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Factorization in Banach algebras

by

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Abstract. Let A be a Banach algebra with bounded left approximate identity and let F be a left Banach A-module. For each sequence $y(n) \in AF^-$ satisfying a certain condition there exists an $a \in A$ and another sequence $z(n) \in AF^-$ such that $y(n) = a^n z(n)$.

Introduction. We present a simple proof of a generalization of the Rudin–Cohen factorization theorem using the method of nondiscrete mathematical induction. This method is based on a simple abstract theorem about families of sets, the so-called *induction theorem*. The induction theorem is closely related to the closed graph theorem and is nothing more than the abstract description of a class of iterative constructions in analysis. One of the advantages of this method consists in the fact that the construction of the sequence of iterations is dealt with by the abstract theorem; this reduces the amount of work required to an investigation of the improvement of the degree of approximation which can be achieved within a given distance from a given point. In this manner, by separating the hard analysis part from the construction this approach not only yields considerable simplifications of proofs but also evidences more clearly the substance of the problem.

1. Preliminaries. Given a positive number r and a set M in a metric space (E, d), we define $U(M, r) = \{y \in E;\ d(y, M) < r\}$. Let T be an interval of the form $\{t;\ 0 < t < t_0\}$, where t_0 is positive or ∞. If $A(t)$, $t \in T$ is a family of subsets of E, we define its limit $A(0)$ as follows

$$A(0) = \bigcap_{0<r} \Big(\bigcup_{s\leqslant r} A(s)\Big)^-.$$

A mapping ω transforming T into itself is called a *small function* or a *rate of convergence on T* if $\sigma(t) = t + \omega(t) + \omega(\omega(t)) + \ldots$ is finite for each $t \in T$. The method of nondiscrete mathematical induction is based on the following simple result.

(1.1) THEOREM. *Let $Z(r)$, $r \in T$ be a family of subsets of a complete metric space (E, d). Let ω be a rate of convergence on T. If*

$$Z(r) \subset U\big(Z(\omega(r)), r\big) \quad \text{for each } r \in T,$$

then $Z(r) \subset U\big(Z(0), \sigma(r)\big)$ for each $r \in T$.

The proof is an exercise; the principles of the method of nondiscrete mathematical induction are expounded in the Gatlinburg Lecture [12].

Let A be a Banach algebra without a unit. We shall say that A possesses a *left approximate unit of norm β* if A satisfies one of the following two equivalent conditions

(1) *for every $a \in A$ and every $\varepsilon > 0$ there exists an $e \in A$ such that $|e| \leqslant \beta$ and*

$$|ea - a| < \varepsilon;$$

(2) *for every finite sequence $a_1, \ldots, a_n \in A$ and every $\varepsilon > 0$ there exists an $e \in A$ such that $|e| \leqslant \beta$ and*

$$|ea_i - a_i| < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots, n.$$

A proof of this equivalence is given in [2].

Let F be a Banach space which is a left A-module, $|ax| \leqslant |a|\,|x|$ for $a \in A$, $x \in F$. We shall denote by F_0 the closure in F of AF. If condition (1) or (2) is satisfied, it is easy to show that

(3) *for every finite sequence $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_m \in F_0$ and every $\varepsilon > 0$ there exists an $e \in A$, $|e| \leqslant \beta$ such that*

$$|ea_i - a_i| < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots, n,$$
$$|ex_j - x_j| < \varepsilon \quad \text{for} \quad j = 1, 2, \ldots, m.$$

We shall need the following simple lemma, the proof of which may be left to the reader:

(1.2) *For every $n \in N$ and all complex a, b*

$$|a+b|^{1/n} \leqslant |a|^{1/n} + |b|^{1/n}.$$

As usual, N denotes the set of all natural numbers.

We shall assing to each Banach algebra W and each Banach left W-module E a new structure $(W, E)^\sigma = (W, E^\sigma, d^\sigma, \circ)$ which consists of W, a linear space E^σ, a metric d^σ on E^σ such that (E^σ, d^σ) is a complete metric space and a mapping $\circ$ of $W \times E^\sigma$ into E^σ. This is done as follows. Let a_n be an arbitrary sequence of positive numbers. We shall denote by E^σ the set of all functions $z\colon N \to WE^-$ such that

$$\|z\| = \sup a_n^{-1} |z(n)|^{1/n} < \infty.$$

It follows from Lemma (1.2) that E^σ is a linear space; equipped with the distance function $d^\sigma = \|z_1 - z_2\|$ it becomes a complete metric space.

For each $a \in W$ and each $z \in E^\sigma$ we define $u = a \circ z$ by setting $u(n) = a^n z(n)$. Clearly, $u \in E^\sigma$ and $\|a \circ z\| \leqslant |a|\,\|z\|$. If λ is a scalar, we define $v = \lambda \circ z$ by setting $v(n) = \lambda^n z(n)$. Clearly, $\lambda \circ z \in E^\sigma$ and $\|\lambda \circ z\| = |\lambda|\,\|z\|$. Also, $\lambda \circ a \circ z = (\lambda a) \circ z$.

Let A be a Banach algebra without a unit. We shall denote by B its unitization. The multiplicative linear functional on B which has A as its kernel will be denoted by f. The mapping P defined by $Pb = b - f(b)$ is a projection of B onto A. The set of all invertible elements of B will be denoted by $G(B)$.

If F is a Banach left A-module with $|xy| \leqslant |x|\,|y|$ for $x \in A$ and $y \in F$, then F is also a left B-module in an obvious manner; the above inequality remains valid for $x \in B$ as well. If A has a bounded approximate unit, it is easy to see that $(By)^- = (Ay)^-$ for each $y \in F$.

(1.3) *Let W be a Banach algebra; for each pair $u, v \in W$ and each $n \in N$*

$$v^n - u^n = \sum_{1}^{n} v^{n-k}(v-u)u^{k-1}.$$

Proof. By induction.

The following technical result will be used in the sequel.

(1.4) *Suppose b, c, w, e are elements of a unital Banach algebra B and α a complex number which satisfy the following relations*

$$|w| \leqslant 3/4, \qquad c = (1+w)^{-1}, \qquad w = \alpha(e-1)b.$$

Then $|c| \leqslant 4$ and $c - 1 = -cw = -\alpha c(e-1)b$.

If F is a Banach space which is a left B-module, then, for each $n \in N$ and each $y \in F$, the following estimates hold

$$\big|\big((bc)^n - b^n\big)y\big| \leqslant |\alpha| \sum_{1}^{n} (4|b|)^j \max_{1 \leqslant k \leqslant n} |(e-1)b^k y|,$$

$$\big|\big((bc)^n - b^n\big)y\big| \leqslant (5|b|)^n |y|.$$

Proof. The second estimate is immediate since

$$\big|\big((bc)^n - b^n\big)y\big| \leqslant (|bc|^n + |b|^n)|y| \leqslant |b|^n(4^n+1)|y|.$$

The first estimate is a consequence of Lemma (1.3) and the relation $c - 1 = -cw$:

$$(bc)^n - b^n = \sum_{1}^{n} (bc)^{n-k} b(c-1)b^{k-1} = -\alpha \sum_{1}^{n} (bc)^{n-k} bc(e-1)b^k$$
$$= -\alpha \sum_{1}^{n} (bc)^{n+1-k}(e-1)b^k.$$

2. Power factorizations.

(2.1) THEOREM. *Let A be a Banach algebra with a left approximate unit of norm β. Let F be a Banach space which is a left A-module. Let a_n be an arbitrary sequence of positive numbers. Let $y(1), y(2), \ldots$ be a sequence of elements of AF^- such that*

$$\lim a_n^{-1}|y(n)|^{1/n} = 0$$

and let $\varepsilon > 0$ be given.

Then there exists a sequence $z(n) \in Ay(n)^-$ and an element $a \in A$ such that

$$y(n) = a^n z(n) \quad \text{for all } n \in \boldsymbol{N},$$

$$|a| \leqslant \beta,$$

$$|z(n) - y(n)| \leqslant a_n^n \varepsilon^n \quad \text{for all } n \in \boldsymbol{N}.$$

Proof. I. Let B be the unitization of A; the letters f and P will have the same meaning as in the preceding section. Construct $(B, F)^\sigma$. The letter U will stand for the set $\{x \in A;\ |x| \leqslant \beta\}$. Let E be the complete metric space obtained by equipping the set $A \times (B \circ y)^-$ with the distance

$$d(p_1, p_2) = \frac{1}{1-\omega} \max\left\{\frac{1}{\beta}|a_1 - a_2|, \frac{1}{\varepsilon}\|z_1 - z_2\|\right\}$$

if $p_1 = [a_1, z_1]$, and $p_2 = [a_2, z_2]$; the closure $(B \circ y)^-$ is taken in the metric d^σ, the number ω is a constant to be chosen later, $0 < \omega < 1$. (We shall see that $\omega = \dfrac{2\beta+1}{2\beta+2}$ is a possible choice.) For each $b \in G(B)$ set

$$p(b) = [P(b^{-1}), b \circ y] \in E.$$

For each positive $r \leqslant 1$ set

$$W(r) = \left\{p(b);\ b \in G(B),\ |f(b^{-1})| \leqslant r,\ d\big(p(b), p(1)\big) \leqslant \frac{1}{1-\omega}(1-r)\right\}.$$

In particular, $[0, y] = p(1) \in W(1)$.

II. Fix $n \in \boldsymbol{N}$ and consider the nth coordinate of $a \circ z - y$ if $[a, z] \in W(r)$. We have

$$\big(b^{-1} - f(b^{-1})\big)^n z(n) - y(n)$$
$$= b^{-n} z(n) + \left(\sum_1^n \binom{n}{k} (b^{-1})^{n-k}\big(-f(b^{-1})\big)^k\right) z(n) - y(n)$$
$$= z(n) \sum_1^n (-1)^k \binom{n}{k} (b^{-1})^{n-k} f(b)^{-k}$$

whence

$$|a^n z(n) - y(n)| \leqslant |z(n)| \sum_1^n \binom{n}{k} |b^{-1}|^{n-k} r^k$$
$$= |z(n)|\big((|b^{-1}| + r)^n - |b^{-1}|^n\big) \leqslant |z(n)| rn (|b^{-1}| + r)^{n-1}.$$

Since $|b^{-1}| \leqslant |Pb^{-1}| + |f(b^{-1})| \leqslant |a| + r$ and since $[a, z] \in W(r)$ implies $|a| \leqslant \beta(1-r)$, we have

$$|a^n z(n) - y(n)| \leqslant r|z(n)| n(\beta + 2r)^{n-1}.$$

It follows that $[a, z] \in W(0)$ implies $a \circ z = y$.

III. We intend to show there exists an ω such that

$$W(r) \subset U\big(W(\omega r), r\big) \quad \text{for each } r > 0.$$

Having proved that, it will follow from the Induction Theorem that $W(1) \subset U\big(W(0), 1/(1-\omega)\big)$, this means that there exists a $p = [a, z] \in W(0)$ with $d\big(p, p(1)\big) \leqslant 1/(1-\omega)$, in other words, $a \circ z = y$, $|a| \leqslant \beta$, $\|z - y\| \leqslant \varepsilon$.

IV. Now let $p(b) \in W(r)$. We intend to show that the pair $p(b')$ corresponding to a slightly perturbed $b' = bc$ will satisfy

$$p(b') \in W(\omega r) \cap U\big(p(b), r\big).$$

For this it suffices clearly to construct c in such a manner that

(1) $|Pb'^{-1} - Pb^{-1}| \leqslant (1-\omega)\beta r$,

(2) $f(c^{-1}) = \omega$,

(3) $\|b' \circ y - b \circ y\| \leqslant (1-\omega)\varepsilon r$.

We shall see that it is possible to satisfy these three conditions by constructing a c for which

(4) $b'^{-1} - b^{-1}$ is a scalar multiple of $e - 1$ for a suitable $e \in U$.

Such a choice — if possible — has the following consequences: assuming $b'^{-1} - b^{-1} = \alpha(e-1)$ for some scalar α, we have

$$-\alpha = f\big(\alpha(e-1)\big) = f(b'^{-1} - b^{-1}) = f(c^{-1} - 1) f(b^{-1}) = (\omega - 1) f(b^{-1})$$

whence

$$P(b'^{-1} - b^{-1}) = P\big(\alpha(e-1)\big) = \alpha e,$$
$$b' - b = -b'(b'^{-1} - b^{-1})b = -\alpha b'(e-1)b,$$
$$\alpha = (1-\omega) f(b^{-1}).$$

For shortness, set $w = \alpha(e-1)b$. We have thus $b' - b = -b'w$, whence $b'(1+w) = b$. It follows that a suitable choice of c will be $c = (1+w)^{-1}$ provided $|w| < 1$. Now $w = \alpha(e-1)b = \alpha(e-1)Pb + \alpha(e-1)f(b) =$

$= a(e-1)Pb + (1-\omega)(e-1)$, whence

$$|w| \leqslant |a|\,|(e-1)Pb| + (1-\omega)(\beta+1).$$

Now choose $\omega = (2\beta+1)/(2\beta+2)$ so that $(1-\omega)(\beta+1) = 1/2$.

An $e \in U$ may be chosen so as to have

(5) $|a|\,|(e-1)Pb| \leqslant 1/4$

so that $|w| \leqslant 3/4$ and $|c| \leqslant 4$.

Now choose an $m \in \boldsymbol{N}$ such that

$$a_n^{-1} 5\,|b|\,|y(n)|^{1/n} \leqslant (1-\omega)\,\varepsilon r$$

for all $n > m$. Having chosen m, choose $e \in U$ which satisfies (5) and at the same time

$$\sum_{1}^{m} (4\,|b|)^j \max_{1 \leqslant k,n \leqslant m} |(e-1)\,b^k y(n)| \leqslant \min_{1 \leqslant t \leqslant m} \big(a_t (1-\omega)\,\varepsilon r\big)^t.$$

According to Lemma (1.4) this implies $\|b c \circ y - b \circ y\| \leqslant (1-\omega) r\varepsilon$. The proof is complete.

As a corollary, let us prove a theorem obtained recently by G.R. Allan and A.M. Sinclair [1].

(2.2) THEOREM. *Let A be a Banach algebra with a bounded left approximate identity bounded by β and let F be a left Banach A-module. Let a_n be a sequence of real numbers such that $a_n > 1$ for all n and $a_n \to \infty$, let $\delta > 0$ and let $m \in \boldsymbol{N}$.*

If y lies in the closed linear span of the set AF^-, then there are an $a \in A$ and $z_1, z_2, \ldots$ in F such that

(1) $y = a^n z_n$,

(2) $|a| \leqslant \beta$

(3) $z_j \in Ay^-$,

(4) $|y - z_h| \leqslant \delta$ *for* $h = 1, 2, \ldots, m$,

(5) $|z_j| \leqslant a_j^j |y|$ *for all* $j \in \boldsymbol{N}$.

Proof. Let $y(n)$ be the sequence obtained by setting $y(n) = y$ for all $n \in \boldsymbol{N}$. Since $a_n \to \infty$, we have $\lim a_n^{-1} |y(n)|^{1/n} = 0$. According to Theorem (2.1) there exists, for each $\varepsilon > 0$, a sequence $z(n) \in Ay^-$ and an element $a \in A$, $|a| \leqslant \beta$ such that $y = y(n) = a^n z(n)$ and $|z(n) - y| \leqslant a_n^n \varepsilon^n$ for all $n \in \boldsymbol{N}$. To satisfy (4) of the present theorem, it suffices to take ε such that $a_n^n \varepsilon^n \leqslant \delta$ for $n = 1, 2, \ldots, m$. To satisfy (5) of the present theorem, it suffices to take ε such that $\varepsilon \leqslant (1 - a_n^{-n})^{1/n} |y|^{1/n}$ for all $n \in \boldsymbol{N}$.

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