduced by Connett and Schwartz in connection with ultraspherical multipliers and characterized for integer $\gamma$ in terms of sequence spaces. Analogous results are obtained in this paper for all real $\gamma>1/q$, where $1<q<\infty$. These results are then used to derive best possible multiplier criteria of Marcinkiewicz type for Jacobi expansions by interpolating between end-point results due to Askey and to the authors and to derive analogous multiplier criteria for Hankel transforms.

**1. Introduction.** In [11] Connett and Schwartz showed that localized Bessel potential spaces $S(q,\gamma)$ are useful in the theory of ultraspherical multipliers. However, one disadvantage of these spaces is that it is hard to verify when a sequence is the restriction (to the positive integers) of an element in $S(q,\gamma)$. In case of $\gamma$ being a positive integer, Connett and Schwartz characterized $S(q,\gamma)$ by means of (finite) difference conditions upon the sequence. The main result of this paper, Theorem 1, extends this characterization to all $\gamma>1/q$ for $1<q<\infty$. We also give a neat description (Theorems 4 and 5) of the imbedding behavior of the wbv and WBV-spaces (defined below), which are important in multiplier theory. These results are then used to derive various multiplier criteria for Jacobi expansions (Theorem 6) and Hankel transforms (Theorem 7).

To define the localized Bessel potential spaces we first recall that the standard space of Bessel potentials $L^q_\gamma(\boldsymbol{R})$, $\gamma>0$, $1\leqslant q\leqslant\infty$, is defined by (see [20], p. 134)

$$L^q_\gamma = \{g\in L^q(0,\infty)\colon g = G_\gamma * h,\ \|g\|_{q,\gamma} = \|h\|_q<\infty\},$$

where the *Bessel kernel* $G_\gamma(x)$ is a function whose Fourier transform is given by

$$G_\gamma^{\wedge}(v) = \int_{-\infty}^{\infty} G_\gamma(x)e^{-ivx}dx = (1+|v|^2)^{-\gamma/2}.$$

* Supported in part by NSF Grant MCS 76-06635.

$L^q_\gamma$ is localized as follows: Let $\Phi(t) \in C^\infty(-\infty, \infty)$ be monotone decreasing for $t \geqslant 0$ with

$$\Phi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1/3, \\ 0, & t \geqslant 2/3, \end{cases} \qquad \Phi(t) = \begin{cases} 1, & -1/3 \leqslant t \leqslant 0, \\ 1-\Phi(t+1), & -2/3 \leqslant t \leqslant -1/3, \\ 0, & t \leqslant -2/3, \end{cases}$$

so that, obviously, $\sum_{k=-\infty}^{\infty} \Phi(t-k) = 1$ for all $t \in \boldsymbol{R}$. Let $\varphi_k(t) = \big(\Phi(t-k)\big)^{1/2}$; then $\varphi_k \in C^\infty(-\infty, \infty)$, $\varphi_k(t) \geqslant 0$, $\varphi_k(t)$ is monotone decreasing for $t > k$ and $\sum_{k=-\infty}^{\infty} \varphi_k^2(t) = 1$. For any measurable function $g$ on $(0, \infty)$, let $g^*(x) = g(e^x)$. Then, for $\gamma > 0$ and $1 < q < \infty$, the *space of localized Bessel potentials* is defined by

$$S(q, \gamma) = \{g\colon \|g\|_{S(q,\gamma)} = \sup_{k\in\boldsymbol{Z}} \|\varphi_k g^*\|_{q,\gamma} < \infty\}, \tag{1.1}$$

where $\boldsymbol{Z}$ is the set of all integers. We also let $\boldsymbol{N} = \{1, 2, \ldots\}$, $l^\infty$ denote the space of bounded sequences $\eta = \{\eta_k\}_{k=0}^\infty$ with the supremum norm $\|\eta\|_\infty$, $L^\infty = L^\infty(0, \infty)$, $l^\infty_C$ and $L^\infty_C$ denote the corresponding spaces of sequences and (bounded) functions with compact support in $(0, \infty)$, and let $C(0, \infty)$ denote the space of continuous functions on $(0, \infty)$. Generic positive constants will be denoted by $C$.

As is customary we shall identify a function which coincides a.e. with a continuous function with that continuous function. When $\gamma > 1/q$, $1 < q < \infty$, each function in $S(q, \gamma)$ can be identified with a continuous function [11] and so we can consider the restriction of the continuous function to the natural numbers and state the Connett and Schwartz characterization of $S(q, \gamma)$ for $\gamma \in \boldsymbol{N}$ as follows.

THEOREM A. *Let $1 < q < \infty$ and $\gamma \in \boldsymbol{N}$.*

(a) *If $g \in S(q, \gamma)$, $|\eta_0| \leqslant C\|g\|_{S(q,\gamma)}$ and $\eta_k = g(k)$, $k \in \boldsymbol{N}$, then*

$$\|\eta\|_\infty + \sup_{m\geqslant 1} \Big(\sum_{2^{m-1}}^{2^m-1} k^{-1} |A_k^\gamma \Delta^\gamma \eta_k|^q\Big)^{1/q} \leqslant C\|g\|_{S(q,\gamma)}, \tag{1.2}$$

*where, as usual, $C$ is a constant independent of $g$.*

(b) *Conversely, to each sequence $\eta$ for which the left side of* (1.2) *is finite there exists a $g \in S(q, \gamma)$ with $g(k) = \eta_k$, $k \in \boldsymbol{N}$, so that* (1.2) *also holds with the inequality reversed* (*and a different positive constant $C$*).

Here the (fractional) difference operator $\Delta^\lambda$ is defined by [5]

$$\Delta^\lambda \eta_k = \sum_{j=k}^{\infty} A_{j-k}^{-\lambda-1} \eta_j, \qquad A_k^\lambda = \binom{k+\lambda}{k} = \frac{\Gamma(k+\lambda+1)}{\Gamma(k+1)\Gamma(\lambda+1)},$$



whenever the series converges. To keep the notation compact we also define the following (weak bounded variation) sequence spaces: for $\gamma > 0$, $1 \leqslant q < \infty$,

$$\mathrm{wbv}_{q,\gamma} = \Big\{\eta \in l^\infty\colon \|\eta\|_{q,\gamma;w} = \|\eta\|_\infty + \sup_{m\geqslant 1} \Big(\sum_{2^{m-1}}^{2^m-1} k^{-1} |A_k^\gamma \Delta^\gamma \eta_k|^q\Big)^{1/q} < \infty\Big\},$$

and for $\gamma \geqslant 0$, $q = \infty$,

$$\mathrm{wbv}_{\infty,\gamma} = \{\eta \in l^\infty\colon \|\eta\|_{\infty,\gamma;w} = \|\eta\|_\infty + \sup_{k\geqslant 0} |A_k^\gamma \Delta^\gamma \eta_k| < \infty\}.$$

Corresponding to $\mathrm{wbv}_{q,\gamma}$ we also consider the space $s(q, \gamma)$, $\gamma > 1/q$, $1 < q < \infty$, of all sequences $\eta$ for which there is a (continuous) $g \in S(q, \gamma)$ such that $\eta_k = g(k)$, $k \in \boldsymbol{N}$. The norm is defined by

$$\|\eta\|_{s(q,\gamma)} = \inf\{\|g\|_{S(q,\gamma)}\colon g \in S(q, \gamma) \text{ and } g(k) = \eta_k,\ k \in \boldsymbol{N}\} + |\eta_0|.$$

Then Theorem A is equivalent to the statement that

$$\mathrm{wbv}_{q,\gamma} = s(q, \gamma), \qquad 1 < q < \infty,\ \gamma \in \boldsymbol{N},$$

with equivalent norms, and our extension of Theorem A to each $\gamma > 1/q$ can be stated in the form

THEOREM 1. *If $1 < q < \infty$ and $\gamma > 1/q$, then*

$$\mathrm{wbv}_{q,\gamma} = s(q, \gamma)$$

*with equivalent norms.*

An intermediate step in the proof of Theorem A uses the fact that (on account of Theorem 3 in [20], p. 135) when $\gamma \in \boldsymbol{N}$

$$\|g\|_\infty + \sum_{j=0}^{\gamma} \sup_{m\in\boldsymbol{Z}} \Big(\int_{2^{m-1}}^{2^m} |t^j g^{(j)}(t)|^q \frac{dt}{t}\Big)^{1/q}$$

is an equivalent norm for $S(q, \gamma)$. To generalize this to fractional $\gamma$ we first define for $0 < \delta < 1$ and a locally integrable function $g$ the fractional integral

$$I_\omega^\delta(g)(t) = \begin{cases} \dfrac{1}{\Gamma(\delta)} \displaystyle\int_t^\omega (s-t)^{\delta-1} g(s)\,ds, & 0 < t < \omega, \\ 0, & t \geqslant \omega \end{cases} \tag{1.3}$$

and the fractional derivatives

$$\begin{aligned} g^{(\delta)}(t) &= \lim_{\omega\to\infty} -\frac{d}{dt} I_\omega^{1-\delta}(g)(t) \\ g^{(\gamma)}(t) &= \Big(\frac{d}{dt}\Big)^{[\gamma]} g^{(\gamma-[\gamma])}(t), \qquad t, \gamma > 0, \end{aligned} \tag{1.4}$$

whenever the right sides exist ($[\gamma]$ denotes the integer part of $\gamma$). See Cossar [12] and [26], § 3.3.1. Then we form the function analog of the wbv-spaces defined for $\gamma > 0$, $1 \leqslant q \leqslant \infty$, by

$$\begin{aligned} \mathrm{WBV}_{q,\gamma} = \{g \in L^\infty \cap C(0,\infty)\colon\ & I_\omega^{1-\delta}(g) \in \mathrm{AC}_{\mathrm{loc}},\ \omega > 0, && \text{if } \delta > 0;\\ & g^{(\delta)}, \ldots, g^{(\gamma-1)} \in \mathrm{AC}_{\mathrm{loc}} && \text{if } \gamma \geqslant 1,\\ & && \text{and } \|g\|_{q,\gamma;W} < \infty\}, \end{aligned}$$

where $\delta = \gamma - [\gamma]$,

$$\|g\|_{q,\gamma;W} = \|g\|_\infty + \sup_{m\in\mathbf{Z}} \Big(\int_{2^{m-1}}^{2^m} |t^\gamma g^{(\gamma)}(t)|^q \frac{dt}{t}\Big)^{1/q}, \qquad 1 \leqslant q < \infty,$$

and

$$\|g\|_{\infty,\gamma;W} = \|g\|_\infty + \|t^\gamma g^{(\gamma)}(t)\|_\infty.$$

In the above and what follows $\mathrm{AC}_{\mathrm{loc}}$ ($L^1_{\mathrm{loc}}$) is the space of functions which are absolutely continuous (integrable) on every compact subinterval of $(0,\infty)$.

Analogously to the definition of $s(q,\gamma)$ we define $\mathrm{wbv}(q,\gamma)$, $\gamma > 0$, $1 \leqslant q \leqslant \infty$, to be the space of all sequences $\eta$ for which there is a $g \in \mathrm{WBV}_{q,\gamma}$ such that $\eta_k = g(k)$, $k \in \mathbf{N}$. The norm is defined by

$$\|\eta\|_{\mathrm{wbv}(q,\gamma)} = \inf\{\|g\|_{q,\gamma;W}\colon\ g \in \mathrm{WBV}_{q,\gamma} \text{ and } g(k) = \eta_k,\ k \in \mathbf{N}\} + |\eta_0|.$$

We shall prove in Section 5 that $\mathrm{wbv}_{q,\gamma}$ and $\mathrm{wbv}(q,\gamma)$ are related as follows:

THEOREM 2. *If $\gamma > 1/q$, $1 < q \leqslant \infty$ or $\gamma \geqslant 1$, $q = 1$, then*

$$\mathrm{wbv}_{q,\gamma} = \mathrm{wbv}(q,\gamma)$$

*with equivalent norms.*

Then, to complete the proof of Theorem 1 it only remains to prove (in Section 6)

THEOREM 3. *If $1 < q < \infty$ and $\gamma > 1/q$, then*

$$\mathrm{WBV}_{q,\gamma} = S(q,\gamma)$$

*and hence*

$$\mathrm{wbv}(q,\gamma) = s(q,\gamma)$$

*with equivalent norms.*

Before proving these theorems we shall first derive some required fundamental properties of the $\mathrm{wbv}_{q,\gamma}$, $\mathrm{WBV}_{q,\gamma}$, and $L^q_\gamma$ spaces in Sections 2–4. The imbedding behavior of the $\mathrm{wbv}_{q,\gamma}$ and $\mathrm{WBV}_{q,\gamma}$ spaces is presented in Section 7 and then these results are used in Section 8 to derive our multiplier criteria for Jacobi expansions. Our multiplier criteria for Hankel transforms are presented in Section 9 along with several remarks concerning related results.



**2. $\mathrm{wbv}_{q,\gamma}$ spaces.** In this section we present some basic properties of the $\mathrm{wbv}_{q,\gamma}$ spaces which are needed for the proof of Theorem 2 and for the applications.

LEMMA 1. *If $0 < \mu < \gamma$, $1 \leqslant q \leqslant \infty$, then*

$$\mathrm{wbv}_{q,\gamma} \subset \mathrm{wbv}_{q,\mu},$$

*where, as elsewhere, the inclusion sign means that the identity map is continuous.*

Proof. The case $q = 1$ is essentially proved in [13], § 2. A slight modification of that proof shows that the Lemma also holds for $1 < q \leqslant \infty$: First note that for $\eta \in l^\infty$ we have [5], Lemma 1

$$\Delta^a(\Delta^b \eta_k) = \Delta^{a+b}\eta_k, \qquad a > -1,\ b \geqslant 0,\ a+b > 0. \tag{2.1}$$

Thus, for $1 < q < \infty$,

$$\begin{aligned} \Big(\sum_{k=2^{m-1}}^{2^m-1} k^{-1}|A_k^\mu \Delta^\mu \eta_k|^q\Big)^{1/q} \leqslant{} & C\Big(\sum_{2^{m-1}}^{2^m-1} k^{-1}\Big|A_k^\mu \sum_{j=k}^{2^m-1} A_{j-k}^{\gamma-\mu-1}\Delta^\gamma \eta_j\Big|^q\Big)^{1/q} + \\ & + C\Big(\sum_{2^{m-1}}^{2^m-1} k^{-1}\Big|A_k^\mu \sum_{j=2^m}^{2^{m+1}-1} A_{j-k}^{\gamma-\mu-1}\Delta^\gamma \eta_j\Big|^q\Big)^{1/q} + \\ & + C\sum_{l=m+2}^{\infty}\Big(\sum_{2^{m-1}}^{2^m-1} k^{-1}\Big|A_k^\mu \sum_{j=2^{l-1}}^{2^l-1} A_{j-k}^{\gamma-\mu-1}\Delta^\gamma \eta_j\Big|^q\Big)^{1/q}. \\ & \equiv I_1 + I_2 + I_3. \end{aligned}$$

Hölder's inequality gives

$$\Big|\sum_{j=k}^{2^m-1} A_{j-k}^{\gamma-\mu-1}\Delta^\gamma \eta_j\Big|^q \leqslant \sum_{j=k}^{2^m-1} A_{j-k}^{\gamma-\mu-1}|\Delta^\gamma \eta_j|^q \Big(\sum_{j=k}^{2^m-1} A_{j-k}^{\gamma-\mu-1}\Big)^{q/p}$$

with $p = q/(q-1)$. Observing that the last factor on the right side is majorized by $C(A_j^{\gamma-\mu})^{q/p}$, $2^{m-1} \leqslant j < 2^m$, we can estimate $I_1$, after an interchange of summation, by

$$\begin{aligned} I_1 &\leqslant C\Big(\sum_{j=2^{m-1}}^{2^m-1} j^{-1}|\Delta^\gamma \eta_j|^q \sum_{k=2^{m-1}}^{j} A_k^{q\mu} A_{j-k}^{\gamma-\mu-1} A_j^{(\gamma-\mu)q/p}\Big)^{1/q} \\ &\leqslant C\Big(\sum_{j=2^{m-1}}^{2^m-1} j^{-1}|\Delta^\gamma \eta_j|^q A_j^{q\mu+\gamma-\mu+(\gamma-\mu)q/p}\Big)^{1/q} \leqslant C\|\eta\|_{q,\gamma;w} \end{aligned}$$

uniformly in $m$. The sums $I_2$ and $I_3$ are estimated analogously (use also the method in [13]), which gives the case $1 < q < \infty$. Since the case $q = \infty$ is even simpler to treat, it is omitted.

LEMMA 2. *Let $\gamma > 0$, $1 \leqslant q \leqslant \infty$, $G \in C^\infty[0,\infty)$ be monotone decreasing with*

$$G(t) = \begin{cases} 1, & t \leqslant 1, \\ 0, & t \geqslant d, \end{cases}$$

*where $d > 1$ is fixed, and let $G_u(t) = G(t/u)$, $u > 0$. Then there exist constants $C_1$, $C_2$ independent of $\eta$ and $u > 0$ such that*

$$C_1 \|\eta\|_{q,\gamma;w} \leqslant \sup_{u>0} \|\{G_u(k)\eta_k\}\|_{q,\gamma;w} \leqslant C_2 \|\eta\|_{q,\gamma;w}.$$

*The left inequality also holds with $\sup\limits_{u>0}$ replaced by $\sup\limits_{u>a}$, where $a > 0$.*

Proof. We first prove the right-side inequality. Let $\eta \in \mathrm{wbv}_{q,\gamma}$. If $\gamma$ is a natural number, Leibniz' formula gives

$$\Delta^\gamma(G_u(k)\eta_k) = \sum_{j=0}^{\gamma} \binom{\gamma}{j} \Delta^j \eta_k \Delta^{\gamma-j} G_u(j+k),$$

and the inequality follows as in the proof of [13], Lemma 2.

So now let $\gamma$ be strictly fractional. Note that by a formula due to Peyerimhoff [18], p. 3 we have

$$\Delta^\gamma(G_u(k)\eta_k) = \eta_k \Delta^\gamma G_u(k) + \sum_{j=0}^{[\gamma]} \binom{\gamma}{j} \Delta^j G_u(k) \Delta^{\gamma-j}\eta_{k+j} + R_k, \tag{2.2}$$

where the remainder term is given by

$$R_k = (-1)^{[\gamma]} \sum_{n=k+1+[\gamma]}^{\infty} A_{n-k}^{-\gamma-1}(\eta_n - \eta_k) \sum_{j=k+1}^{n-[\gamma]} A_{n-[\gamma]-j}^{[\gamma]-1}\left(\Delta^{[\gamma]}G_u(j) - \Delta^{[\gamma]}G_u(k)\right).$$

Also note that, by [26], p. 39, p. 37, we have for any $\delta > 0$

$$\begin{gathered} |A_k^\delta \Delta^\delta G_u(k)| \leqslant \sum_{i=0}^{\infty} A_i^\delta |\Delta^{\delta+1} G_u(i)| \leqslant C_\delta, \\ \left| t^\delta \left(\frac{d}{dt}\right)^\delta G_u(t)\right| \leqslant \int_0^\infty s^\delta \left|\left(\frac{d}{ds}\right)^{\delta+1} G_u(s)\right| ds \leqslant C'_\delta. \end{gathered} \tag{2.3}$$

We shall show that

$$\sup_k |A_k^\gamma R_k| \leqslant C\|\eta\|_\infty. \tag{2.4}$$

For the case $0 < \gamma < 1$, $2^{m-1} \leqslant k < 2^m$, we have from (2.3) (observe that $A_{n-k}^{-\gamma-1}$, $n \geqslant k+1$, and $\Delta G_u(i)$ do not change sign)

$$|A_k^\gamma R_k| \leqslant 2\|\eta\|_\infty A_k^\gamma \left| \sum_{n=k+1}^{2^{m+1}-1} A_{n-k}^{-\gamma-1} \sum_{i=k}^{n-1} \Delta G_u(i)\right| +$$



$$\begin{aligned} &+2\|\eta\|_\infty A_k^\gamma \left| \sum_{n=2^{m+1}}^{\infty} A_{n-k}^{-\gamma-1} (G_u(n) - G_u(k))\right| \\ &\leqslant C\|\eta\|_\infty A_k^{\gamma-1} \left| \sum_{n=k+1}^{2^{m+1}-1} A_{n-k}^{-\gamma-1} \sum_{i=k}^{n-1} 1 \right| + C\|\eta\|_\infty \|G\|_\infty A_k^\gamma \left| \sum_{n=2^{m+1}}^{\infty} A_{n-k}^{-\gamma-1}\right| \\ &\leqslant C\|\eta\|_\infty. \end{aligned}$$

Analogously, for $\gamma > 1$, $\gamma \notin N$, $2^{m-1} \leqslant k < 2^m$, we have

$$\begin{aligned} |A_k^\gamma R_k| &\leqslant 2\|\eta\|_\infty A_k^\gamma \left| \sum_{n=k+1+[\gamma]}^{2^{m+1}-1} A_{n-k}^{-\gamma-1} \sum_{j=k+1}^{n-[\gamma]} A_{n-[\gamma]-j}^{[\gamma]-1} \sum_{i=k}^{j-1} |\Delta^{[\gamma]+1} G_u(i)| \right| + \\ &\quad + C\|\eta\|_\infty A_k^{\gamma-[\gamma]} \left| \sum_{n=2^{m+1}}^{\infty} A_{n-k}^{-\gamma-1} \sum_{j=k+1}^{n-[\gamma]} A_{n-[\gamma]-j}^{[\gamma]-1} \right| \\ &\leqslant C\|\eta\|_\infty A_k^{\gamma-[\gamma]-1} \left| \sum_{n=k+1+[\gamma]}^{2^{m+1}-1} A_{n-k}^{-\gamma-1} A_{n-[\gamma]-k-1}^{[\gamma]+1} \right| + \\ &\quad + C\|\eta\|_\infty A_k^{\gamma-[\gamma]} \left| \sum_{n=2^{m+1}}^{\infty} A_{n-k}^{-\gamma-1} A_{n-[\gamma]-k-1}^{[\gamma]} \right| \\ &\leqslant C\|\eta\|_\infty \end{aligned}$$

uniformly in $k$. Thus (2.4) holds. The right-side inequality of Lemma 2 now follows easily. By (2.2)–(2.4), it is immediately obvious in the case $q = \infty$. If $1 \leqslant q < \infty$, then we need but observe that from (2.2)–(2.4) and Lemma 1 we have

$$\begin{aligned} \Big(\sum_{k=2^{m-1}}^{2^m-1} k^{-1} \left| A_k^\gamma \Delta^\gamma (\eta_k G_u(k))\right|^q\Big)^{1/q} &\leqslant C\Big(\|\eta\|_\infty + \sum_{j=0}^{[\gamma]} \binom{\gamma}{j} C_j \|\eta\|_{q,\gamma-j;w} + \|\eta\|_\infty\Big) \\ &\leqslant C\|\eta\|_{q,\gamma;w}. \end{aligned}$$

Conversely, let $\|\{G_u(k)\eta_k\}\|_{\gamma,q;w}$ be uniformly bounded in $u > 0$. Fix $m$ and choose $u \geqslant 2^{m+1}$, $2^{m-1} \leqslant k < 2^m$. Then

$$\begin{aligned} |A_k^\gamma \Delta^\gamma(\eta_k - G_u(k)\eta_k)| &= \left| A_k^\gamma \sum_{j=2^{m+1}}^{\infty} A_{j-k}^{-\gamma-1} (\eta_j - G_u(j)\eta_j)\right| \\ &\leqslant C\|\{G_u(j)\eta_j\}\|_\infty \end{aligned}$$

for $u$ sufficiently large. By the triangle inequality,

$$\begin{aligned} \Big(\sum_{k=2^{m-1}}^{2^m-1} k^{-1} |A_k^\gamma \Delta^\gamma \eta_k|^q\Big)^{1/q} &\leqslant \sup_m \Big(\sum_{k=2^{m-1}}^{2^m-1} k^{-1} |A_k^\gamma \Delta^\gamma (G_u(k)\eta_k)|^q\Big)^{1/q} + C\|\{G_u(j)\eta_j\}\|_\infty \\ &\leqslant C\sup_{u>0} \|\{G_u(k)\eta_k\}\|_{q,\gamma;w} \end{aligned}$$

uniformly in $m$, which gives the left inequality in the lemma. The case $q=\infty$ follows analogously.

**3.** $\mathrm{WBV}_{q,\gamma}$ **spaces.** Suppose that $\gamma>0$, $1\leqslant q\leqslant\infty$ and $g\in\mathrm{WBV}_{q,\gamma}$. If $\gamma\geqslant 1$ and $g$ has compact support in $(0,\infty)$, it follows from [26], Lemma 3.14 and [27] that

$$g(t)=\frac{\pm 1}{\Gamma(\gamma)}\int_t^\infty (s-t)^{\gamma-1}g^{(\gamma)}(s)\,ds\quad\text{a.e.,}\tag{3.1}$$

which can easily be used to show that

$$g^{(\mu)}(t)=\frac{\pm 1}{\Gamma(\gamma-\mu)}\int_t^\infty (s-t)^{\gamma-\mu-1}g^{(\gamma)}(s)\,ds\quad\text{a.e.,}\quad 0<\mu<\gamma.\tag{3.2}$$

Since formula (3.2) will be needed for all $g\in\mathrm{WBV}_{q,\gamma}$ to derive the imbedding properties of these spaces, it is natural to try to prove (3.2) by first proving (3.1) for these spaces. Unfortunately, if $g$ does not have compact support, then the integral in (3.1) might not converge a.e. even though the integral in (3.2) clearly converges a.e. since, for $2^j>t$,

$$\begin{aligned}\int_{2^j}^\infty (s-t)^{\gamma-\mu-1}|g^{(\gamma)}(s)|\,ds &\leqslant C\sum_j^\infty 2^{-k\mu}\int_{2^k}^{2^{k+1}} s^\gamma|g^{(\gamma)}(s)|\frac{ds}{s}\\ &\leqslant C\sum_j^\infty 2^{-k\mu}\Big(\int_{2^k}^{2^{k+1}}|s^\gamma g^{(\gamma)}(s)|^q\frac{ds}{s}\Big)^{1/q}<\infty\end{aligned}\tag{3.3}$$

and $\int_t^{2^j}\ldots ds$ converges a.e. Therefore for functions without compact support we shall have to use a more delicate approach which requires the following preliminary results:

Lemma 3. *Suppose* $0<\mu<\delta<1$, $g\in L^\infty\cap C(0,\infty)$, *and* $I_\omega^{1-\delta}(g)\in\mathrm{AC}_{\mathrm{loc}}$ *for each* $\omega>0$. *Then*

$$\|I_\omega^\delta(g^{(\delta)})\|_\infty\leqslant C\|g\|_\infty,\qquad \omega>0,\tag{3.4}$$

*where* $C$ *depends only on* $\delta$, *and* $I_\omega^{1-\mu}(g)\in\mathrm{AC}_{\mathrm{loc}}$, $\omega>0$.

Proof. From the hypotheses and the definition of $g^{(\delta)}$ it follows for a.e. $t$ in $(0,\omega)$ that

$$\begin{aligned}g^{(\delta)}(t)&=\frac{-1}{\Gamma(1-\delta)}\frac{d}{dt}\int_t^\infty (s-t)^{-\delta}g(s)\,ds-\frac{\delta}{\Gamma(1-\delta)}\int_\omega^\infty (s-t)^{-\delta-1}g(s)\,ds\\ &=I_1(t)+I_2(t).\end{aligned}\tag{3.5}$$



Clearly,

$$|I_\omega^\delta(I_2)(x)|\leqslant C\|g\|_\infty\int_x^\omega (t-x)^{\delta-1}\int_\omega^\infty (s-t)^{-\delta-1}\,ds\,dt\leqslant C\|g\|_\infty$$

and from

$$\begin{aligned}\int_x^\omega I_\omega^\delta(I_1)(u)\,du&=\frac{-1}{\Gamma(\delta)\Gamma(1-\delta)}\int_x^\omega\int_u^\omega (t-u)^{\delta-1}\Big(\frac{d}{dt}\int_t^\infty (s-t)^{-\delta}g(s)\,ds\Big)dt\,du\\ &=\frac{-1}{\Gamma(\delta)\Gamma(1-\delta)}\int_x^\omega\Big(\frac{d}{dt}\int_t^\infty (s-t)^{-\delta}g(s)\,ds\Big)\int_x^t (t-u)^{\delta-1}\,du\,dt\\ &=\frac{-1}{\Gamma(\delta+1)\Gamma(1-\delta)}\int_x^\omega (t-x)^\delta\frac{d}{dt}\int_t^\infty (s-t)^{-\delta}g(s)\,ds\,dt\\ &=\frac{1}{\Gamma(\delta)\Gamma(1-\delta)}\int_x^\omega (t-x)^{\delta-1}\int_t^\infty (s-t)^{-\delta}g(s)\,ds\,dt=\int_x^\omega g(s)\,ds,\end{aligned}$$

we have that $I_\omega^\delta(I_1)(x)=g(x)$ for a.e. $x$ in $(0,\omega)$, which gives (3.4).

Since $I_\omega^{1-\mu}(g)(x)=0$ for $x\geqslant\omega$ and is continuous at $x=\omega$ it only remains to show that if $0<a<\omega$, then this function is absolutely continuous on $[a,\omega]$. By hypothesis there is an $f\in L^1[a,\omega]$ such that

$$I_\omega^{1-\delta}(g)(x)=\int_x^\omega f(t)\,dt,\qquad a\leqslant x\leqslant\omega.$$

Then, for $a\leqslant x\leqslant\omega$,

$$\begin{aligned}I_\omega^{1-\mu}(g)(x)&=\frac{1}{\Gamma(1-\mu)}\int_x^\omega (s-x)^{-\mu}g(s)\,ds\\ &=\frac{1}{\Gamma(\delta-\mu)}\int_x^\omega (t-x)^{\delta-\mu-1}I_\omega^{1-\delta}(g)(t)\,dt\\ &=\frac{1}{\Gamma(\delta-\mu)}\int_x^\omega (t-x)^{\delta-\mu}f(t)\,dt\\ &=\frac{1}{\Gamma(\delta-\mu)}\int_x^\omega\int_s^\omega (t-s)^{\delta-\mu-1}f(t)\,dt\,ds,\end{aligned}$$

which shows that $I_\omega^{1-\mu}(g)\in\mathrm{AC}[a,\omega]$, since the inner integral is in $L^1[a,\omega]$.

LEMMA 4. *Suppose* $1<\gamma<2$ *and* $g\in \mathrm{WBV}_{1,\gamma}$. *Then* $g^{(\lambda-1)}\in \mathrm{AC}_{\mathrm{loc}}$, $1\leqslant\lambda<\gamma$, *and formula* (3.2) *holds. Also* $I_\omega^{1-\mu}(g)\in \mathrm{AC}_{\mathrm{loc}}$ *for* $0<\mu<1$, $\omega>0$.

Proof. Let $\delta=\gamma-1$ and define

$$(t-x)_+^{\alpha}=\begin{cases}(t-x)^{\alpha}, & t>x.\\ 0, & t\leqslant x.\end{cases}$$

Use the decomposition in (3.5) to write

$$I_\omega^\delta(g^{(\delta)})(x)-I_\omega^\delta(g^{(\delta)})(y)=g(x)-g(y)+R_\omega(x,y)$$

for a.e. $x, y$ in $(0,\omega)$. Then from

$$|R_\omega(x,y)|=C\left|\int_0^\omega\{(t-x)_+^{\delta-1}-(t-y)_+^{\delta-1}\}\int_\omega^\infty(s-t)^{-\delta-1}g(s)\,ds\,dt\right|$$

$$\leqslant C\|g\|_\infty\int_0^{\omega-1}|(t-x)_+^{\delta-1}-(t-y)_+^{\delta-1}|(\omega-t)^{-\delta}\,dt+$$

$$+C\|g\|_\infty\int_{\omega-1}^{\omega}|(t-x)_+^{\delta-1}-(t-y)_+^{\delta-1}|(\omega-t)^{-\delta}\,dt$$

it is easily seen that $R_\omega(x,y)\to 0$ as $\omega\to\infty$. For, by the dominated convergence theorem, the penultimate term tends to zero as $\omega\to\infty$, and the last integral also tends to zero since, for fixed $x, y$ with $0<x,\ y<3\omega/4$, $|\cdots|<\frac{2}{5}\omega^{\delta-1}\to 0$ as $\omega\to\infty$. Therefore we have the representation $(0<\delta<1)$

(3.6) $$g(x)-g(y)=\frac{1}{\Gamma(\delta)}\int_0^\infty\{(t-x)_+^{\delta-1}-(t-y)_+^{\delta-1}\}g^{(\delta)}(t)\,dt$$

for a.e. $x, y$ in $(0,\omega)$ and hence for all $x, y>0$ since both sides are continuous functions and the integral is absolutely convergent for $x, y>0$.

Since $g^{(\delta)}\in \mathrm{AC}_{\mathrm{loc}}$ and, by (3.3),

$$g^{(\delta)}(y)-g^{(\delta)}(x)=\int_x^y g^{(\gamma)}(t)\,dt$$

tends to zero as $x, y\to\infty$, $\lim\limits_{y\to\infty}g^{(\delta)}(y)=l$ exists and

(3.7) $$g^{(\delta)}(x)=l-\int_x^\infty g^{(\gamma)}(t)\,dt.$$

In fact, $l=0$. For if $l>0$ and $g^{(\delta)}(x)-l=h(x)$, where $|h(x)|<l/2$ when $x\geqslant X$, then



$$I_{2X}^\delta(g^{(\delta)})(X)=C\left(\frac{l}{\delta}X^\delta+\int_X^{2X}(t-X)^{\delta-1}h(t)\,dt\right)$$

$$\geqslant C\left(\frac{l}{\delta}X^\delta-\frac{l}{2}\int_X^{2X}(t-X)^{\delta-1}\,dt\right)$$

$$=ClX^\delta/2\delta\to\infty \quad\text{as}\quad X\to\infty,$$

in contradiction to Lemma 3. Similarly, $l$ cannot be negative. Substitution of (3.7) (with $l=0$) into (3.6) and some obvious manipulations yield

(3.8) $$g(x)-g(y)=\frac{1}{\Gamma(\gamma-1)}\int_y^x\int_t^\infty(s-t)^{\gamma-2}g^{(\gamma)}(s)\,ds\,dt$$

which shows that $g\in \mathrm{AC}_{\mathrm{loc}}$ since the inner integral is in $L^1_{\mathrm{loc}}$. Hence the right side of (3.8) also equals $\int_y^x g'(t)\,dt$, so

(3.9) $$g'(t)=\frac{1}{\Gamma(\gamma-1)}\int_t^\infty(s-t)^{\gamma-2}g^{(\gamma)}(s)\,ds,\quad\text{a.e.}$$

If $0<\mu<1$, then $\lim\limits_{\omega\to\infty}g(\omega)(\omega-x)^{-\mu}=0$ since $g$ is bounded, and by an integration by parts and (3.9) we have

(3.10) $$g^{(\mu)}(x)=\lim_{\omega\to\infty}\left(-\frac{d}{dx}I_\omega^{1-\mu}(g)(x)\right)$$

$$=\frac{1}{\Gamma(2-\mu)}\lim_{\omega\to\infty}\frac{d}{dx}\int_x^\omega(t-x)^{1-\mu}g'(t)\,dt$$

$$=\frac{1}{\Gamma(1-\mu)}\lim_{\omega\to\infty}\frac{d}{dx}\int_x^\omega g'(t)\int_x^t(t-s)^{-\mu}\,ds\,dt$$

$$=\frac{1}{\Gamma(1-\mu)}\lim_{\omega\to\infty}\frac{d}{dx}\int_x^\omega\int_s^\omega(t-s)^{-\mu}g'(t)\,dt\,ds$$

$$=\frac{1}{\Gamma(1-\mu)}\int_x^\infty(t-x)^{-\mu}g'(t)\,dt$$

$$=\frac{1}{\Gamma(1-\mu)\Gamma(\gamma-1)}\int_x^\infty(t-x)^{-\mu}\int_t^\infty(s-t)^{\gamma-2}g^{(\gamma)}(s)\,ds\,dt$$

$$=\frac{1}{\Gamma(1-\mu)\Gamma(\gamma-1)}\int_x^\infty g^{(\gamma)}(s)\int_x^s(t-x)^{-\mu}(s-t)^{\gamma-2}\,dt\,ds$$

$$=\frac{1}{\Gamma(\gamma-\mu)}\int_x^\infty(s-x)^{\gamma-\mu-1}g^{(\gamma)}(s)\,ds,\quad\text{a.e.}$$

Comparison of the first and fourth lines in (3.10) also shows that $I_\omega^{1-\mu}(g) \in \mathrm{AC}_{\mathrm{loc}}$, $\omega > 0$.

If $1 < \mu < \gamma$, then it follows from (3.10) that

$$g^{(\mu-1)}(x) = \frac{1}{\Gamma(\gamma-\mu)} \int_x^\infty \int_t^\infty (s-t)^{\gamma-\mu-1} g^{(\gamma)}(s)\, ds\, dt.$$

Since the inner integral is in $L^1_{\mathrm{loc}}$, $g^{(\mu-1)} \in \mathrm{AC}_{\mathrm{loc}}$ and (3.2) holds, which completes the proof.

Lemma 5. *If $\gamma = 1$ or $2$ and $g \in \mathrm{WBV}_{1,\gamma}$, then $I_\omega^{1-\mu}(g) \in \mathrm{AC}_{\mathrm{loc}}$ for $0 < \mu < 1$, $\omega > 0$, and (3.2) holds.*

The proof is analogous to that of Lemma 4, except that when $\gamma = 2$ it is shown that if $l > 0$ in the formula

$$g'(x) = l - \int_x^\infty g''(t)\, dt$$

and $g'(x) - l = h(x)$, where $|h(x)| < l/2$ for $x \geqslant X$, then

$$g(2X) - g(X) = \int_X^{2X} g'(t)\, dt = lX + \int_X^{2X} h(t)\, dt$$
$$\geqslant lX/2 \to \infty \quad \text{as} \quad X \to \infty$$

which contradicts the boundedness of $g$.

Lemma 6. *If $0 < \gamma < 1$ and $g \in \mathrm{WBV}_{1,\gamma}$, then (3.2) holds.*

Proof. Since the hypotheses do not imply that $g$ is locally absolutely continuous, we have to proceed in a way which is quite different from the proof of Lemma 4. If $0 < x < \omega$, then

$$\int_x^\infty \int_t^\infty (s-t)^{\gamma-\mu-1} \frac{d}{ds} I_\omega^{1-\gamma}(g)(s)\, ds\, dt$$
$$= \frac{1}{\gamma-\mu} \int_x^\infty (s-x)^{\gamma-\mu} \frac{d}{ds} I_\omega^{1-\gamma}(g)(s)\, ds$$
$$= \frac{-1}{\Gamma(1-\gamma)} \int_x^\infty (s-x)^{\gamma-\mu-1} \int_s^\omega (y-t)^{-\gamma} g(y)\, dy\, ds$$
$$= -\frac{\Gamma(\gamma-\mu)}{\Gamma(1-\mu)} \int_x^\omega (y-x)^{-\mu} g(y)\, dy$$
$$= \Gamma(\gamma-\mu) \int_x^\infty \frac{d}{dt} I_\omega^{1-\mu}(g)(t)\, dt.$$



Hence

$$\frac{d}{dt} I_\omega^{1-\mu}(g)(t) = \frac{1}{\Gamma(\gamma-\mu)} \int_t^\infty (s-t)^{\gamma-\mu-1} \frac{d}{ds} I_\omega^{1-\gamma}(g)(s)\, ds$$

for a.e. $t$ in $(0, \omega)$ and the argument on p. 32 of [26] shows that if $a > 0$ and $\Omega$ is a countable set dense in $(a, \infty)$, then this formula also holds for all $\omega \in \Omega$ for a.e. $t$ in $(0, a)$. In addition, for $\omega \in \Omega$ and a.e. $t$ in $(0, a)$

$$(3.11) \quad \left| \frac{1}{\Gamma(\gamma-\mu)} \int_t^\infty (s-t)^{\gamma-\mu-1} g^{(\gamma)}(s)\, ds + \frac{d}{dt} I_\omega^{1-\mu}(g)(t) \right|$$
$$= C \left| \int_t^\infty (s-t)^{\gamma-\mu-1} \int_\omega^\infty (y-s)^{-\gamma-1} g(y)\, dy\, ds \right|$$
$$\leqslant C \|g\|_\infty (\omega - t)^{-\mu} \to 0 \quad \text{as} \quad \omega \to \infty \ (\omega \in \Omega).$$

Since the first integral in (3.11) is a continuous function of $\omega$ $(0 < t < a < \omega)$ and for a.e. $t$ in $(0, a)$ it converges as $\omega \to \infty$, it follows that $g^{(\mu)}(t)$ exists for a.e. $t$ in $(0, a)$ and that (3.2) holds a.e. in $(0, a)$. But $a$ is arbitrary, so (3.2) holds a.e. in $(0, \infty)$.

Lemma 7. *If $\gamma \geqslant 2$, then*

$$\mathrm{WBV}_{1,\gamma} \subset \mathrm{WBV}_{1,\gamma-k}, \quad k \leqslant \gamma - 1,\ k \in \mathbf{N}.$$

Lemma 8. *If $\gamma > 0$, $1 \leqslant q \leqslant \infty$ and $g \in \mathrm{WBV}_{q,\gamma}$, then (3.2) holds.*

Lemmas 7 and 8 follow by the standard arguments in [26], pp. 36, 37, and the trivial observation that $\mathrm{WBV}_{q,\gamma} \subset \mathrm{WBV}_{p,\gamma}$ when $1 \leqslant p < q$, $\gamma > 0$.

Lemma 9. *If $\gamma > 0$ and $1 \leqslant q \leqslant \infty$, then*

$$\mathrm{WBV}_{q,\gamma} \subset \mathrm{WBV}_{q,\mu}, \quad 0 < \mu < \gamma.$$

The proof is analogous to that of Lemma 1, with (3.2) being used instead of (2.1) and, naturally, the sums being replaced by integrals; therefore we omit it. The next result is an analog of Lemma 2.

Lemma 10. *Let $\gamma > 0$, $1 \leqslant q \leqslant \infty$, $g \in L^\infty$ and $g_u = g(G_u - G_{1/u})$, $u > 0$, where $G_u$ is the function defined in Lemma 2. Then $g \in \mathrm{WBV}_{q,\gamma}$ if and only if $g_u \in \mathrm{WBV}_{q,\gamma}$ for each $u > 0$ and $\sup_{u>} \|g_u\|_{q,\gamma;W} < \infty$. Moreover, there exist constants $C_1$, $C_2$ independent of $g$ and $u > 0$ such that*

$$(3.12) \quad C_1 \|g\|_{q,\gamma;W} \leqslant \sup_{u>0} \|g_u\|_{q,\gamma;W} \leqslant C_2 \|g\|_{q,\gamma;W}$$

*and the left inequality also holds with $\sup_{u>0}$ replaced by $\sup_{u>a}$, where $a > 0$.*

Proof. Suppose $0<\gamma<1$, $1\leqslant q\leqslant\infty$ and $g\in \mathrm{WBV}_{q,\gamma}$. Then formula (3.6) holds with $\delta=\gamma$ and setting

$$h(t)=\frac{1}{\Gamma(-\gamma)}\int_0^\infty s^{-1-\gamma}\big(g(s+t)-g(t)\big)\,ds, \tag{3.13}$$

we have for $0<a<b$ that

$$\int_a^b h(t)\,dt$$

$$=\frac{1}{\Gamma(-\gamma)\Gamma(\gamma)}\int_a^b\int_0^\infty s^{-1-\gamma}\int_0^\infty \{(w-s-t)_+^{\gamma-1}-(w-t)_+^{\gamma-1}\}g^{(\gamma)}(w)\,dw\,ds\,dt$$

$$=\frac{1}{\Gamma(-\gamma)\Gamma(\gamma)}\int_0^\infty g^{(\gamma)}(w)\Big[\int_0^\infty s^{-1-\gamma}\int_a^b\{\ldots\}\,dt\,ds\Big]\,dw.$$

Clearly, $\{\ldots\}=0$ if $w<a$, so $\int_0^\infty\ldots dw=\int_a^\infty\ldots dw$. By considering the intervals $(0,w-b)$, $(w-b,w-a)$, $(w-a,\infty)$ we find that the above double integral in brackets equals zero for $w>b$ and equals $\Gamma(\gamma)\Gamma(-\gamma)$ for $a<w<b$. Hence

$$\int_a^b h(t)\,dt=\int_a^b g^{(\gamma)}(w)\,dw$$

and then $g^{(\gamma)}(t)=h(t)$, a.e., since $a$ and $b$ are arbitrary.

Now let $u>0$, $\Delta_s g(t)=g(s+t)-g(t)$, and

$$h_u(t)=\frac{1}{\Gamma(-\gamma)}\int_0^\infty s^{-1-\gamma}\Delta_s g_u(t)\,ds.$$

Then, setting $H_u(t)=G_u(t)-G_{1/u}(t)$ and

$$\Gamma(-\gamma)h_u(t)=\int_0^\infty s^{-1-\gamma}\Delta_s g(t)\,\Delta_s H_u(t)\,ds+$$

$$+H_u(t)\int_0^\infty s^{-1-\gamma}\Delta_s g(t)\,ds+g(t)\int_0^\infty s^{-1-\gamma}\Delta_s H_u(t)\,ds,$$

we have

$$\Big(\int_{2^{m-1}}^{2^m}|t^\gamma h_u(t)|^q\frac{dt}{t}\Big)^{1/q}$$

$$\leqslant 2\|g\|_\infty\Big(\int_{2^{m-1}}^{2^m}\Big|t^\gamma\frac{1}{\Gamma(-\gamma)}\int_0^\infty s^{-1-\gamma}\Big|\Delta_s H_u(t)\,ds|^q\frac{dt}{t}\Big)^{1/q}+$$



$$+\|H_u\|_\infty\Big(\int_{2^{m-1}}^{2^m}|t^\gamma g^{(\gamma)}(t)|^q\frac{dt}{t}\Big)^{1/q}+\|g\|_\infty\Big(\int_{2^{m-1}}^{2^m}|t^\gamma H_u^{(\gamma)}(t)|^q\frac{dt}{t}\Big)^{1/q}$$

$$\leqslant C\|g\|_{q,\gamma;W}$$

uniformly in $m$, where we used the fact that $|\Delta_s H_u(t)|\leqslant|\Delta_s G_u(t)|+|\Delta_s G_{1/u}(t)|$ and $G_u$ is monotone decreasing. Extending $h_u$ to $t\leqslant 0$ by letting $g_u(t)=0$ for $t\leqslant 0$ and using the fact that $g_u$ is a bounded function with compact support in $(0,\infty)$, it can easily be shown that $h_u\in L^1(-\infty,\infty)$ and, by taking Fourier transforms,

$$(h_u)^\wedge(v)=(-iv)^\gamma(g_u)^\wedge(v) \tag{3.14}$$

where $(-iv)^\gamma$ is defined by

$$(-iv)^\gamma=|v|^\gamma\Big(\cos\frac{\pi}{2}\gamma-i(\operatorname{sgn}v)\sin\frac{\pi}{2}\gamma\Big). \tag{3.15}$$

Since the right side of (3.14) equals the Fourier distributional transform of $(g_u)^{(\gamma)}\in S'$, where $S'$ is the set of tempered distributions, it follows that $(g_u)^{(\gamma)}(t)=h_u(t)$ a.e. and that the right-side inequality in (3.12) holds.

Since, for $0<a\leqslant x\leqslant b\leqslant\omega<\omega'$,

$$I_{\omega'}^{1-\gamma}(g_u)(x)=I_\omega^{1-\gamma}(g_u)(x)+\frac{1}{\Gamma(1-\gamma)}\int_\omega^{\omega'}(t-x)^{-\gamma}g_u(t)\,dt$$

and, by means of [25], p. 371, Ex. 6, the integral on the right side is in $\mathrm{AC}[a,b]$, to show that $I_\omega^{1-\gamma}(g_u)\in \mathrm{AC}_{\mathrm{loc}}$, $\omega>0$, it suffices to show this for $\omega$ so large that $g_u(t)=0$ for $t\geqslant\omega-1$. Then

$$I_\omega^{1-\gamma}(g_u)(x)=\frac{1}{\Gamma(1-\gamma)}\int_x^\infty(t-x)^{-\gamma}g_u(t)\,dt$$

and extending this function to $x\leqslant 0$ by letting $g_u(t)=0$ for $t\leqslant 0$, we see that the function

$$f(x)=I_\omega^{1-\gamma}(g_u)(x)-I_\omega^{1-\gamma}(g_u)(x+\omega)$$

belongs to $L^1(-\infty,\infty)$ and

$$f^\wedge(v)=(-iv)^{\gamma-1}(1-e^{i\omega v})(g_u)^\wedge(v),$$

where $(-iv)^{\gamma-1}$ is defined by (3.15). Thus, from (3.14),

$$(-iv)f^\wedge(v)=(1-e^{i\omega v})(h_u)^\wedge(v)$$
$$=(h_u)^\wedge(v)-\big(h_u(\cdot+\omega)\big)^\wedge(v)$$

which, by [6], Prop. 5.1.15, implies that $f \in AC_{loc}(-\infty, \infty)$, and hence $I_\omega^{1-\gamma}(g_u) \in AC_{loc}(0, \infty)$ since $f(x) = I_\omega^{1-\gamma}(g_u)(x)$ for $x > 0$.

To show that $I_\omega^{1-\gamma}(g) \in AC_{loc}$ if $g_u \in WBV_{q,\gamma}$ for each $u > 0$, one need but observe that if $0 < a \leqslant x \leqslant b \leqslant \omega$ and if $u$ is chosen so large that $g_u = g$ on $[a, \omega]$, then $I_\omega^{1-\gamma}(g)(x) = I_\omega^{1-\gamma}(g_u)(x) \in AC[a, b]$. For $\gamma \geqslant 1$ it easily follows by applying (3.5) to $g_u$ that $g^{(\delta)}, \ldots, g^{(\gamma-1)} \in AC_{loc}$ if $g_u \in WBV_{q,\gamma}$, $u > 0$. The left-side inequality in (3.12) is proved for $\gamma > 0$ just as in the corresponding case of Lemma 2, with sums replaced by integrals.

The remaining part of the proof for the case $\gamma \geqslant 1$ is a slight modification of the proof of relation (1.2) in [27] and so it is omitted.

Furthermore, it should be pointed out that, as in [26], p. 37, if $g \in WBV_{1,\gamma}$, $\gamma > 0$, then

(3.16) $$g^{(\gamma)}(t) = \big(g^{([\gamma])}(t)\big)^{(\gamma-[\gamma])}, \text{ a.e.}$$

**4. Bessel potential spaces.** Before turning to the proofs of the theorems we shall also need the following characterization of the Bessel potential space $L_\gamma^p$ by an appropriate space of hypersingular integrals.

LEMMA 11. *Let* $1 < q < \infty$.

(a) *If* $\gamma \in N$, *then*

$$\|g\|_{q,\gamma} \cong \sum_{k=0}^{\gamma} \|g^{(k)}\|_q.$$

(b) *If* $\gamma > 0$ *and* $\delta = \gamma - [\gamma] > 0$, *then*

$$\|g\|_{q,\gamma} \cong \sum_{k=0}^{[\gamma]} \|g^{(k)}\|_q + \left\| \frac{1}{\Gamma(-\delta)} \int_0^\infty s^{-1-\gamma} \Delta_s g^{([\gamma])}(t)\, ds \right\|_q \equiv |||g|||_{q,\gamma},$$

*where* $\Delta_s$ *is as defined in Section* 3.

Proof. Part (a) is due to Calderón; for a proof see Stein [20], p. 135. Consider (b). From (a) and [20], p. 136,

$$\|g\|_{q,\gamma} \cong \sum_{k=0}^{[\gamma]} \|g^{(k)}\|_q + \|g^{([\gamma])}\|_{q,\gamma-[\gamma]}$$

which implies that we may restrict ourselves to the case $0 < \gamma < 1$. Let $g \in L_\gamma^q$, $1 < q < \infty$. Then from Wheeden [31] we have

$$2 \left\| \int_0^\infty s^{-1-\gamma} \Delta_s g(t)\, ds \right\|_q \leqslant \left\| \int_{-\infty}^\infty |s|^{-1-\gamma} \Delta_s g(t)\, ds \right\|_q + \left\| \int_{-\infty}^\infty (\operatorname{sgn} s)\, |s|^{-1-\gamma} \Delta_s g(t)\, ds \right\|_q \leqslant C\|g\|_{q,\gamma};$$

so $|||g|||_{q,\gamma} \leqslant C\|g\|_{q,\gamma}$.



Conversely, let $\varphi \in S$, $S$ being the set of all infinitely differentiable functions on $(-\infty, \infty)$ that are rapidly decreasing at infinity. By [20], pp. 133, 134 there is a bounded measure $\mu$ on $(-\infty, \infty)$ with

$$(1+|v|^2)^{\gamma/2} = (d\mu)\hat{}\,(v)(1+|v|^\gamma).$$

From (3.15),

$$|v|^\gamma = (-iv)^\gamma \Big(\cos\frac{\pi}{2}\gamma + i(\operatorname{sgn} v)\sin\frac{\pi}{2}\gamma\Big)$$

and so letting $g\tilde{}$ denote the Hilbert transform of $g$ defined by $(g\tilde{})\hat{}\,(v) = -i(\operatorname{sgn} v) g\hat{}\,(v)$ and letting $F^{-1}$ denote the inverse Fourier transform, we have

$$\begin{aligned}\|\varphi\|_{q,\gamma} &= \big\| F^{-1}\big((1+|v|^2)^{\gamma/2}\varphi\hat{}\,\big) \big\|_q \\ &\leqslant C\big\{\|\varphi\|_q + \big\|F^{-1}\big((-iv)^\gamma\varphi\hat{}\,\big)\big\|_q + \big\|F^{-1}\big((-iv)^\gamma\varphi\tilde{}\,\hat{}\,\big)\big\|_q\big\} \\ &\leqslant C\big\{\|\varphi\|_q + \big\|F^{-1}\big((-iv)^\gamma\varphi\hat{}\,\big)\big\|_q\big\} \leqslant C\Big\{\|\varphi\|_q + \Big\|\int_0^\infty s^{-1-\gamma}\Delta_s\varphi(t)\,ds\Big\|_q\Big\}\end{aligned}$$

with $C$ independent of $\varphi \in S$. The desired result then follows by applying the density argument in Wheeden [31], I, p. 432.

**5. Proof of Theorem 2.** Since, by Lemma 9, the case $\gamma \in N$ is already proved in [11], we shall only consider the case $\gamma > 0$, $\gamma \notin N$. Without loss of generality we may assume that $\eta_0 = 0$ for any sequence $\eta$ under consideration.

We shall first show that if $\gamma > 1/q$, $1 < q \leqslant \infty$ or $\gamma \geqslant 1$, $q = 1$, and if $\eta_k = g(k)$, $k \in N$, where $g \in WBV_{q,\gamma} \cap L_c^\infty$, then

(5.1) $$\|\eta\|_{q,\gamma;w} \leqslant C\|g\|_{q,\gamma;W},$$

which implies, by Lemmas 2 and 10, that $wbv(q,\gamma) \subset wbv_{q,\gamma}$.

Consider the case $q = \infty$ and let $2^{m-1} \leqslant k < 2^m$. Looking at the proof of Lemma 3 it is clear that (3.1) even holds for $\gamma > 0$ provided $g$ has compact support. Thus

$$\begin{aligned}|\Delta^\gamma\eta_k| &\leqslant C\Big|\sum_{i=k}^\infty A_{i-k}^{-\gamma-1}\sum_{l=i}^\infty\int_l^{l+1}(s-i)^{\gamma-1}g^{(\gamma)}(s)\,ds\Big| \\ &\leqslant C\sum_{l=k}^\infty\int_l^{l+1}|g^{(\gamma)}(s)|\Big|\sum_{i=k}^{l}(s-i)^{\gamma-1}A_{i-k}^{-\gamma-1}\Big|\,ds.\end{aligned}$$

For $\gamma > 0$, $l < s < l+1$,

(5.2) $$\Big|\sum_{i=k}^{l}(s-i)^{\gamma-1}A_{i-k}^{-\gamma-1}\Big| \leqslant C(l+1-k)^{-\gamma-1}\big((s-l)^{\gamma-1}+1\big),$$

(use $\alpha = \gamma$, $n = [2\gamma]$ in [30], pp. 560–561). Hence

$$|A_k^\gamma \Delta^\gamma \eta_k| \leqslant C A_k^\gamma \sum_{l=k}^{\infty} (l+1-k)^{-\gamma-1} \int_l^{l+1} |g^{(\gamma)}(s)| \big((s-l)^{\gamma-1}+1\big)\, ds$$

$$\leqslant C \|g\|_{\infty,\gamma;W} k^{-\gamma} A_k^\gamma \sum_{l=k}^{\infty} (l+1-k)^{-\gamma-1} \leqslant C \|g\|_{\infty,\gamma;W}.$$

Now let $1 \leqslant q < \infty$ and $1/p + 1/q = 1$. Then (5.2) gives

$$|\Delta^\gamma \eta_k| \leqslant C \sum_{l=k}^{\infty} (l+1-k)^{-\gamma-1} \Big(\int_l^{l+1} |g^{(\gamma)}(s)|^q ds\Big)^{1/q} \Big(\int_l^{l+1} (s-l)^{p(\gamma-1)} ds\Big)^{1/p},$$

where the last factor is uniformly bounded if $\gamma > 1/q$, $1 < q < \infty$, or if $\gamma \geqslant 1$, $q = 1$. Hence

$$\Big(\sum_{k=2^{m-1}}^{2^m-1} k^{-1} |A_k^\gamma \Delta^\gamma \eta_k|^q\Big)^{1/q}$$

$$\leqslant C \Big\{ \sum_{k=2^{m-1}}^{2^m-1} k^{-1} A_k^{\gamma q} \sum_{l=k}^{\infty} (l+1-k)^{-\gamma-1} \int_l^{l+1} |g^{(\gamma)}(s)|^q ds \Big(\sum_{j=k}^{\infty} (j+1-k)^{-\gamma-1}\Big)^{q/p} \Big\}^{1/q}$$

$$\leqslant C \Big\{ \sum_{k=2^{m-1}}^{2^m-1} \cdots \sum_{l=k}^{2^m-1} \cdots \int_l^{l+1} \cdots ds \Big\}^{1/q} + C \Big\{ \sum_{k=2^{m-1}}^{2^m-1} \cdots \sum_{l=2^m}^{2^{m+1}-1} \cdots \int_l^{l+1} \cdots ds \Big\}^{1/q} +$$

$$+ C \sum_{i=m+2}^{\infty} \Big\{ \sum_{k=2^{m-1}}^{2^m-1} \cdots \sum_{l=2^{i-1}}^{2^i-1} \cdots \int_l^{l+1} \cdots ds \Big\}^{1/q} = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

A change in order of summation gives

$$\Sigma_1 \leqslant C \Big( \sum_{l=2^{m-1}}^{2^m-1} l^{-1} A_l^{\gamma q} \int_l^{l+1} |g^{(\gamma)}(s)|^q ds \sum_{k=2^{m-1}}^{l} (l+1-k)^{-\gamma-1} \Big)^{1/q} \leqslant C \|g\|_{q,\gamma;W}$$

uniformly in $m$. $\Sigma_2$ and $\Sigma_3$ are estimated analogously and so (5.1) holds.

To prove the converse for $\gamma > 1/q$, $1 < q \leqslant \infty$, and $\gamma \geqslant 1$, $q = 1$, we first suppose that $\eta \in \mathrm{wbv}_{q,\gamma}$ has compact support. Fix $j = [\gamma]+2$ and as in [11] (note that $j \geqslant 2$ and our $\Delta^i$ operator differs from that in [11] by the factor $(-1)^i$) let

$$P_k(t) = \sum_{i=0}^{j-1} \binom{t-k}{i} (-1)^i \Delta^i \eta_k$$

and

$$(5.3) \qquad g(t) = P_k(t) + \frac{(-1)^j}{2} \begin{cases} -\psi_k(t)\, \Delta^j \eta_{k-1}, & k \leqslant t \leqslant k+1/2, \\ \psi_{k+1}(t)\, \Delta^j \eta_k, & k+1/2 \leqslant t \leqslant k+1, \end{cases}$$

where $\psi_k(t) = \psi(t-k)$ and $\psi(t)$ is an infinitely differentiable function with

$$\psi(t) := \begin{cases} \binom{t}{j-1}, & |t| < 1/6, \\ 0, & |t| \geqslant 1/3. \end{cases}$$



Then $g(t)$ is a bounded function which vanishes identically for large $t$ and satisfies $g(k) = \eta_k$ for $k \in \mathbf{N}$. Also $g, g^{(1)}, \ldots, g^{(j-1)} \in \mathrm{AC}[0, \infty)$ and

$$(5.4) \qquad g^{(j)}(t) = \frac{(-1)^j}{2} \begin{cases} -\psi_k^{(j)}(t)\, \Delta^j \eta_{k-1}, & k \leqslant t \leqslant k+1/2, \\ \psi_{k+1}^{(j)}(t)\, \Delta^j \eta_k, & k+1/2 \leqslant t \leqslant k+1, \end{cases}$$

for $k = 0, 1, \ldots$ Since it is clear that $g \in \mathrm{WBV}_{1,j} \subset \mathrm{WBV}_{q,\gamma}$, we now have to show that $\|g\|_{q,\gamma;W} \leqslant C \|\eta\|_{q,\gamma;w}$ with $C$ independent of $\eta$. By (3.2)

$$g^{(\gamma)}(t) = C \int_t^{\infty} (s-t)^{\mu} g^{(j)}(s)\, ds, \qquad \mu = [\gamma]+1-\gamma,$$

and hence, for $k \leqslant t \leqslant k+1$,

$$(5.5) \qquad g^{(\gamma)}(t) = C \sum_{i=k+1}^{\infty} \int_i^{i+1/2} (s-t)^{\mu} \big(-\psi_i^{(j)}(s)\, \Delta^j \eta_{i-1}\big)\, ds +$$

$$+ C \sum_{i=k+1}^{\infty} \int_{i+1/2}^{i+1} (s-t)^{\mu} \big(\psi_{i+1}^{(j)}(s)\, \Delta^j \eta_i\big)\, ds +$$

$$+ C \int_t^{k+1} (s-t)^{\mu} g^{(j)}(s)\, ds = I_1 + I_2 + I_3.$$

Since $\psi_i^{(j)}(i+s) = \psi^{(j)}(s)$ and, by (2.1),

$$\Delta^j \eta_k = \Delta^{\mu+1}(\Delta^\gamma \eta_k) = \sum_{l=0}^{\infty} A_l^{-\mu-2} \Delta^\gamma \eta_{k+l},$$

it follows that

$$|I_1| \leqslant C \int_0^{1/2} \Big| \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (i+k+1+s-t)^{\mu} A_l^{-\mu-2} \Delta^\gamma \eta_{i+k+l} \Big|\, |\psi^{(j)}(s)|\, ds.$$

Now set $i = n-l$ and use (5.2) to obtain

$$|I_1| \leqslant C \int_0^{1/2} \sum_{n=0}^{\infty} |\Delta^\gamma \eta_{n+k}| \Big| \sum_{l=0}^{n} (n-l+k+1+s-t)^{\mu} A_l^{-\mu-2} \Big| |\psi^{(j)}(s)|\, ds$$

$$\leqslant C \sum_{n=0}^{\infty} (n+1)^{-\mu-2} |\Delta^\gamma \eta_{n+k}|$$

since $0 \leqslant k+1+s-t \leqslant 3/2$ and $|\psi^{(j)}(s)| \leqslant C$. Similarly,

$$|I_2| \leqslant C \sum_{n=0}^{\infty} (n+1)^{-\mu-2} |\Delta^\gamma \eta_{n+k+1}|.$$

Since, by (5.4),

$$|g^{(j)}(s+k)| \leqslant C(|\Delta^j \eta_{k-1}| + |\Delta^j \eta_k|), \qquad 0 \leqslant s \leqslant 1,$$

we have

$$|I_3| \leqslant C(|\Delta^j \eta_{k-1}|+|\Delta^j \eta_k|) \tag{5.6}$$

$$\leqslant C\sum_{n=0}^{\infty} |A_n^{-\mu-2}|(|\Delta^\gamma \eta_{k+n-1}|+|\Delta^\gamma \eta_{k+n}|)$$

$$\leqslant C\sum_{n=0}^{\infty} (n+1)^{-\mu-2}(|\Delta^\gamma \eta_{k+n-1}|+|\Delta^\gamma \eta_{k+n}|)$$

and hence for $k \leqslant t \leqslant k+1$, $\gamma > 0$,

$$|g^{(\gamma)}(t)| \leqslant C\sum_{n=0}^{\infty} (n+1)^{-\mu-2}(|\Delta^\gamma \eta_{k+n-1}|+|\Delta^\gamma \eta_{k+n}|+|\Delta^\gamma \eta_{k+n+1}|) \tag{5.7}$$

by (5.5). Therefore $\|g^{(\gamma)}\|_\infty \leqslant C\|\eta\|_\infty$ and if $m$ is a non-positive integer, then

$$\Big(\int_{2^{m-1}}^{2^m} |t^\gamma g^{(\gamma)}(t)|^q \frac{dt}{t}\Big)^{1/q} \leqslant C\|\eta\|_\infty \leqslant C\|\eta\|_{q,\gamma;w}.$$

If $m$ is a positive integer and $q=1$, $\gamma \geqslant 1$, then (5.7) gives

$$\int_{2^{m-1}}^{2^m} t^{\gamma-1}|g^{(\gamma)}(t)|\,dt$$

$$\leqslant C\sum_{k=2^{m-1}}^{2^m-1}\int_k^{k+1} t^{\gamma-1}\sum_{n=0}^{\infty}(n+1)^{-\mu-2}(|\Delta^\gamma \eta_{k+n-1}|+|\Delta^\gamma \eta_{k+n}|+|\Delta^\gamma \eta_{k+n+1}|)\,dt$$

$$\leqslant C\sum_{n=0}^{\infty}(n+1)^{-\mu-2}\sum_{k=2^{m-1}}^{2^m-1}\sum_{l=n-1}^{n+1} A_{k+l}^{\gamma-1}|\Delta^\gamma \eta_{k+l}| \leqslant C\|\eta\|_{1,\gamma;w}$$

since

$$\sum_{k=2^{m-1}}^{2^m-1} A_{k+i}^{\gamma-1}|\Delta^\gamma \eta_{k+i}| \leqslant 2\|\eta\|_{1,\gamma;w}$$

uniformly in $i$. If $q=\infty$, the desired estimate follows immediately from (5.7). If $1<q<\infty$, $\gamma > 1/q$, then $t^{q\gamma-1} \leqslant Ck^{-1}(A_{k+l}^\gamma)^q$, $k \leqslant t \leqslant k+1$, and so application of Hölder's and Minkowski's inequalities to (5.7) gives

$$\Big(\int_{2^{m-1}}^{2^m} |t^\gamma g^{(\gamma)}(t)|^q \frac{dt}{t}\Big)^{1/q} \leqslant C\sum_{n=0}^{\infty}(n+1)^{-\mu-2}\sum_{l=n-1}^{n+1}\Big(\sum_{k=2^{m-1}}^{2^m-1} k^{-1}|A_{k+l}^\gamma \Delta^\gamma \eta_{k+l}|^q\Big)^{1/q}$$

$$\leqslant C\|\eta\|_{q,\gamma;w}$$

which completes the proof for the case where $\eta$ has compact support.

For arbitrary $\eta \in \mathrm{wbv}_{q,\gamma}$, with $\eta_0=0$, let $g$ be the function defined by (5.3). Also let $u \geqslant 1$ and $f_u$ be the function constructed in the same way



as $g$, but with the sequence $\{\eta_k\}$ replaced by the sequence $\{G(k/u)\eta_k\} \in l_c^\infty$, where $G$ is as defined in Lemma 2. Then $g(k)=\eta_k$, $f_u(k)=G(k/u)\eta_k$ for $k \in \boldsymbol{N}$ and $f_u \in L_c^\infty$. Since $\Delta^j \eta_k$ depends only on $\eta_k, \ldots, \eta_{k+j}$ and $G(k/u)=1$, $k \leqslant u$, it follows for $0 \leqslant t \leqslant [u]-j+1$ that both $g(t)$ and $f_u(t)$ depend only on $\eta_0, \ldots, \eta_{[u]}$; hence $g(t)=f_u(t)$, $0 \leqslant t \leqslant [u]-j+1$. With $1 \leqslant v \leqslant ([u]-j+1)/d$ it then follows that

$$g_v(t) = \big(G_v(t)-G_{1/v}(t)\big)g(t) = \big(G_v(t)-G_{1/v}(t)\big)f_u(t), \qquad t \geqslant 0.$$

Thus, by Lemma 2, the above compact support case, and Lemma 10,

$$\|\eta\|_{q,\gamma;w} \geqslant C\|\{G(k/u)\eta_k\}\|_{q,\gamma;w} \geqslant C\|f_u\|_{q,\gamma;W}$$

$$\geqslant C\|(G_v-G_{1/v})f_u\|_{q,\gamma;W} = C\|g_v\|_{q,\gamma;W}$$

uniformly in $v \geqslant 1$, which completes the proof by an application of Lemma 10.

**6. Proof of Theorem 3.** Since the case $\gamma \in \boldsymbol{N}$ is already proved in Connett and Schwartz [11], we shall assume that $\gamma > 1/q$, $1<q<\infty$, $\gamma \notin \boldsymbol{N}$.

First observe that if $g \in S(q,\gamma)$ and we have proved that $g_u^{(\gamma-k)} \in \mathrm{AC}_{\mathrm{loc}}$ for $u>0$, $k=1,\ldots,[\gamma]$, where $g_u$ is the function defined in Lemma 10, then it easily follows that $g^{(\gamma-k)} \in \mathrm{AC}_{\mathrm{loc}}$. For if $0<a\leqslant t\leqslant b$ and $u$ is chosen so large that $g-g_u=0$ on $[a, b+1]$, then

$$(g-g_u)^{(\delta)}(t) = -\frac{\delta}{\Gamma(1-\delta)}\int_{b+1}^{\infty}(s-t)^{-\delta-1}(g-g_u)(s)\,ds$$

for $\delta=\gamma-[\gamma]$ and this function can be repeatedly differentiated for $a \leqslant t \leqslant b$. Thus $g^{(\gamma-k)} = (g_u)^{(\gamma-k)}+(g-g_u)^{(\gamma-k)} \in \mathrm{AC}_{\mathrm{loc}}$. Similarly, if $I_\omega^{1-\delta}(g_u) \in \mathrm{AC}_{\mathrm{loc}}$ for $\omega, u>0$, then $I_\omega^{1-\delta}(g) \in \mathrm{AC}_{\mathrm{loc}}$. Therefore, by Lemma 10 and the fact [11] that

$$\|g\|_{S(q,\gamma)} \cong \sup_{u>0}\|g_u\|_{S(q,\gamma)},$$

in proving Theorem 3 it suffices to consider only functions with compact support in $(0,\infty)$.

Also observe that in the definition of $\mathrm{WBV}_{q,\gamma}$ spaces the fact that we decompose $(0,\infty)$ by intervals of the form $[2^{m-1}, 2^m]$ is unessential; see Remark 6.2.2 in [20], p. 109. In what follows it is more convenient to replace $[2^{m-1}, 2^m]$ by $[e^m, e^{m+1}]$. Furthermore, if $g \in S(q,\gamma)$, $\gamma > 1/q$, then $\|g\|_\infty \leqslant C\|g\|_{S(q,\gamma)}$, [11], while if $g \in \mathrm{WBV}_{q,\gamma}$, then $\|\varphi_k g^*\|_q \leqslant C\|g\|_\infty \leqslant C\|g\|_{q,\gamma;W}$, where $\varphi_k$ and $g^*$ are as defined in Section 1. These observations reduce the proof to showing that if $g \in S(q,\gamma)\cap L_c^\infty$, then $g$ satisfies the $\mathrm{AC}_{\mathrm{loc}}$ conditions in the definition of $\mathrm{WBV}_{q,\gamma}$ and

$$\Big(\int_{e^m}^{e^{m+1}} |t^\gamma g^{(\gamma)}(t)|^q \frac{dt}{t}\Big)^{1/q} \leqslant C\|g\|_{S(q,\gamma)} \tag{6.1}$$

uniformly in $m\in\mathbf{Z}$, and, conversely, by Lemma 11, that if $g\in \mathrm{WBV}_{q,\gamma}\cap L_c^\infty$, then

$$\left\| \frac{1}{\Gamma(-\delta)}\int_0^\infty s^{-1-\delta}\Delta_s(\varphi_k g^*)^{([\gamma])}(t)\,ds\right\|_q \leqslant C\|g\|_{q,\gamma;W}^{\ast} \tag{6.2}$$

uniformly for $k\in\mathbf{Z}$.

We shall first show for $g\in S(q,\gamma)$ that

$$\sum_{k=m}^{\infty}\Bigg(\int_{e^m}^{e^{m+1}}\left|\frac{t^\gamma}{\Gamma(-\gamma)}\int_0^\infty s^{-1-\delta}\Delta_s\big(\varphi_k^2(\log t)g(t)\big)^{([\gamma])}ds\right|^q\frac{dt}{t}\Bigg)^{1/q}\leqslant C\|g\|_{S(q,\gamma)} \tag{6.3}$$

uniformly in $m$. After (6.3) is established we will then show that

$$\big(\varphi_k^2(\log t)g(t)\big)^{(\gamma)}=\frac{1}{\Gamma(-\delta)}\int_0^\infty s^{-1-\delta}\Delta_s\big(\varphi_k^2(\log t)g(t)\big)^{([\gamma])}ds,\quad \text{a.e.} \tag{6.4}$$

which then yields (6.1) because

$$\Bigg(\int_{e^m}^{e^{m+1}}|t^\gamma g^{(\gamma)}(t)|^q\frac{dt}{t}\Bigg)^{1/q}$$
$$\leqslant\sum_{k=m}^{\infty}\Bigg(\int_{e^m}^{e^{m+1}}\big|t^\gamma\varphi_k^2(\log t)g(t))^{(\gamma)}\big|^q\frac{dt}{t}\Bigg)^{1/q}\leqslant C\|g\|_{S(q,\gamma)}.$$

In the following estimates we will omit the factor $1/\Gamma(-\delta)$, keeping in mind that it keeps the constant multiples of $1/\delta$ and $1/(1-\delta)$ occurring in the integrations bounded. With the aid of

$$t^k g^{(k)}(t)=\sum_{j=0}^{k}A_{j,k}\Big(t\frac{d}{dt}\Big)^j g(t) \tag{6.5}$$

and the successive substitutions $s=r-t$, $r=e^v$, $t=e^u$, the left side of (6.3) can be estimated by

$$C\sum_{k=m}^{\infty}\sum_{j=0}^{[\gamma]}\Bigg(\int_m^{m+1}\Big|\int_u^\infty(1-e^{u-v})^{-1-\delta}\{e^{(u-v)\gamma}(\varphi_k^2g^*)^{(j)}(v)-$$
$$-e^{(u-v)\delta}(\varphi_k^2g^*)^{(j)}(u)\}\,dv\Big|^q du\Bigg)^{1/q}\equiv C\sum_k\sum_j I_{j,k}.$$

Consider the single therms of the sum. The most critical ones occurring are those with $k=m$ or $k=m+1$. Let us first discuss

$$I_{j,m}\equiv\Bigg(\int_m^{m+1}\Big|\int_0^\infty(1-e^{-t})^{-1-\delta}\{e^{-t\gamma}(\varphi_m^2g^*)^{(j)}(t+u)-e^{-t\delta}(\varphi_m^2g^*)^{(j)}(u)\}\,dt\Big|^q du\Bigg)^{1/q}.$$



Since $\varphi_m(u)=0$ for $u\geqslant m+2/3$,

$$I_{j,m}\leqslant\Bigg(\int_m^{m+1}\Big|\int_{2/3}^\infty(1-e^{-t})^{-1-\delta}e^{-t\delta}(\varphi_m^2g^*)^{(j)}(u)\,dt\Big|^q du\Bigg)^{1/q}+\Bigg(\int_m^{m+1}\Big|\int_0^{2/3}\ldots\Big|^q du\Bigg)^{1/q}$$
$$=J_{j,m}+K_{j,m}.$$

By the integral Minkowski inequality and a result of Strichartz [22], § 3, it follows for $j\leqslant[\gamma]$ that

$$J_{j,m}\leqslant C\|\varphi_m^2g^*\|_{q,j}\leqslant C\|\varphi_m g^*\|_{q,\gamma}\leqslant C\|g\|_{S(q,\gamma)},$$

where we used the fact that if $h\in L_\lambda^q$, then

$$\|h\|_{q,\mu}\leqslant\|h\|_{q,\lambda},\qquad 0\leqslant\mu<\lambda. \tag{6.6}$$

In view of Lemma 11 we want to replace $\int_0^{2/3}$ in $K_{j,m}$ by

$$\int_0^\infty t^{-1-\delta}\Delta_t(\varphi_m^2g^*)^{(j)}(u)\,dt=\int_0^{2/3}+\int_{2/3}^\infty.$$

As in the estimation of $J_{j,m}$ we have

$$\Bigg(\int_m^{m+1}\Big|\int_{2/3}^\infty t^{-1-\delta}\Delta_t(\varphi_m^2g^*)^{(j)}(u)\,dt\Big|^q du\Bigg)^{1/q}\leqslant C\|g\|_{S(q,\gamma)};$$

so it remains to estimate

$$\Bigg(\int_m^{m+1}\Big|\int_0^{2/3}(1-e^{-t})^{-1-\delta}\{e^{-t\gamma}(\varphi_m^2g^*)^{(j)}(t+u)-e^{-t\delta}(\varphi_m^2g^*)^{(j)}(u)\}\,dt-$$
$$-\int_0^{2/3}t^{-1-\delta}\Delta_t(\varphi_m^2g^*)^{(j)}(u)\,dt\Big|^q du\Bigg)^{1/q}.$$

Since

$$1-e^{-t}\geqslant te^{-1},\qquad e^{-t\gamma}-1>-\gamma t,\qquad (1-e^{-t})^{-1-\delta}-t^{-1-\delta}=O(t^{-\delta})$$

for $0<t<1$, the above formula can be estimated after routine reformulations by

$$C\Bigg(\int_m^{m+1}\Big|\int_0^{2/3}t^{-\delta}\{|(\varphi_m^2g^*)^{(j)}(u)|+|(\varphi_m^2g^*)^{(j)}(t+u)|\}\,dt\Big|^q du\Bigg)^{1/q}$$
$$\leqslant C\|\varphi_m^2g^*\|_{q,j}\int_0^{2/3}t^{-\delta}dt\leqslant C\|g\|_{S(q,\gamma)}.$$

So a combination of these estimates yields

$$I_{j,m}\leqslant C\|g\|_{S(q,\gamma)}+C\Bigg(\int_m^{m+1}\Big|\int_0^\infty t^{-1-\delta}\Delta_t(\varphi_m^2g^*)^{(j)}(u)\,dt\Big|^q du\Bigg)^{1/q}$$
$$\leqslant C\|g\|_{S(q,\gamma)}+C\|\varphi_m^2g^*\|_{q,j+\delta}\leqslant C\|g\|_{S(q,\gamma)}$$

by (6.6) and Lemma 11. Similarly,

$$I_{j,m+1}\leqslant C\,\|g\|_{S(q,\gamma)},$$

so it remains to estimate $\sum\limits_{k=m+2}^{\infty}\sum\limits_{j=0}^{[\gamma]} I_{j,k}$. Since supp $\varphi_k\subset[k-2/3,\,k+2/3]$,

$$\begin{aligned}
&\sum_{k=m+2}^{\infty}\sum_{j=0}^{[\gamma]} I_{j,k}\\
&\leqslant \sum_{k=m+2}^{\infty}\sum_{j=0}^{[\gamma]}(1-e^{-1/3})^{-1-\delta}\Big(\int\limits_{m}^{m+1}\Big|\int\limits_{k-m-5/3}^{k-m+2/3} e^{-t\gamma}|(\varphi_k^2 g^*)^{(j)}(t+u)|\,dt\Big|^q du\Big)^{1/q}\\
&\leqslant C\sum_{k=m+2}^{\infty}\sum_{j=0}^{[\gamma]}\int\limits_{k-m-5/3}^{\infty} e^{-t\gamma}\,\|\varphi_k^2 g^*\|_{q,j}\,dt\\
&\leqslant Ce^{(m+5/3)\gamma}\sum_{k=m+2}^{\infty}\|\varphi_k g^*\|_{q,\gamma}\,e^{-\gamma k}\leqslant C\|g\|_{S(q,\gamma)}
\end{aligned}$$

and hence (6.3) is established.

To consider (6.4), we set

$$g_k(t)=\begin{cases}\varphi_k^2(\log t)\,g(t), & t>0,\\ 0, & t\leqslant 0.\end{cases}$$

Since $S(q,\gamma)\subset S(q,[\gamma])$ and $g_k$ has compact support, it is clear that $g_k,\ g_k^{([\gamma])}\in L^1(-\infty,\infty)$. By (6.3)

$$h_k(t)\equiv\frac{1}{\Gamma(-\delta)}\int^{\infty} s^{-1-\delta}\Delta_s g_k^{([\gamma])}(t)\,ds\in L^1(e^{m-1},\infty),\qquad k\geqslant m$$

for each $m\in\boldsymbol{Z}$, and, by a direct elementary calculation, it also belongs to $L^1(-\infty,e^{m-1})$. Hence we may take the (classical) Fourier transform $(h_k)^\wedge(v)=(-1)^{[\gamma]}(-iv)^\gamma\hat{g_k}(v)$. On the other hand, in $S'$ there also holds

$$\begin{aligned}
(g_k^{(\gamma)})^\wedge(v)&=\left(-\left(\frac{d}{dt}\right)^{[\gamma+1]}\frac{1}{\Gamma(1-\delta)}\int\limits_t^{\infty}(s-t)^{-\delta}g_k(s)\,ds\right)^\wedge(v)\\
&=(-1)^{[\gamma]}(-iv)^\gamma\hat{g_k}(v).
\end{aligned}$$

Hence, by the uniqueness theorem for Fourier transforms, (6.4) holds and to complete the proof of $S(q,\gamma)\subset \mathrm{WBV}_{q,\gamma}$ it only remains to show that $I_\omega^{1-\delta}(g),\ g^{(\delta)},\ldots,g^{(\gamma-1)}\in \mathrm{AC}_{\mathrm{loc}}$.

Since $g$ has compact support, it follows as in the proof of Lemma 10 that $I_\omega^{1-\delta}(g)\in \mathrm{AC}_{\mathrm{loc}}$, $\omega>0$. Also, from the above, if $\gamma>1$, then

$$(-iv)^{[\gamma]}(g_k^{(\delta)})^\wedge(v)=(-iv)^\gamma\hat{g_k}(v)=(-1)^{[\gamma]}\hat{h_k}(v)$$



and hence, by [6], Prop. 5.1.15, $g_k^{(\delta)},\ldots,g_k^{(\gamma-1)}\in \mathrm{AC}_{\mathrm{loc}}$. But $g$ is a finite sum of the $g_k$ and fractional differentiation is linear, so $g^{(\delta)},\ldots,g^{(\gamma-1)}\in \mathrm{AC}_{\mathrm{loc}}$.

As mentioned before, to prove the converse we have to show that if $g\in \mathrm{WBV}_{q,\gamma}\cap L_c^\infty$, $\gamma>1/q$, $1<q<\infty$, then (6.2) holds. By the integer case of the theorem,

$$\begin{aligned}
\Big\|\int\limits_0^\infty s^{-1-\delta}\Delta_s(\varphi_k g^*)^{([\gamma])}(t)\,ds\Big\|_q&\leqslant\Big\|\int\limits_0^1\ldots\,ds\Big\|_q+C\int\limits_1^\infty s^{-1-\delta}\|\varphi_k g^*\|_{q,[\gamma]}\,ds\\
&\leqslant I_1+C\|g\|_{q,\gamma;W}.
\end{aligned}$$

We shall handle the term $I_1$ by essentially reading the proof of (6.3) backwards. So again by the integer case,

$$\begin{aligned}
I_1&\leqslant\Big\|\int\limits_0^1(1-e^{-s})^{-1-\delta}\Delta_s(\varphi_k g^*)^{([\gamma])}(t)\,ds\Big\|_q+\\
&\qquad+\Big\|\int\limits_0^1\Delta_s(\varphi_k g^*)^{([\gamma])}(t)\,\{s^{-1-\delta}-(1-e^{-s})^{-1-\delta}\}\,ds\Big\|_q\\
&=I_2+C\|g\|_{q,\gamma;W},
\end{aligned}$$

where

$$\begin{aligned}
I_2&\leqslant\Big\|\int\limits_0^1(1-e^{-s})^{-1-\delta}(1-e^{-s\gamma})(\varphi_k g^*)^{([\gamma])}(s+t)\,ds\Big\|_q+\\
&\quad+\Big\|\int\limits_0^1(1-e^{-s})^{-1-\delta}(1-e^{-s\delta})(\varphi_k g^*)^{([\gamma])}(t)\,ds\Big\|_q+\\
&\quad+\Big\|\int\limits_0^\infty(1-e^{-s})^{-1-\delta}\{e^{-s\gamma}(\varphi_k g^*)^{([\gamma])}(s+t)-e^{-s\delta}(\varphi_k g^*)^{([\gamma])}(t)\}\,ds\Big\|_q+\\
&\quad+\Big\|\int\limits_1^\infty(1-e^{-s})^{-1-\delta}\{\ldots\}\,ds\Big\|_q=I_3+I_4+I_5+I_6.
\end{aligned}$$

Clearly, by the integral Minkowski inequality,

$$I_3+I_4+I_6\leqslant C\|g\|_{q,\gamma;W};$$

so the essential term is

$$\begin{aligned}
I_5=\Big(\int\limits_{-\infty}^{\infty}\Big|\int\limits_t^{\infty}(1-e^{-(v-t)})^{-1-\delta}\{&e^{-(v-t)\gamma}(\varphi_k g^*)^{([\gamma])}(v)-\\
&-e^{-(v-t)\delta}(\varphi_k g^*)^{([\gamma])}(t)\}\,dv\Big|^q dt\Big)^{1/q}.
\end{aligned}$$

By first substituting $e^t = u$ and then $e^v = s$, observing that $\operatorname{supp}(\Delta_s\varphi_k(\log u)g(u)) \cap [e^{k+1}, \infty)$ is empty, and then letting $s = t+u$, one arrives at

$$I_5 \leqslant C\Bigg(\int\limits_0^{e^{k+1}} \Bigg| u^\gamma \int\limits_0^\infty t^{-1-\delta}\Big\{(t+u)^{-[\gamma]}\Big((t+u)\frac{d}{dt}\Big)^{[\gamma]}\big(\varphi_k(\log(t+u))\,g(t+u)\big) - u^{-[\gamma]}\Big(u\frac{d}{du}\Big)^{[\gamma]}\big(\varphi_k(\log u)\,g(u)\big)\Big\}\,dt\Bigg|^q \frac{du}{u}\Bigg)^{1/q}$$

$$\leqslant C\sum_{j=0}^\infty \Bigg(\int\limits_{e^{k-2j-1}}^{e^{k-2j+1}} \Big| u^\gamma \int\limits_0^\infty \ldots\, dt\Big|^q \frac{du}{u}\Bigg)^{1/q} = \sum_{j=0}^\infty J_j.$$

Setting $f_k(u) = \varphi_k(\log u)\,g(u)$, $u > 0$, and using the fact that

$$\text{(6.7)}\qquad \Big(t\frac{d}{dt}\Big)^n g(t) = \sum_{i=0}^n a_{i,n}t^i g^{(i)}(t), \qquad a_{0,n} = 0 \quad \text{for} \quad n \geqslant 1,$$

we have

$$J_0 \leqslant C\sum_{i=0}^{[\gamma]}\Bigg(\int\limits_{e^{k-1}}^{e^{k+1}} \Big| u^\gamma \int\limits_0^\infty t^{-1-\delta}\Delta_t\big(u^{i-[\gamma]}f_k^{(i)}(u)\big)\,dt\Big|^q\frac{du}{u}\Bigg)^{1/q}$$

$$\leqslant C\sum_{i=0}^{[\gamma]}\Bigg(\int\limits_{e^{k-1}}^{e^{k+1}} \Big| u^{i+\delta} \int\limits_0^\infty t^{-1-\delta}\Delta_t f_k^{(i)}(u)\,dt\Big|^q\frac{du}{u}\Bigg)^{1/q} +$$

$$+ C\sum_{i=1}^{[\gamma]-1}\Bigg(\int\limits_{e^{k-1}}^{e^{k+1}} \Big| u^{i+\delta} \int\limits_0^\infty t^{-1-\delta}\Big\{\Big(\frac{t}{u}+1\Big)^{i-[\gamma]}-1\Big\}f_k^{(i)}(t+u)\,dt\Big|^q\frac{du}{u}\Bigg)^{1/q}.$$

By (3.13), (3.16) and Lemma 9 the first sum on the right side is majorized by $C\|g\|_{q,\gamma;W}$. In the second sum the substitution $t = us$ leads to

$$\sum_{i=1}^{[\gamma]-1}\Bigg(\int\limits_{e^{k-1}}^{e^{k+1}} \Big| u^i \int\limits_0^\infty s^{-1-\delta}\{(s+1)^{i-[\gamma]}-1\}f_k^{(i)}\big(u(s+1)\big)\,ds\Big|^q\frac{du}{u}\Bigg)^{1/q}$$

$$\leqslant C\sum_{i=1}^{[\gamma]-1}\|f_k\|_{q,i;W}\int\limits_0^\infty s^{-1-\delta}(s+1)^{-i}\{(s+1)^{i-[\gamma]}-1\}\,ds \leqslant C\|g\|_{q,\gamma;W}$$

by Lemmas 9 and 10. Thus it remains to estimate $\sum\limits_1^\infty J_j$. Since $\operatorname{supp} f_k \subset \{u\colon e^{k-2/3} \leqslant u \leqslant e^{k+2/3}\}$, setting $m = k-2j$ and using (6.7) we have for $j \geqslant 1$

$$J_j \leqslant C\sum_{i=0}^{[\gamma]}\Bigg(\int\limits_{e^{m-1}}^{e^{m+1}} \Big| u^\delta \int\limits_{e^{k-2/3}-e^{m+1}}^{e^{k+2/3}-e^{m-1}} t^{-1-\delta}(t+u)^i\,|f_k^{(i)}(t+u)|\,dt\Big|^q\frac{du}{u}\Bigg)^{1/q}$$

$$\leqslant C\,e^{(m+1)\delta}(e^k)^{-\delta}\|g\|_{q,\gamma;W}$$

which concludes the proof since $(e^k)^{-\delta}\sum\limits_{j=1}^\infty e^{(k+1-2j)\delta} = O(1)$.



**7. Imbeddings.** Our results lead to some important information concerning the imbedding behavior of the $\mathrm{WBV}_{q,\gamma}$ and $\mathrm{wbv}_{q,\gamma}$ spaces.

THEOREM 4. *Let $\gamma, \mu > 0$.*

(a) *If* $1 \leqslant p \leqslant q \leqslant \infty$, *then*

$$\mathrm{WBV}_{q,\gamma} \subset \mathrm{WBV}_{p,\gamma};$$

(b) *If* $1 < p < q < \infty$, $1/p - 1/q \leqslant \gamma - \mu$, $\gamma > 1/p$, *then*

$$\mathrm{WBV}_{p,\gamma} \subset \mathrm{WBV}_{q,\mu};$$

(c) *If* $1 \leqslant q < \infty$, $1 - 1/q < \gamma - \mu$, *then*

$$\mathrm{WBV}_{1,\gamma} \subset \mathrm{WBV}_{q,\mu};$$

(d) $$\mathrm{WBV}_{1,\gamma} \subset \mathrm{WBV}_{\infty,\gamma-1} \quad \textit{for} \quad \gamma \geqslant 1;$$

(e) *If* $1/q < \gamma - \mu$, $1 < q \leqslant \infty$, *then*

$$\mathrm{WBV}_{q,\gamma} \subset \mathrm{WBV}_{\infty,\mu}.$$

Proof. Part (a) follows directly from Hölder's inequality and (b) follows from the well-known imbedding property of Bessel potential spaces [8]. For (c) first note that by (3.2)

$$\Bigg(\int\limits_{2^{m-1}}^{2^m}\Big|t^\mu g^{(\mu)}(t)\Big|^q\frac{dt}{t}\Bigg)^{1/q}$$

$$\leqslant C\Bigg(\int\limits_{2^{m-1}}^{2^m}\Big|t^\mu\int\limits_t^\infty (s-t)^{\gamma-\mu-1}g^{(\gamma)}(s)\,ds\Big|^q\frac{dt}{t}\Bigg)^{1/q}$$

$$\leqslant C\Bigg(\int\limits_{2^{m-1}}^{2^m}\Big|\int\limits_t^{2^m}\ldots\Big|^q\frac{dt}{t}\Bigg)^{1/q} + C\sum_{k=m+1}^\infty\Bigg(\Big|\int\limits_{2^{k-1}}^{2^k}\ldots\Big|^q\frac{dt}{t}\Bigg)^{1/q} = \Sigma_1 + \Sigma_2.$$

Since $(\gamma-\mu-1)q > -1$, the integral Minkowski inequality gives

$$\Sigma_1 \leqslant C\int\limits_{2^{m-1}}^{2^m}|g^{(\gamma)}(s)|\Bigg(\int\limits_{2^{m-1}}^{s}(s-t)^{(\gamma-\mu-1)q}t^{\mu q-1}\,dt\Bigg)^{1/q}ds$$

$$\leqslant C\int\limits_{2^{m-1}}^{2^m}s^{\gamma-1}|g^{(\gamma)}(s)|\,ds \leqslant C\|g\|_{1,\gamma;W}.$$

The sum $\Sigma_2$ is estimated analogously. Finally parts (d) and (e) follow similarly by straightforward computations and so we omit the proof. Note that in (d), $\mathrm{WBV}_{\infty,0} \equiv L^\infty \cap C(0, \infty)$.

On account of Theorem 2 we also have

THEOREM 5. *Let $\gamma, \mu > 0$, then*

(a) *If* $1 \leqslant p \leqslant q \leqslant \infty$, *then*

$$\mathrm{wbv}_{q,\gamma} \subset \mathrm{wbv}_{p,\gamma};$$

(b) *If* $1<p<q<\infty$, $1/p-1/q\leqslant\gamma-\mu$, $\gamma>1/p$, *then*

$$\mathrm{wbv}_{q,\gamma}\subset\mathrm{wbv}_{q,\mu};$$

(c) *If* $1\leqslant q<\infty$, $1-1/q<\gamma-\mu$, $\gamma\geqslant 1$, *then*

$$\mathrm{wbv}_{1,\gamma}\subset\mathrm{wbv}_{q,\mu};$$

(d) *We have*

$$\mathrm{wbv}_{1,\gamma}\subset\mathrm{wbv}_{\infty,\gamma-1}\quad\textit{for}\quad\gamma\geqslant 1;$$

(e) *If* $1/q<\gamma-\mu$, $1<q\leqslant\infty$, *then*

$$\mathrm{wbv}_{q,\gamma}\subset\mathrm{wbv}_{\infty,\mu}.$$

**8. Multipliers for Jacobi expansions.** Fix $\alpha\geqslant\beta\geqslant-1/2$ and let $L^p=L^p_{(\alpha,\beta)}$, $1\leqslant p<\infty$, denote the space of measurable functions $f(x)$ on $[-1,1]$ for which

$$\|f\|_p=\Big(\int_{-1}^{1}|f(x)|^p(1-x)^\alpha(1+x)^\beta\,dx\Big)^{1/p}<\infty.$$

To each $f\in L^p$ there can be associated the formal expansion [24], Chap. 9

$$f(x)\sim\sum_{k=0}^{\infty}a_kh_kP_k^{(\alpha,\beta)}(x),$$

where $P_k^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order $(\alpha,\beta)$,

$$h_k=h_k^{(\alpha,\beta)}=(\|P_k^{(\alpha,\beta)}(x)\|_2)^{-2},$$

and

$$a_k=\int_{-1}^{1}f(x)P_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\,dx.$$

A sequence $\eta\in l^\infty$ is called a *multiplier of type* $(p,q)$, notation $\eta\in M_p^q$, if for each $f\in L^p$ there exists a function $f^\eta\in L^q$ with

$$\text{(8.1)}\qquad f^\eta(x)\sim\sum_{k=0}^{\infty}\eta_ka_kh_kP_k^{(\alpha,\beta)}(x),\qquad\|f^\eta\|_q\leqslant C\|f\|_p,$$

where $C$ is independent of $f$. The smallest constant $C$ independent of $f$ for which (8.1) holds is called the *multiplier norm* of $\eta$, and it is denoted by $\|\eta\|_{M_p^q}$. We note that in deriving multiplier criteria we may assume that $\eta_0=0$.

The following end-point results concerning Jacobi multipliers are known:



THEOREM B. (a) $l^\infty=M_2^2$;

(b) $\mathrm{wbv}_{1,1}\subset M_p^p$ *if* $1\leqslant\dfrac{2\alpha+2}{\alpha+3/2}<p<\dfrac{2\alpha+2}{\alpha+1/2}\leqslant\infty$;

(c) $\mathrm{wbv}_{1,\gamma}\subset M_p^p$ *if* $1<p<\infty$, $\gamma>\alpha+3/2$;

(d) $\mathrm{wbv}_{2,\gamma}\subset M_p^p$ *if* $1<p<\infty$, $\gamma=[\alpha+2]$.

Part (a) is well-known and a consequence of the Parseval formula for complete orthogonal expansions. Part (b) in the case $\alpha=\beta=-1/2$ is the classical Marcinkiewicz multiplier criterion for Fourier series (see, e.g. [32], p. 232); in the ultraspherical case $\alpha=\beta>-1/2$ it is due to Muckenhoupt and Stein [17], and the general case is due to Askey [1]. Part (c) is proved in Gasper and Trebels [13]. Both (b) and (c) are best possible in the sense that in (b) the $p$-range cannot be enlarged, and (c) is not true for $\gamma<\alpha+3/2$. Theorem 3(a) below shows that (c) also holds for $\gamma=\alpha+3/2$. For (d) see Connett and Schwartz [9] and Gasper and Trebels [13].

Our aim here is to derive new multiplier criteria for Jacobi expansions by interpolating between the various end-point results collected in Theorem B. This will be accomplished by using Theorem 1 and the following theorem, established by Connett and Schwartz [11] in connection with ultraspherical multipliers but which immediately carries over to Jacobi polynomials.

THEOREM C. *Define the operator* $T\colon S(q,\gamma)\to M_2^2$ *by*

$$Tg(x)\sim\sum_{k=1}^{\infty}\eta_kh_kP_k^{(\alpha,\beta)}(x),\qquad g\in S(q,\gamma),\qquad g(k)=\eta_k,\qquad k\in\boldsymbol{N},$$

*and suppose that* $T\colon S(q_i,\gamma_i)\to M_{p_i}^{p_i}$, $1<p_i,q_i<\infty$, $\gamma_i>0$, *is continuous for* $i=0,1$. *If* $0<s<1$ *and*

$$(\gamma,1/p,1/q)=(1-s)(\gamma_0,1/p_0,1/q_0)+s(\gamma_1,1/p_1,1/q_1),$$

*then* $T\colon S(q,\gamma)\to M_p^p$ *is also continuous.*

Our multiplier results are contained in

THEOREM 6. *Let* $\alpha\geqslant\beta\geqslant-1/2$, $1<p<\infty$, *and* $\gamma>1/q$, $1\leqslant q\leqslant\infty$.

(a) *If* $\gamma>(2\alpha+2)|1/p-1/2|+1/2$ *and* $\gamma>1$, *then* $\mathrm{wbv}_{1,\gamma}\subset M_p^p$. *This result is best possible in the sense that* $\mathrm{wbv}_{1,\mu}\not\subset M_p^p$ *when* $\mu\leqslant(2\alpha+2)|1/p-1/2|+1/2$.

(b) *If* $\gamma>(4\alpha+4)|1/p-1/2|>1/q$, $\dfrac{2\alpha+2}{\alpha+3/2}<p<\dfrac{2\alpha+2}{\alpha+1/2}$, *then* $\mathrm{wbv}_{q,\gamma}\subset M_p^p$.

(c) *If* $\gamma>(2\alpha+3)|1/p-1/2|$, $q>p/|2-p|$, *then* $\mathrm{wbv}_{q,\gamma}\subset M_p^p$.

(We shall omit the statements of the various criteria arising when Theorem B(d) is one end-point of the interpolation since this end-point is not sharp.)

Proof. Part(a) follows by interpolating between parts (b) and (c) of Theorem B. First let $\varepsilon$, $\delta > 0$ be small. By Theorem 5(a) and Theorems 1–3 the hypotheses of Theorem C are satisfied if we choose

$$\gamma_0 = 3/2+\varepsilon, \quad \gamma_1 = 1, \quad q_0 = q_1 = 1+\delta,$$
$$p_0 = 1+\delta, \quad p_1 = (2\alpha+2)/(\alpha+3/2)+\delta.$$

Hence $T$: $S(q, \gamma') \to M_p^p$ is continuous when

$$0 < s < 1, \quad q = 1+\delta, \quad \gamma' = (1-s)(\alpha+3/2+\varepsilon)+s,$$
$$1/p = (1-s)/(1+\delta)+s/\left(\frac{2\alpha+2}{\alpha+3/2}+\delta\right).$$

Note that $\gamma'-(2\alpha+2)(1/p-1/2)-1/2 > 0$ and tends to zero as $\varepsilon$, $\delta \to 0$. Similarly, with $\gamma_0$, $\gamma_1$, $q_0$, $q_1$ defined as above and $p_0 = \delta^{-1}$, $p_1 = (2\alpha+2)/(\alpha+1/2)-\delta$ we find that $T$: $S(q, \gamma') \to M_p^p$ is continuous when

$$0 < s < 1, \quad q = 1+\delta, \quad \gamma' = (1-s)(\alpha+3/2+\varepsilon)+s,$$
$$1/p = (1-s)\,\delta+s/\left(\frac{2\alpha+2}{\alpha+1/2}-\delta\right),$$

where $\gamma'-(2\alpha+2)(1/2-1/p)-1/2 > 0$ and tends to zero as $\varepsilon$, $\delta \to 0$. By Theorem 5 (c),

$$\mathrm{wbv}_{1,\gamma} \subset \mathrm{wbv}_{q,\gamma'} \subset M_p^p,$$

where $\gamma > \gamma'$ may be chosen arbitrarily near $\gamma'$ since $q = 1+\delta$ may be chosen arbitrarily near 1; hence $\mathrm{wbv}_{1,\gamma} \subset M_p^p$ for $\gamma > (2\alpha+2)|1/p-1/2|+1/2$.

The fact that the $(p, \gamma)$-range is best possible can be seen by the counterexample of the Cesàro kernel. Askey and Hirschman [2] have shown that the sequence $\eta_{\mu,n}$

$$\eta_{\mu,n}(k) = \begin{cases} A_{n-k}^{\mu}/A_n^{\mu}, & 0 \leqslant k \leqslant n,\\ 0, & k > n \end{cases}$$

is not a uniformly (in $n$) bounded $M_p^p$-multiplier family for expansions in ultraspherical polynomials ($\alpha = \beta \geqslant -1/2$) provided $0 \leqslant \mu \leqslant (2\alpha+2)|1/p-1/2|-1/2$. Their proof carries over to the general Jacobi polynomial case (use [24], § 9.41 and Ex. 91) and so $\eta_{\mu,n}$ is not a uniformly bounded family of Jacobi $M_p^p$-multipliers in the same $(p, \mu)$-range. But $\eta_{\mu,n}$ satisfies

$$\sum_{k=0}^{\infty} A_k^{\mu}|\Delta^{\mu+1}\eta_{\mu,n}(k)| \leqslant C$$



uniformly in $n > 0$ (see the Remark in [26], p. 44), so $\eta_{\mu,n} \in \mathrm{wbv}_{1,\mu+1}$ with $\mu+1 \leqslant (2\alpha+2)|1/p-1/2|+1/2$; which completes the proof of (a). Also see Bonami and Clerc [4].

Parts (b) and (c) of Theorem 6 follow easily by interpolating between (a), (b) and (a), (c) of Theorem B, respectively.

Remarks. The special case $\alpha = \beta > -1/2$, $\gamma \in \boldsymbol{N}$, of Theorem 6 (a) is contained in Bonami [3], which also gives an extension to weighted $L^p$-spaces. Also Theorem 6 supplements some results in Connett and Schwartz. What seems necessary now is to extend Theorem 6 by replacing the Marcinkiewicz type conditions by Hörmander ones: $\mathrm{wbv}_{2,\varkappa}$, where the parameter $\varkappa$ is governed by the imbedding relations in Theorem 5 applied to Theorem 6(a), i.e. $\varkappa > (2\alpha+2)|1/p-1/2|$.

Looking at Theorem 6 there naturally arises the question if one can give similar conditions for $M_p^q$-multipliers, $1 < p < q < \infty$. This can indeed be accomplished by the following idea. Write $\{\eta_k\} = \{k^{\sigma}\eta_k\}\{k^{-\sigma}\}$; then for appropriate $\sigma > 0$ the multiplier sequence $\{k^{-\sigma}\}$ transforms $f \in L^p$ into a corresponding $f^{\sigma} \in L^q$. Now one has only to check if $\{k^{\sigma}\eta_k\}$ satisfies the sufficient $M_q^q$-condition of Theorem 6. This again leads to "sharp" multiplier conditions in terms of modified weak bounded variation spaces. Details will be given in a subsequent paper.

**9. Multipliers for Hankel transforms.** Fix $\alpha > -1$ and let $L^p = L^p_{(\alpha)}$, $1 \leqslant p < \infty$, denote the space of measurable functions $f(x)$ on $(0, \infty)$ for which

$$\|f\|_p = \Big(\int_0^{\infty} |f(x)|^p x^{2\alpha+1}\,dx\Big)^{1/p}$$

is finite. Following Hirschman [15], we define for $f \in L^1$ the (modified) Hankel transform of order $\alpha$ by

$$H_{\alpha}[f](y) = f^{\wedge}(y) = \int_0^{\infty} f(x)(xy)^{-\alpha}J_{\alpha}(xy)x^{2\alpha+1}\,dx,$$

where $J_{\alpha}(x)$ is the Bessel function of the first kind; thus, the Hankel transform of order $\alpha = (n-2)/2$ coincides with the Fourier transform of a radial function integrable on $\boldsymbol{R}^n$ (see [21], p. 155). The multiplier transformation associated with a function $\psi(y)$ is defined formally by

$$U_{\psi}f(x) = \int_0^{\infty} \psi(y)f^{\wedge}(y)(xy)^{-\alpha}J_{\alpha}(xy)y^{2\alpha+1}\,dy.$$

$\psi \in L^{\infty}(0, \infty)$ is called a *multiplier of type* $(p, p)$, notation $\psi \in M_p^p(H)$ if to each $f \in L^1 \cap L^p$ there exists a function $U_{\psi}f = f^{\psi} \in L^p$ such that

$$\|U_{\psi}f\|_p \leqslant C\|f\|_p \quad (f \in L^1 \cap L^p); \tag{9.1}$$

the least constant $C$ for which (9.1) holds defines the operator norm of the multiplier $\psi$, notation $|\psi|_{M_p^p}$. Igari [16] has proved the following interesting connection between Hankel and Jacobi multipliers of strong type $(p, p)$; an analogous result for weak type multipliers can be found in Connett and Schwartz [10].

THEOREM D. *Let $1 \leqslant p < \infty$ and $\alpha, \beta > -1$. Let $\psi$ be a continuous function on $(0, \infty)$, set $\psi^\varepsilon(y) = \psi(\varepsilon y)$, and denote by $\{\psi_k^\varepsilon\}_{k=0}^\infty$ the sequence for which $\psi_k^\varepsilon = \psi(\varepsilon k)$, $\psi_0^\varepsilon = 0$. Assume $\psi^\varepsilon$ to be a uniformly bounded family of Jacobi multipliers (for small $\varepsilon > 0$). Then $|\psi|_{M_p^p}$ is finite and*

$$|\psi|_{M_p^p} \leqslant \liminf_{\varepsilon\to 0+} \|\{\psi_k^\varepsilon\}\|_{M_p^p},$$

*where $\|\cdot\|_{M_p^p}$ denotes the Jacobi multiplier norm.*

This enables us to use Theorem 6 to obtain analogous results for Hankel multipliers. We restrict ourselves to the case $q = 1$ (part (a) of Theorem 6).

THEOREM 7. *Let $1 < p < \infty$, $\alpha \geqslant -1/2$. Then*

$$\mathrm{WBV}_{1,\gamma} \subset M_p^p(H) \subset \mathrm{WBV}_{\infty,\mu}, \tag{9.2}$$

*where* $0 \leqslant \mu < (2\alpha+2)|1/p-1/2|-1/2 < \gamma-1$.

Proof. The right hand-side inclusion is proved in [28] for $\alpha = (n-2)/2$; the extension to arbitrary $\alpha \geqslant -1/2$ is completely analogous. For the left hand-inclusion apply Theorem D, Theorem 6(a), Lemma 2, and Theorem 2 to an arbitrary $\psi \in \mathrm{WBV}_{1,\gamma}$:

$$\begin{aligned}|\psi|_{M_p^p} &\leqslant \liminf_{\varepsilon\to 0+} \|\{\psi_k^\varepsilon\}\|_{M_p^p} \leqslant C \liminf_{\varepsilon\to 0+} \|\{\psi_k^\varepsilon\}\|_{1,\gamma;w}\\ &\leqslant C \liminf_{\varepsilon\to 0+} \sup_{u>0} \|\{(\psi_u^\varepsilon)_k\}\|_{1,\gamma;w} \leqslant C \liminf_{\varepsilon\to 0+} \sup_{u>0} \|\psi_u^\varepsilon\|_{1,\gamma;W},\end{aligned}$$

where $\psi_u^\varepsilon = (G_u^\varepsilon - G_{1/u}^\varepsilon)\psi^\varepsilon$. But by direct computation (cf. [26], p. 41)

$$\left(\frac{d}{dt}\right)^\gamma \psi_u(\varepsilon t) = \varepsilon^\gamma \psi_u^{(\gamma)}(\varepsilon t)$$

so that for arbitrary integer $m$, by Lemma 10,

$$\int_{2^m}^{2^{m+1}} \left| t^\gamma \left(\frac{d}{dt}\right)^\gamma \psi_u^\varepsilon(t) \right| \frac{dt}{t} = \int_{\varepsilon 2^m}^{\varepsilon 2^{m+1}} |t^\gamma \psi_u^{(\gamma)}(t)| \frac{dt}{t} \leqslant C\|\psi\|_{1,\gamma;W}$$

which completes the proof.

Remarks. (i) On account of Theorem 3 and Theorem 4(c), Theorem 7 is equivalent to a result (in terms of localized Bessel potential spaces) in Connett and Schwartz [11]. In the special case $\alpha = (n-2)/2$, $\psi(t) = (1-t^2)_+^\gamma$, Theorem 7 yields Welland's result [29] on the boundedness of the Bochner–Riesz means for radial functions on $L^p(\boldsymbol{R}^n)$.



To show that the $(p, \gamma)$-range in Theorem 7 is best possible it suffices to consider the Bochner–Riesz kernel

$$m(y) = m_\gamma(y) = (1-y^2)_+^{\gamma-1}, \qquad \gamma > 1,$$

and use a modification of Welland's proof of the special case $\alpha = (n-2)/2$ (which utilized formula (3.1) in [29]; personal communication). In place of [29], (3.1) we shall use the fact that if $f$ is the characteristic function of an interval $[0, \varepsilon]$ and $\mathscr{I}_\alpha(x) = x^{-\alpha}J_\alpha(x)$, then

$$\begin{aligned}V_m f(z) &= \int_0^1 (1-y^2)^{\gamma-1} f^\wedge(y)\mathscr{I}_\alpha(yz)y^{2\alpha+1}dy \\ &= C_{\alpha,\gamma}\int_0^\varepsilon f(x)x^{2\alpha+1}\Big(\int_0^\pi \mathscr{I}_{\alpha+\gamma}\big((x^2+z^2-2xz\cos\varphi)^{1/2}\big)\sin^{2\alpha}\varphi d\varphi\Big)\,dx.\end{aligned} \tag{9.3}$$

This formula follows easily by using the product formula ($\alpha > -1/2$)

$$\begin{aligned}\mathscr{I}_\alpha(xt)\mathscr{I}_\alpha(yt) &= \int_0^\infty \mathscr{I}_\alpha(zt)\mathscr{K}_\alpha(x, y, z)z^{2\alpha+1}dz \\ &= C_\alpha\int_0^\pi \mathscr{I}_\alpha\big((x^2t^2+y^2t^2-2xyt^2\cos\varphi)^{1/2}\big)\sin^{2\alpha}\varphi d\varphi,\end{aligned}$$

where the kernel $\mathscr{K}_\alpha(x, y, z)$ is symmetric in $x, y, z$, and Lemma 4.13 in [21]. In view of the asymptotic formula [24], (1.71.7)

$$\mathscr{I}_\alpha(x) = c_1x^{-\alpha-1/2}\cos(x+c_2)+O(x^{-\alpha-3/2}), \qquad x\to+\infty,$$

if $[a, b]$, $0 < a < b$, is an interval such that

$$\cos(x+c_2) \geqslant 1/2, \qquad a \leqslant x \leqslant b,$$

and if $\varepsilon = (b-a)/4$, then, by (9.3),

$$V_m f(z) \geqslant Cz^{-\alpha-\gamma-1/2}, \qquad a+\varepsilon+2\pi k \leqslant z \leqslant b-\varepsilon+2\pi k,$$

for $k \geqslant k_0$, where $k_0$ is sufficiently large. Hence

$$\|V_m f\|_p^p \geqslant C\sum_{k=k_0}^\infty \int_{a+\varepsilon+2\pi k}^{b-\varepsilon+2\pi k} z^{(-\alpha-\gamma-1/2)p}z^{2\alpha+1}dz = \infty$$

when $p \leqslant 4(\alpha+1)/(2\alpha+2\gamma+1)$.

(ii) By Theorem 2, Theorem D, and Theorem 7 we also obtain a necessary condition for a family of sequences $\psi^\varepsilon$ with $\psi_k^\varepsilon = \psi(\varepsilon k)$, $\varepsilon > 0$, to be a uniformly bounded Jacobi multiplier family.

(iii) The case $\gamma = 1$ follows analogously by Theorem B(b) and Theorem D, and reads as follows.

If $\psi \in L^\infty(0, \infty)$ is locally of bounded variation with

$$\|\psi\|_\infty + \sup_m \int_{2^m}^{2^{m+1}} |d\psi(t)| < \infty,$$

then $\psi \in M_p^p(H)$ provided $1 \leqslant \frac{2a+2}{a+3/2} < p < \frac{2a+2}{a+1/2} \leqslant \infty$.

This method is due to Igari [16]; for another proof giving multiplier criteria even in weighted $L^p$-spaces see Guy [14], p. 187; a heuristic proof using Littlewood–Paley functions can be found in Sunouchi [23].

(iv) It is interesting to note what happens for $p = 1$ (or $p = \infty$). To this end introduce the space of functions of bounded variation of order $\gamma$ (cf. [26], p. 36)

$$\mathrm{BV}_\gamma = \{g \in C_0[0, \infty)\colon I_\omega^{1-\delta}(g) \in \mathrm{AC}_{\mathrm{loc}},\ \omega > 0 \text{ if } \delta > 0;$$
$$g^{(\delta)}, \ldots, g^{(\gamma-1)} \in \mathrm{AC}_{\mathrm{loc}} \text{ and } \|g\|_{\mathrm{BV}_\gamma} < \infty\},$$

where $\delta = \gamma - [\gamma]$,

$$\|g\|_{\mathrm{BV}_\gamma} = \|g\|_\infty + \int_0^\infty t^{\gamma-1} |g^{(\gamma)}(t)|\, dt$$

and $C_0[0, \infty)$ is the set of all continuous functions on $[0, \infty)$ which vanish at infinity. Denote by $[L^1]^\wedge$ the set of all functions which are Hankel transforms of some $L^1$-function. Then there holds

$$\mathrm{BV}_\gamma \subset [L^1]^\wedge \subset \mathrm{BV}_\mu, \quad 0 < \mu < a+1/2 < \gamma - 1.$$

The left-side inclusion is proved in Butzer, Nessel, and Trebels [7], the right-side inclusion is only a simple modification of [28], noting that the Riemann–Lebesgue Lemma holds for Hankel transforms (see [19]). By the same technique as in [28] one can also show

$$[L^1]^\wedge \subset M_1^1(H) \subset \mathrm{WBV}_{\infty,\mu}, \quad 0 \leqslant \mu < a+1/2.$$

(v) There is a differentiation gap of about 1 $(\mu+1 < \gamma)$ between necessary $(\mu)$ and sufficient $(\gamma)$ Hankel multiplier criteria. Theorem 4(d) and the example $(1-t)_+^\mu$ show that this gap cannot essentially be diminished as long as we stick to $\mathrm{WBV}_{1,\gamma}$ and $\mathrm{WBV}_{\infty,\mu}$-classes. As in the Jacobi multiplier case the conjecture for an improvement is

$$\mathrm{WBV}_{2,\varkappa} \subset M_p^p(H), \quad \varkappa > (2a+2)|1/p - 1/2|,\ 1 < p < \infty.$$

One result in this direction is due to Connett and Schwartz [10] who verified the embedding for $\varkappa = [a+1]+1$, $a > -1/2$.



**Added in proof** (Oct. 4, 1979): For recent results concerning the remarks on p. 273 and p. 276 (v) see: G. Gasper and W. Trebels, *Multiplier criteria of Hörmander type for Fourier series and applications to Jacobi series and Hankel transforms*, Math. Ann. 242 (1979), pp. 225–240.

**References**

[1] R. Askey, *A transplantation theorem for Jacobi series*, Illinois J. Math. 13 (1969), pp. 583–590.
[2] R. Askey and I. I. Hirschman, Jr., *Mean summability for ultraspherical polynomials*, Math. Scand. 12 (1963), pp. 167–177.
[3] A. Bonami, *Multiplicateurs des séries ultrasphériques*, C. R. Acad. Sci. Paris, Sér. A 273 (1971), pp. 148–150.
[4] A. Bonami and J.-L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc. 183 (1973), pp. 223–263.
[5] L. S. Bosanquet, *Note on the Bohr–Hardy theorem*, J. London Math. Soc. 17 (1942), pp. 166–173.
[6] P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation*, Vol. 1, Academic Press, N. Y. 1971.
[7] P. L. Butzer, R. J. Nessel and W. Trebels, *Multipliers with respect to spectral measures in Banach spaces and approximation I*, J. Approximation Theory 8 (1973), pp. 335–356.
[8] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), pp. 113–190.
[9] W. C. Connett and A. L. Schwartz, *A multiplier theorem for Jacobi expansions*, ibid. 52 (1975), pp. 243–261; Correction, ibid. 54 (1975), p. 107.
[10] —, — *Weak type multipliers for Hankel transforms*, Pacific J. Math. 63 (1976), pp. 125–129.
[11] —, — *The theory of ultraspherical multipliers*, Mem. Amer. Math. Soc. 9 (1977), no. 183.
[12] J. Cossar, *A theorem on Cesàro summability*, J. London Math. Soc. 16 (1941), pp. 56–68.
[13] G. Gasper and W. Trebels, *Multiplier criteria of Marcinkiewicz type for Jacobi expansions*, Trans. Amer. Math. Soc. 231 (1977), pp. 117–132.
[14] D. L. Guy, *Hankel multiplier transformations and weighted p-norms*, ibid. 95 (1960), pp. 137–189.
[15] I. I. Hirschman, Jr., *Variation diminishing Hankel transforms*, J. Analyse Math. 8 (1960/61), pp. 307–336.
[16] S. Igari, *On the multipliers of Hankel transforms*, Tôhoku Math. J. (2) 24 (1972), pp. 201–206.
[17] B. Muckenhoupt and E. M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. 118 (1965), pp. 17–92.
[18] A. Peyerimhoff, *Über Summierbarkeitsfaktoren und verwandte Fragen bei Cesàroverfahren II*, Acad. Serbe Sci., Publ. Inst. Math. 10 (1956), pp. 1–18.
[19] A. L. Schwartz, *The smoothness of Hankel transforms*, J. Math. Anal. Appl. 28 (1969), pp. 500–507.
[20] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.
[21] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton 1971.

[22] R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. 16 (1967), pp. 1031–1060.
[23] G. Sunouchi, *On the Littlewood–Paley function $g^*$ of multiple Fourier integrals and Hankel multiplier transformations*, Tôhoku Math. J. 19 (1967), pp. 496–511.
[24] G. Szegö, *Orthogonal polynomials*, AMS Colloq. Publ., vol. 23, AMS Providence, R. I., 1975.
[25] E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, London 1960.
[26] W. Trebels, *Multipliers for $(C, a)$-bounded Fourier expansions in Banach spaces and approximation theory*, Lecture Notes Math. 329, Springer, Berlin 1973.
[27] — *Some Fourier multiplier criteria and the spherical Bochner–Riesz kernel*, Rev. Roumaine Math. Pures Appl. 20 (1975), pp. 1173–1185.
[28] — *Some necessary conditions for radial Fourier multipliers*, Proc. Amer. Math. Soc. 58 (1976), pp. 97–103.
[29] G. V. Welland, *Norm convergence of Riesz–Bochner means for radial functions*, Canad. J. Math. 27 (1975), pp. 176–185.
[30] U. Westphal, *An approach to fractional powers of operators via fractional differences*, Proc. London Math. Soc. (3) 29 (1974), pp. 557–576.
[31] R. L. Wheeden, *On hypersingular integrals and Lebesgue spaces of differentiable functions, I*, Trans. Amer. Math. Soc. 134 (1968), pp. 421–435, II, ibid. 139 (1969), pp. 37–53.
[32] A. Zygmund, *Trigonometric series*, vol. 2, Cambridge Univ. Press, 1968.

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*Received February 9, 1977* (1261)

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STUDIA MATHEMATICA, T. LXV. (1979)

# Factorization in Banach algebras

by

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**Abstract.** Let $A$ be a Banach algebra with bounded left approximate identity and let $F$ be a left Banach $A$-module. For each sequence $y(n) \in AF^-$ satisfying a certain condition there exists an $a \in A$ and another sequence $z(n) \in AF^-$ such that $y(n) = a^n z(n)$.

**Introduction.** We present a simple proof of a generalization of the Rudin–Cohen factorization theorem using the method of nondiscrete mathematical induction. This method is based on a simple abstract theorem about families of sets, the so-called *induction theorem*. The induction theorem is closely related to the closed graph theorem and is nothing more than the abstract description of a class of iterative constructions in analysis. One of the advantages of this method consists in the fact that the construction of the sequence of iterations is dealt with by the abstract theorem; this reduces the amount of work required to an investigation of the improvement of the degree of approximation which can be achieved within a given distance from a given point. In this manner, by separating the hard analysis part from the construction this approach not only yields considerable simplifications of proofs but also evidences more clearly the substance of the problem.

**1. Preliminaries.** Given a positive number $r$ and a set $M$ in a metric space $(E, d)$, we define $U(M, r) = \{y \in E;\ d(y, M) < r\}$. Let $T$ be an interval of the form $\{t;\ 0 < t < t_0\}$, where $t_0$ is positive or $\infty$. If $A(t)$, $t \in T$ is a family of subsets of $E$, we define its limit $A(0)$ as follows

$$A(0) = \bigcap_{0<r} \Big(\bigcup_{s\leqslant r} A(s)\Big)^- .$$

A mapping $\omega$ transforming $T$ into itself is called a *small function* or a *rate of convergence on $T$* if $\sigma(t) = t + \omega(t) + \omega(\omega(t)) + \ldots$ is finite for each $t \in T$. The method of nondiscrete mathematical induction is based on the following simple result.