STUDIA MATHEMATICA, T. LXV. (1979)

Singular integral operators with complex homogeneity

by

J. M. ASH, P. F. ASH, C. L. FEFFERMAN, and R. L. JONES (Chicago, Ill.)

Abstract. We consider various approximations to singular integral operators. For example, we consider approximating the Hilbert transform by convolving with $x^{-1}|x|^{-i\gamma}$ and then letting $\gamma \to 0^+$. For this procedure norm convergence holds but, surprisingly, pointwise convergence fails. We also consider higher dimensional generalizations and more successful approximating procedures.

1. Introduction. Let f be an integrable function on the real line $\boldsymbol{R}$. The Hilbert transform $\tilde{f}$ is the principle value convolution of f with the singular kernel $(\pi t)^{-1}$. At almost every point, the integral defining $\tilde{f}$ converges, and hence $\tilde{f}$ exists [18]. The existence of $\tilde{f}$ is a central fact in the theory of real and complex analysis, and one which has been susceptible to a wide variety of generalizations.

A. Calderón suggested a possible new approach to the problem of establishing the conjugate functions existence. In one dimension his idea was the following. Each positive real number γ gives rise to a kernel $(\pi t)^{-1}|t|^{-i\gamma}$.

If (1) for each γ, convolution with the associated kernel would converge to a function (in some sense) and (2) the functions $f_\gamma(x)$ would tend to $\tilde{f}(x)$ as $\gamma \to 0$, then we would be on our way to getting a novel perspective on $\tilde{f}$ and its generalizations.

Step (1) of this program was carried out successfully in [12] where B. Muckenhoupt showed that f_γ exists almost everywhere for each fixed γ, for $f \in L^p$, $1 \leqslant p < \infty$.

Step (2) of the program started out well when B. Muckenhoupt also showed that $f_\gamma \to \tilde{f}$ in L^2 norm, i.e., that

$$\|f_\gamma - \tilde{f}\|_2 = \Big(\int_{-\infty}^{\infty} |f_\gamma(x) - \tilde{f}(x)|^2\,dx\Big)^{1/2} \to 0 \quad \text{as} \quad \gamma \to 0. \ [12]$$

We were further encouraged when we found that rather weak conditions on $\hat{f}$ — the Fourier transform of f — guaranteed that f_γ tended to $\tilde{f}$ at almost every point. (Theorem 2.1.) Unfortunately Theorem 2.4 shows there are reasonably good (L^p, for all p, $0 < p < \infty$) functions f such that

$\overline{\lim_{\gamma\to 0}}|f_\gamma| = +\infty$ at every single point (and hence f_γ certainly does not go to $\tilde{f}$ pointwise).

The above program can also be carried out in the Fourier series setting. Part (1) of the program was carried out for series by Mary Weiss and Zygmund [15]. Since our results, both positive and negative, are more transparent in this setting; we first present them for series (Theorem 2.2) and then transfer them to the line. Theorem 2.2 is the major result of this paper.

In Section 2 we study the above program in one dimension. In Section 3 we generalize the positive norm convergent aspects of these results to higher dimensions. Finally in Section 4 we examine other approximations to the Hilbert transform. In particular convolution with either $(\pi x)^{-1}|x|^{(\delta+i\gamma)\operatorname{sgn}(1-|x|)}$, $\delta > 0$, or with the $(C, 2)$ mean of $(\pi x)^{-1}|x|^{-i\gamma}$ yields a suitable approximating process. Whether convolution with $(\pi x)^{-1}(1-|x|^{-i\gamma})/(i\gamma \ln|x|)$ which is the $(C, 1)$ mean of $(\pi x)^{-1}|x|^{-i\gamma}$ yields a suitable approximation process we leave as an open question.

The letter C will denote a positive real number, not necessarily the same from line to line.

2. One dimensional results

THEOREM 2.1. *If $\hat{f} \in L^1(\boldsymbol{R})$ or if $\hat{f}$ is of bounded variation on $\boldsymbol{R}$ and tending to 0 at $\pm\infty$, then $f_\gamma(x) \to \tilde{f}(x)$ as $\gamma \to 0$ for almost every x.*

We postpone the proof to the end of this section for dramatic effect.

All the operators mentioned above have direct analogues in the Fourier series setting. The analogue of the Hilbert transform $\tilde{f}$ is the conjugate function — principal value convolution with $\left(2\pi \cot\frac{t}{2}\right)^{-1}$ — and will also be denoted by $\tilde{f}$. Here

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f|^p\,dt\right)^{1/p},$$

and if $f \in L^2(\boldsymbol{T})$ has Fourier series

$$S[f](x) = \sum \hat{f}_n e^{inx},$$

the analogue of f_γ — which we will again denote by f_γ — is the function [15] with Fourier series

$$S[f_\gamma](x) = \sum \hat{f}_n(-i\operatorname{sgn} n)|n|^{i\gamma} e^{inx}.$$

Observe that formally $f_\gamma \to f$ as $\gamma \to 0$ in the sense that the Fourier series of $S[f_\gamma]$ becomes $\sum \hat{f}_n(-i\operatorname{sgn} n)e^{inx}$, which is the Fourier series of the conjugate function $\tilde{f}$.

The following theorem discusses the extent to which this formal interchange of summation and limit is justified.

THEOREM 2.2. *If $f \in L^2(\boldsymbol{T})$, $\|f_\gamma - \tilde{f}\|_2 \to 0$. However, there is a function $g \in L^p(\boldsymbol{T})$ for each $p < \infty$ with the property that $g_\gamma(x)$ diverges as $\gamma \to 0$ at every point x.*

Proof. The positive part of the assertion is easy. One writes $f = p + b$ where p is a trigonometric polynomial and b has small L^2 norm. Since both b_γ and $\tilde{b}$ are obtained from b by multipliers which shrink the modulus of every Fourier coefficient ($\left|-i\operatorname{sgn} n\,|n|^{i\gamma}\right| = 1$, $n \neq 0$, $\operatorname{sgn} 0 = 0$), from Plancherel's theorem we have

$$\|b_\gamma - \tilde{b}\|_2 \leqslant \|b_\gamma\|_2 + \|\tilde{b}\|_2 \leqslant 2\|b\|_2.$$

Also it is immediate that the p_γ converge uniformly to $\tilde{p}$ so that $\|p_\gamma - \tilde{p}\|_2 \leqslant \sup|p_\gamma - \tilde{p}|$ is small if γ is small.

We now pass to the more substantive and surprising part of the proof. Take the function g to be

$$g(x) = \sum_{n=1}^{\infty} \frac{e^{i2^{8^n}x}}{n}.$$

Since $\sum n^{-2} < \infty$, $g(x) \in L^2(\boldsymbol{T})$, and the series is $S[g]$. Further g is in every L^p, $p < \infty$, since $S[g]$ is (quite) lacunary ([18], vol. 1, p. 215). Since $S[g]$ has only terms with positive indices, $\tilde{S}[g] = -iS[g]$. The series $\tilde{S}[g]$, being lacunary and L^2 is consequently easily seen to converge a.e. ([18], vol. 1, p. 203, Theorem 6.3).

Similarly for each γ the series

$$S_\gamma[g](x) = -i\sum_{n=1}^{\infty} \frac{e^{i2^{8^n}x}}{n}(2^{8^n})^{i\gamma}$$

converges to the L^2 function g_γ a.e. To complete the counterexample we must show that $\lim_{\gamma\to 0} g_\gamma(x)$ does not exist.

Here is the central idea. Keep x fixed! Then $g_\gamma(x)$ is immediately seen to be a lacunary Fourier series *in the variable* $\theta = \gamma \ln 2$. In fact,

$$g_\gamma(x) = -i\sum_{n=1}^{\infty}\left(\frac{e^{i2^{8^n}x}}{n}\right)e^{i8^n\theta}.$$

But now our series has the form

$$h(\theta) = \sum_{n=1}^{\infty} \varrho_n e^{i8^n\theta}$$

where $\sum|\varrho_n| = \sum n^{-1} = \infty$. On a dense set of θ's, and in particular for a sequence of θ's tending to 0, $\mathrm{Re}\,h(\theta) = +\infty$ by the lemma below. Thus $\lim_{\theta\to 0} h(\theta) = \lim_{\gamma\to 0} g_\gamma(x)$ does not exist. This is what was to be shown, since x was arbitrary.

LEMMA 2.3. *If $\sum|\varrho_n| = \infty$, then for a dense set of θ's* $\mathrm{Re}\left(\sum \varrho_n e^{i8^n\theta}\right) = +\infty$.

Proof. Given any interval I we will find a $\theta^* \in I$ so that $\mathrm{Re}\left(\sum \varrho_n e^{i8^n\theta^*}\right) = \infty$. Choose N so large that $e^{i8^N\theta}$ has a full period as θ varies over some interval $J \subseteq I$. For some subinterval $I_N \subset J$ we have

$$\{\arg(\varrho_N e^{i8^N\theta}):\ \theta \in I_N\} \subset (-\pi/4, \pi/4) \quad \text{and} \quad |I_N| = \tfrac{1}{8}|J|.$$

In other words we have aligned the Nth term of the series to be "mostly real and positive". Next $e^{i8^{N+1}\theta}$ has a full period as θ varies over I_N so choose I_{N+1} to be a subinterval of I_N with $|I_{N+1}| = \frac{1}{8}|I_N|$ and $\arg(\varrho_{N+1}e^{i8^{N+1}\theta}) \in (-\pi/4, \pi/4)$ for $\theta \in I_{N+1}$. This "aligns" the $(N+1)$ term. Proceed inductively and finally let θ^* be the unique member of $\bigcap_{n\geqslant N} I_n$. Then

$$\mathrm{Re}\left(\sum_{n=N}^{\infty} \varrho_n e^{i8^n\theta^*}\right) \geqslant \sum_{n\geqslant N} |\varrho_n|\cos\frac{\pi}{4} = 2^{-1/2}\sum_{n\geqslant N}|\varrho_n| = \infty.$$

The above lemma is well known ([7], p. 397, [17], pp. 77–78). The terms 8^n were chosen to make the lemma easy; actually the lemma is true with 8^n replaced by k_n, where $k_{n+1}/k_n \geqslant q > 1$, although then the proof lies deeper ([17], [18], vol. 1, pp. 247–250).

Theorem 2.2 has an analogue when the circle $\boldsymbol{T}$ is replaced by the real line $\boldsymbol{R}$. In this setting the Hilbert transform $\tilde{f}(x)$ is now the principal value convolution of f with the singular kernel $1/x$. Here the analogue of the map $S[f] \to S_\gamma[f]$ is given by

$$K_\gamma\colon f \to f_\gamma = f * \{k_\gamma\} = \lim_{\varepsilon\to 0}\int_{\varepsilon^{-1}>|y|>\varepsilon} f(x-y)\left\{\frac{1}{\pi}\,\frac{\operatorname{sgn} y}{|y|^{1+i\gamma}}\right\}dy.$$

THEOREM 2.4. *If $f \in L^2(\boldsymbol{R})$,*

$$\|\tilde{f} - f_\gamma\|_2 = \left(\int_{-\infty}^{\infty} |\tilde{f}(x) - f_\gamma(x)|^2\,dx\right)^{1/2} \to 0 \quad as \quad \gamma \to 0.$$

However, there is a function $h(x) \in L^p(\boldsymbol{R})$ for each p, $0 < p < \infty$, with the property that $h_\gamma(x)$ diverges as $\gamma \to 0$ at every point x.

Proof. We begin with the identity

$$\|f_\gamma\|_2 = \sqrt{\frac{2}{\pi\gamma}\tanh\frac{\gamma\pi}{2}}\,\|f\|_2 = A_\gamma\|f\|_2 \tag{2.1}$$

which is equivalent to proving that $A_\gamma^{-1}K_\gamma$ is an isometry of $L^2(\boldsymbol{R})$. For this it suffices to show that $A_\gamma^{-1}K_\gamma$ fixes the L^2 norm of the characteristic function of each interval $[a, b]$ ([2]). For such a characteristic function χ

$$\|K_\gamma\chi\|_2^2 = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\left|\int_a^b \frac{\operatorname{sgn}(y-x)}{|y-x|^{1+i\gamma}}\,dx\right|^2 dy \tag{2.2}$$
$$= \frac{1}{\pi^2\gamma^2}\int_{-\infty}^{\infty}\left|\,|y-b|^{-i\gamma} - |y-a|^{-i\gamma}\right|^2 dy$$
$$= \frac{4}{\pi^2\gamma^2}\int_{-\infty}^{\infty}\sin^2\left(\frac{\gamma}{2}\ln\left|\frac{y-b}{y-a}\right|\right)dy$$
$$= \frac{4(b-a)}{\pi^2\gamma^2}\int_{-\infty}^{\infty}\sin^2\left(\frac{\gamma}{2}\ln\left|\frac{x-1}{x}\right|\right)dx$$

where the last step follows from substituting $x = \dfrac{y-a}{b-a}$. Let $u = \ln\left|\dfrac{x-1}{x}\right|$. Then $du = \dfrac{1}{x(x-1)}dx$. If $x \in (0, 1)$, $e^u = -\dfrac{x-1}{x}$, so $x = \dfrac{1}{e^u+1}$, and $dx = -\dfrac{e^u}{(e^u+1)^2}du$. However, if $x \notin [0, 1]$, $e^u = \dfrac{x-1}{x}$, so $dx = \dfrac{e^u}{(e^u-1)^2}du$. Thus

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{\gamma}{2}\ln\left|\frac{x-1}{x}\right|\right)dx = \int_{-\infty}^{0} + \int_0^1 + \int_1^{\infty}\sin^2\left(\frac{\gamma}{2}\ln\left|\frac{x-1}{x}\right|\right)dx \tag{2.3}$$
$$= \int_0^{\infty}\sin^2\left(\frac{\gamma}{2}u\right)\frac{e^u}{(e^u-1)^2}\,du + \int_{+\infty}^{-\infty}\sin^2\left(\frac{\gamma}{2}u\right)\frac{-e^u}{(e^u+1)^2}\,du +$$
$$+\int_{-\infty}^{0}\sin^2\left(\frac{\gamma}{2}u\right)\frac{e^u}{(e^u-1)^2}\,du$$
$$= \int_{-\infty}^{\infty}\sin^2\left(\frac{\gamma}{2}u\right)\left[\frac{e^u}{(e^u-1)^2} + \frac{e^u}{(e^u+1)^2}\right]du$$
$$= \int_{-\infty}^{\infty}\sin^2\left(\frac{\gamma}{2}u\right)\frac{\cosh u}{\sinh^2 u}\,du$$

$$= -\frac{\sin^2\left(\frac{\gamma}{2}u\right)}{\sinh u}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left[2\sin\frac{\gamma}{2}u\cdot\cos\frac{\gamma}{2}u\cdot\frac{\gamma}{2}\right]\frac{1}{\sinh u}\,du$$

$$= \frac{\gamma}{2}\int_{-\infty}^{\infty}\frac{\sin\gamma u}{\sinh u}\,du = \left(\frac{\gamma\pi}{2}\right)\tanh\frac{\gamma\pi}{2}$$

([9], p. 503).

From (2.2), (2.3), and the definition of A_γ, we find $\|A_\gamma^{-1}K_\gamma\chi\|_2^2 = (b-a)$, which proves equality (2.1), since $b-a = \int\chi^2$ is the square of the L^2 norm of χ.

Now let $f\in L^2$. Write $f = g+b$ where b has small L^2 norm and $g\in C_0^\infty(\boldsymbol{R})$, the infinitely differentiable compactly supported functions. It will be shown in the proof of Theorem 3.6 below that $\|\tilde g - g_\gamma\|_2\to 0$ as $\gamma\to 0$. From (2.1),

$$\|\tilde b - b_\gamma\|_2 \leqslant \|\tilde b\|_2 + \|b_\gamma\|_2 \leqslant (1+A_\gamma)\|b\|_2$$

and $A_\gamma = 1+O(\gamma) = O(1)$ as $\gamma\to 0$, which proves the positive part of Theorem 2.4.

We transfer our example from the circle to the line. Define f by letting its Fourier Transform $\hat f(\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}\,dx$ be a series of blips centered at $-a_n$, where $a_n = 2^{8^n}$. More precisely let $\hat f(\xi)$ have height $1/n$ where ξ is in the interval $[-a_n-\frac12, -a_n+\frac12]$, $n = 1, 2, \ldots$, and let $\hat f(\xi) = 0$ otherwise. Not only is f in L^2 (immediate from Plancherel's theorem); it is also in L^p for $1<p<\infty$ as the following argument shows. We first find $f(x)$ explicitly:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(\xi)e^{-ix\xi}\,d\xi = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-a_n-1/2}^{-a_n+1/2} e^{-ix\xi}\,d\xi$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{e^{ia_nx}}{n}\cdot\frac{e^{-ix/2}-e^{ix/2}}{-2i\dfrac{x}{2}}$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{e^{ia_nx}}{n}\cdot\frac{\left(\sin\dfrac{x}{2}\right)}{\left(\dfrac{x}{2}\right)}.$$

Then

$$\|f\|_p^p = \left(\frac{1}{2\pi}\right)^p\sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)}\left|\frac{\sin(x/2)}{x/2}\right|^p\cdot\left|\sum_{n=1}^{\infty}\frac{e^{ia_nx}}{n}\right|^p dx,$$

$\left|\dfrac{\sin(x/2)}{x/2}\right|$ is bounded by $2/|k|$ on $[2\pi k, 2\pi(k+1)]$ if $k\neq 0, -1$ and by 1 on $[-2\pi, 0]$ and $[0, 2\pi]$, and $\int_{2\pi k}^{2\pi(k+1)}\left|\sum_{n=1}^{\infty}\dfrac{e^{ia_nx}}{n}\right|^p dx$ is a constant b_p independent of k; so that

$$\|f\|_p^p \leqslant \left(\frac{1}{2\pi}\right)^p\left(2+2\sum_{k=1}^{\infty}\frac{2^p}{|k|^p}\right)b_p < \infty.$$

We now show that for almost every real x,

$$\limsup_{\gamma\to 0}|\mathrm{Re}(K_\gamma * f)(x)| = +\infty.$$

It is well known that

$$\hat K_\gamma(\xi) = c_\gamma|\xi|^{i\gamma}\operatorname{sgn}(\xi)$$

where

$$\operatorname{sgn}(\xi) = \begin{cases} 1, & \xi>0,\\ 0, & \xi=0,\\ -1, & \xi<0,\end{cases} \quad\text{and}\quad |c_\gamma| = \sqrt{\frac{\tanh(\pi\gamma/2)}{\pi\gamma/2}}.$$

(See [8], [1].) Note that c_γ is bounded for γ small. Hence,

$$(K_\gamma * f)(x) = (\hat K_\gamma\cdot\hat f)\check{}\,(x) = \frac{c_\gamma}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-a_n-1/2}^{-a_n+1/2}|\xi|^{i\gamma}\operatorname{sgn}(\xi)e^{-ix\xi}\,d\xi.$$

Let $|\xi|^{i\gamma} = a_n^{i\gamma} + (|\xi|^{i\gamma} - a_n^{i\gamma})$ and note $\operatorname{sgn}(\xi) = -1$ to obtain

$$(K_\gamma * f)(x) = \frac{-c_\gamma}{2\pi}\left\{\sum_{n=1}^{\infty}\frac{e^{ia_nx}}{n}a_n^{i\gamma}\right\}\left(\frac{\sin(x/2)}{x/2}\right) + E, \tag{2.4}$$

where

$$|E| \leqslant \frac{|c_\gamma|}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-a_n-1/2}^{-a_n+1/2}\left||\xi|^{i\gamma} - a_n^{i\gamma}\right|d\xi.$$

By the mean value theorem,

$$|E| \leqslant \frac{|c_\gamma|}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sup_{\xi\in[a_n-1/2,\, a_n+1/2]}|(e^{i\gamma\ln\xi})'|\cdot\frac{1}{2}$$

$$\leqslant \frac{|c_\gamma||\gamma|}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n(a_n-\frac12)} = O(\gamma) \quad\text{as}\quad \gamma\to 0.$$

As in the counterexample of Theorem 1 the curly bracketed term of (2.4) has the limsup of its real part infinite as $\gamma \to 0$. Since E tends to 0, $f_\gamma(x)$ diverges as $\gamma \to 0$ whenever $x \notin \{2n\pi: n = 1, -1, 2, -2, \ldots\}$.

To produce the required counterexample for Theorem 2.4 we replace $\frac{\sin(x/2)}{x/2}$ (which is the Fourier transform of the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$) by a function φ which is strictly positive, rapidly decreasing (i.e., $\xi^\eta \varphi(\xi) \to 0$ as $|\xi| \to \infty$ for every integer η), and whose inverse Fourier transform is supported in $[-\frac{1}{2}, \frac{1}{2}]$. To construct φ, let $\check{\psi}(\xi) \not\equiv 0$ be any even C^∞ function supported in $[-\frac{1}{4}, \frac{1}{4}]$. Then $\check{\psi} * \check{\psi}$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$, $(\check{\psi} * \check{\psi})^\wedge = \psi^2$ is non-negative and analytic. Let $\{x_n\} = X$ be the zeroes of ψ^2 and let $D = \{x_n - x_m\}_{n,m=1,2,\ldots}$. Since D is countable, there is a real number $\tau \notin D$. Then $\varphi = \psi^2(x) + \psi^2(x-\tau)$ is rapidly decreasing, strictly positive (if $x \in X$, then $x - \tau \notin X$ since $x - (x-\tau) = \tau \notin D$), and has inverse Fourier transform $\check{\varphi}(\xi) = (1 + e^{-i\xi\tau})(\check{\psi} * \check{\psi})(\xi)$ supported in $[-\frac{1}{2}, \frac{1}{2}]$.

Finally, define

$$h(x) = \varphi(x) \sum_{n=1}^{\infty} \frac{e^{ia_n x}}{n} \quad \text{so that} \quad \hat{h}(\xi) = \sum_{n=1}^{\infty} \frac{\hat{\varphi}(\xi + a_n)}{n}.$$

The argument that showed $f \in L^p$, $1 < p < \infty$ shows $h \in L^p$, $0 < p < \infty$ since $\sum |\varphi(n)|^p < \infty$ for each $p > 0$. Furthermore, the expansion of $K_\gamma * h(x)$ consist of an error term which tends to 0 with γ and a main term which is just like the first term on the right hand side of (2.4) with $\frac{\sin(x/2)}{x/2}$ replaced by $\varphi(x)$. Since φ is nowhere 0, there are no points where $K_\gamma * h(x)$ might converge.

In order to present the postponed proof of Theorem 2.1 we need a technical lemma.

Lemma 2.5. *For fixed real $y \neq 0$ the function $v_\gamma(\xi, y) = \int_1^\xi e^{-ity} t^{i\gamma} dt$ satisfies $|v_\gamma(\xi, y)| \leqslant c(y) < \infty$ where $c(y)$ depends neither on γ, $0 \leqslant \gamma \leqslant 1$ nor on ξ, $\xi \geqslant 1$.*

Proof. Integrate by parts twice:

$$v_\gamma(\xi, y) = \frac{t^{i\gamma} e^{-ity}}{-iy}\Big|_1^\xi + \frac{\gamma}{y}\left[\frac{t^{i\gamma-1} e^{-ity}}{-iy}\Big|_1^\xi - \frac{(i\gamma - 1)}{-iy} \int_1^\xi t^{i\gamma-2} e^{-ity} dt\right].$$

Since $|\xi^{i\gamma}| = |e^{iy}| = |i| = 1$,

$$|v_\gamma(\xi, y)| \leqslant \frac{2}{|y|} + \frac{\gamma}{|y|}\left[\left(\frac{1}{\xi|y|} + \frac{1}{|y|}\right) + \frac{|i\gamma - 1|}{|y|} \int_1^\xi \frac{dt}{t^2}\right] \leqslant \frac{2}{|y|} + \frac{2 + \sqrt{2}}{y^2}.$$

Proof of Theorem 2.1. Write

$$f_\gamma(y) = \left\{\left(\frac{1}{\pi} \frac{\operatorname{sgn}(\xi)}{|\xi|^{1+i\gamma}}\right)^\wedge \hat{f}(\xi)\right\}^\vee (y) = \frac{c_\gamma}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \operatorname{sgn}(\xi) |\xi|^{i\gamma} e^{-i\xi y} d\xi.$$

If $\hat{f} \in L^1$, let $\gamma \to 0$, using Lebesgue's dominated convergence theorem to justify the interchange of limit and integral. Now assume the other hypothesis and decompose the domain of integration into $[-1, 1] \cup (1, \infty) \cup (-\infty, -1)$. On $[-1, 1]$, $\hat{f}$ is of bounded variation, hence bounded, hence integrable. The integral over the second domain may be written as

$$\int_1^\infty \hat{f}(\xi) dv_\gamma(\xi, y) = \hat{f}(\xi) v_\gamma(\xi, y)\Big|_1^\infty - \int_1^\infty v_\gamma(\xi, y) d\hat{f}(\xi) = -\int_1^\infty v_\gamma(\xi, y) d\hat{f}(\xi)$$

and limit and integral may be interchanged since v_γ is uniformly bounded by Lemma 2.5. The third part is similarly treated.

3. Higher dimensional results. In n dimensions, we can study extensions of the type considered by Calderón and Zygmund for the Hilbert transform.

Following Muckenhoupt [12], we define

$$K_\gamma(x) = \frac{\Omega(x)}{|x|^{n+i\gamma}}$$

where Ω is homogeneous of degree 0. As usual, the type of theorem proved will depend upon the assumptions placed on Ω. Under the mere assumption that $\Omega \in L^1(\Sigma)$, Muckenhoupt [12] is able to prove that K_γ defines a bounded operator from L^p to L^p, $1 < p < \infty$. However, his bound depends on γ, and grows to $+\infty$ as γ converges to 0. By placing additional restrictions on Ω, we will show that the operator K_γ converges in L^p to K_0 — the associated Calderón–Zygmund operator of real homogeneity.

Let

$$K_\gamma^{\varepsilon,\eta}(x) = \begin{cases} K_\gamma(x), & \varepsilon < |x| < \eta, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following result:

Theorem 3.1. *Suppose $\int_\Sigma \Omega(t) dt = 0$ and ω_1, the integral modulus of continuity of Ω (defined in [3]), satisfies the Dini condition*

$$\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta < \infty.$$

Then $\sup_{\varepsilon,\eta} |K_\gamma^{\varepsilon,\eta}|$ is weak type $(1, 1)$ and strong type (p, p) $(1 < p < \infty)$, *with constants independent of* γ $(|\gamma|<1)$. *As* $\eta\to\infty$, *and* $\varepsilon\to 0$ *successively,* $K_\gamma^{\varepsilon,\eta}$ *tends to a limit pointwise almost everywhere and in* L^p *norm. Call this limit* K_γ. *Further we have*

(1) $|\{K_\gamma * f>\lambda\}|\leqslant \frac{c}{\lambda}\|f\|_1,$

(2) $\|K_\gamma * f\|_p\leqslant c_p\|f\|_p,\ 1<p<\infty,$ *where* $c_p\leqslant c\left(\frac{1}{p-1}+p\right)$.

Proof. The conclusions of the theorem follow from well known results (see for example [13]) provided the following conditions are satisfied.

(a) $|\int_{\varepsilon<|x|<\eta} K_\gamma(x)\,dx|<c$, c independent of ε,η,γ,

(b) $\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<\eta} K_\gamma(x)\,dx$ exists,

(c) $\int_{a<|x|<2a}|K_\gamma(x)|\,dx<c$, c independent of a,γ,

(d) $\int_{|x|>2|y|}|K_\gamma(x-y)-K_\gamma(x)|\,dx<c$, c independent of γ.

Due to the mean value zero on Ω, (a) and (b) are automatic. Condition (c) follows as in the case of real homogeneity. Condition (d) is the subject of the following lemma.

Lemma 3.2. *Suppose* $\int_\Sigma \Omega(t)\,dt=0$, $|\gamma|\leqslant 1$, *and* ω_1, *the integral modulus of continuity of* Ω, *satisfies the Dini condition. Then there exists a constant* C *such that, for any* $b>0$,

$$\int_{|x|\geqslant 2b}|K_\gamma(x-y)-K_\gamma(x)|\,dx<C \tag{3.1}$$

whenever $|y|<b$. C *is independent of* b *and* γ.

Proof.

$$\int_{|x|\geqslant 2b}|K_\gamma(x-y)-K_\gamma(x)|\,dx=\int_{|x|\geqslant 2b}|K_\gamma(x-by_0)-K_\gamma(x)|\,dx$$

where $|y_0|<1$. Let $x=bu$, so $dx=b^n du$, and the integral equals

$$\int_{|u|\geqslant 2}|K_\gamma(b(u-y_0))-K_\gamma(bu)|b^n\,du$$

$$=\int_{|u|\geqslant 2}|K_\gamma(u-y_0)-K_\gamma(u)|\,du=\int_{|u|\geqslant 2}\left|\frac{\Omega(u-y_0)}{|u-y_0|^{n+i\gamma}}-\frac{\Omega(u)}{|u|^{n+i\gamma}}\right|du$$

$$\leqslant\int_{|u|\geqslant 2}\left|\frac{\Omega(u-y_0)-\Omega(u)}{|u-y_0|^{n+i\gamma}}\right|du+\int_{|u|\geqslant 2}|\Omega(u)|\left|\frac{1}{|u-y_0|^{n+i\gamma}}-\frac{1}{|u|^{n+i\gamma}}\right|du$$

$$=\mathrm{I}+\mathrm{II}.$$

To estimate II let $|u|=r$, so that $|u-y_0|\geqslant|u|-|y_0|>r-1$, while $|u-y_0|\leqslant|u|+|y_0|<r+1$. Define d by $|u-y_0|=r+d$, so $|d|<1$. Then, expressing II in polar coordinates,

$$\mathrm{II}=\int_2^\infty\int_\Sigma|\Omega(u')|\left|\frac{1}{|ru'-y_0|^{n+i\gamma}}-\frac{1}{r^{n+i\gamma}}\right|r^{n-1}du'\,dr$$

$$=\int_2^\infty\int_\Sigma|\Omega(u')|\,|(r+d)^{-n-i\gamma}-r^{-n-i\gamma}|r^{n-1}du'\,dr$$

and

$$|(r+d)^{-n-i\gamma}-r^{-n-i\gamma}|\leqslant|d|(n+|\gamma|)(r-1)^{-n-1}<(n+1)(r-1)^{-n-1}.$$

Thus,

$$\mathrm{II}\leqslant\int_\Sigma|\Omega(u')|\,du'\int_2^\infty(n+1)(r-1)^{-n-1}r^{n-1}dr=c_n\Omega_1.$$

The second integral is finite because the integrand is continuous for $r\geqslant 2$ and is $O(r^{-2})$ as $r\to\infty$. But

$$\mathrm{I}\leqslant\int_{|v|\geqslant 1}\frac{|\Omega(v)-\Omega(v+y_0)|}{|v|^n}\,dv,$$

and so by a result in [3], p. 65, $\mathrm{I}\leqslant C=C(\Omega)$, completing the proof of Lemma 3.2, and hence Theorem 3.1.

By modifying the conditions on Ω, various extentions of Theorem 3.1 are possible.

The non-negative function ω is said to be in A_∞ if there are positive constants C,δ such that given any cube Q and any measurable subset $E\subset Q$, $\frac{\omega(E)}{\omega(Q)}\leqslant C\left(\frac{|E|}{|Q|}\right)^\delta$, where $\omega(A)=\int_A\omega(x)\,dx$ and $|A|$ denotes Lebesgue measure of A.

Theorem 3.3. *If* $\omega\in A_\infty$, *and* Ω *has mean value* 0, *and satisfies the smoothness condition*

$$|\Omega(x-y)-\Omega(x)|\leqslant\frac{c|y|}{|x|},$$

then

$$\int_{R^n}|K_\gamma * f(x)|^p\omega(x)\,dx\leqslant c_p\int_{R^n}|f^*(x)|^p\omega(x)\,dx,\qquad 0<p<\infty,$$

where f^* *represents the Hardy-Littlewood maximal function.*

Proof. The kernel K_γ satisfies conditions (a), (b), and (c) of Theorem III in [6].

THEOREM 3.4. *Suppose Ω is odd and $\Omega \in L^1(\Sigma)$. Then for $1 < p < \infty$ and $|\gamma| \leqslant 1$, we have*

(i) $\|K_\gamma^{\varepsilon,\eta} * f\|_p \leqslant C\|f\|_p$,

(ii) *as $\eta \to \infty$ and $\varepsilon \to 0$ successively, $K_\gamma^{\varepsilon,\eta} * f$ tends to a limit $K_\gamma f$ pointwise and in L^p and $\|K_\gamma f\|_p \leqslant C\|f\|_p$, where C depends only on p, n, and Ω.*

Proof. Apply the standard method of rotations argument, using the 1 dimensional result from Theorem 3.1. Note that the smoothness assumption there is automatic in the case $n = 1$.

THEOREM 3.5. *Suppose $\Omega \in L^1(\Sigma)$, Ω has mean value 0, and*

$$\Omega(x) + \Omega(-x) \in L\log^+ L(\Sigma).$$

Then for $1 < p < \infty$ and $|\gamma| < 1$, we have

(i) $\|K_\gamma^{\varepsilon,\eta} * f\|_p \leqslant c\|f\|_p$, *$c$ depending only on p, n and Ω,*

(ii) *as $\eta \to \infty$ and $\varepsilon \to 0$, $K_\gamma^{\varepsilon,\eta} * f$ tends to a limit $K_\gamma f$ pointwise and in L^p, and $\|K_\gamma f\|_p \leqslant c\|f\|_p$, c independent of γ.*

Proof. Decompose Ω as a sum of even and odd functions,

$$\Omega(x) = \tfrac{1}{2}[\Omega(x) + \Omega(-x)] + \tfrac{1}{2}[\Omega(x) - \Omega(-x)].$$

The odd part is treated by Theorem 3.4.

For Ω even, use the Riesz transforms, and follow the standard arguments, as for the case of real homogenity. See for example [4]. For more details see [1].

THEOREM 3.6. *If $f \in L^p$, $1 < p < \infty$, and Ω satisfies the conditions of Theorem 3.1, 3.4 or 3.5 then $K_\gamma f$ converges to $K_0 f$ in L^p norm, as $\gamma \to 0$.*

Proof. For each positive integer n, we may select $f_n \in C_0^1$ so that

$$\|f - f_n\|_p \leqslant \frac{1}{n}.$$

Then

$$\begin{aligned}\|(K_\gamma - K_0)f\|_p &\leqslant \|K_\gamma(f - f_n)\|_p + \|(K_\gamma - K_0)f_n\|_p + \|K_0(f_n - f)\|_p \\ &\leqslant 2C_p \cdot \frac{1}{n} + \|(K_\gamma - K_0)f_n\|_p.\end{aligned}$$

Thus, it suffices to prove that for $f \in C_0^1$,

$$\|(K_\gamma - K_0)f\|_p \to 0 \quad \text{as} \quad \gamma \to 0.$$

Now choose R such that $\operatorname{supp} f \subset \{x\colon |x| \leqslant R\}$. Suppose $|x| \leqslant 2R$. Then if $|x - y| > 3R$, $|y| > R$, so $f(y) = 0$. Thus,

$$\begin{aligned}|(K_\gamma - K_0)f(x)| &= \Big|\int_{|x-y|\leqslant 3R} (K_\gamma - K_0)(x-y)f(y)\,dy\Big| \\ &= \Big|\int_{|x-y|\leqslant 3R} (K_\gamma - K_0)(x-y)[f(y) - f(x)]\,dy\Big|,\end{aligned}$$

because $\int_\Sigma \Omega(t)\,dt = 0$. Since $f \in C_0^1$, $|f(y) - f(x)| \leqslant C|y - x|$ and we have

$$\begin{aligned}|(K_\gamma - K_0)f(x)| &\leqslant C\int_{|x-y|\leqslant 3R} |(K_\gamma - K_0)(x-y)|\,|x-y|\,dy \\ &= C\int_{|u|\leqslant 3R} \big||u|^{-n-i\gamma} - |u|^{-n}\big|\,|\Omega(u')|\,|u|\,du \\ &= C\int_\Sigma |\Omega(u')|\,du' \int_0^{3R} |r^{-i\gamma} - 1|\,dr.\end{aligned}$$

The integrand of the second integral in the last expression tends to zero pointwise as $\gamma \to 0$ (except for $r = 0$) and is uniformly bounded. Thus, by the bounded convergence theorem, $|(K_\gamma - K_0)f(x)| \to 0$ as $\gamma \to 0$, uniformly for $|x| \leqslant 2R$. Now to treat $|x| > 2R$, write $(K_\gamma - K_0)f = f_1 + f_2$ where

$$f_1(x) = \begin{cases}(K_\gamma - K_0)f(x) & \text{if} \quad |x| \leqslant 2R, \\ 0 & \text{otherwise}.\end{cases}$$

We have proved $f_1(x)$ converges uniformly to 0, and since it has compact support, $\|f_1\|_p \to 0$ as $\gamma \to 0$. But

$$|f_2(x)| = \Big|\int (K_\gamma - K_0)(y)f(x-y)\,dy\Big| \leqslant C\int_{|x-y|\leqslant R} |(K_\gamma - K_0)(y)|\,dy,$$

$|x| \geqslant 2R$, where $C = \sup|f(x)|$. Since $|(K_\gamma - K_0)(y)| = |\Omega(y')|\,|y|^{-n}\big||y|^{-i\gamma} - 1\big|$, and $|y| \geqslant |x| - R$, we have

$$|f_2(x)| \leqslant C(|x| - R)^{-n}\int_{|x-y|\leqslant R} |\Omega(y')|\,\big|1 - |y|^{-i\gamma}\big|\,dy. \tag{3.2}$$

Since $\Omega(y')$ is integrable over compact sets, $\big|1 - |y|^{-i\gamma}\big| \leqslant 2$, and $1 - |y|^{-i\gamma} \to 0$ as $\gamma \to 0$, the right-hand side of (3.2) converges to zero pointwise as $\gamma \to 0$. Moreover,

$$|f_2(x)| \leqslant 2C(|x| - R)^{-n}\int_{|x-y|\leqslant R} |\Omega(y')|\,dy = g(x),$$

and if we show $\|g\|_p$ (norm over $|x| \geqslant 2R$) is finite, by the Lebesgue dominated convergence theorem, we can conclude $\|f_2(x)\|_p \to 0$ as $\gamma \to 0$. Now

$$|g(x)| \leqslant C|x|^{-n}\int_{|x-y|\leqslant R} |\Omega(y')|\,dy \leqslant 2RC|x|^{-n}\int_\Sigma |\Omega(y')|\,dy' \leqslant C|x|^{-n},$$

since $\int_{|x-y|\leqslant R.} |\Omega(y')|\,dy$ may be calculated by integrating first over lines through the origin, the interval of integration on each line being at most $2R$ in length, and Ω being constant; and then over the portion of Σ met by such lines. But $C|x|^{-n}$ is in $L^p(\{x\colon |x| \geqslant 2R\})$.

Remark. The kernels K_γ seem to be somewhat nicer than their classical counterparts. Recall, for example, that K_γ is a bounded operator if Ω is merely in $L^1(\Sigma)$; furthermore, Ω does not even have to have mean value zero. As $\gamma\to 0$ the bounds on the operator K_γ sometimes blow up. It is easy to see that the bounds of K_γ always blow up if we do not assume Ω has mean value zero. The following argument shows that the operator bounds also may blow up if $\Omega\notin L\log^+L$, even if we demand that Ω have mean value zero.

More precisely, if $\varphi(u), u\geqslant 0$ is a non-negative, non-decreasing function of u which is $o(u\log^+u)$ as $u\to\infty$, then there is an Ω in $\varphi(L)$ of mean value zero such that the norms (as operators on L^p) of the associated K_γ's are not uniformly bounded on $(0,\gamma_0)$ for any positive γ_0. To see this choose Ω as in [16], so that the associated operator $-K_0-$ is unbounded. Assume the K_γ's are uniformly bounded for $0<\gamma<\gamma_0$ and let $f\in L^p$. Then $K_\gamma f$ is a Cauchy net as $\gamma\to 0$. This follows exactly as in the proof of Theorem 3.6. Thus as $\gamma\to 0$, K_γ converges as an operator on L^p to some operator $\overline{K}$, which by continuity is bounded on L^p. However if $f\in C_0^1$, the second half of the argument of Theorem 3.6 shows $K_\gamma f\to K_0 f$ in L^p norm. Thus $K_0f=\overline{K}f$ for $f\in C_0^1$. Since C_0^1 is dense in L^p, $K_0=\overline{K}$ as operators on L^p. Hence, K_0 is bounded on L^p, a contradiction.

4. Other approximations. In Section 2 we saw that $\dfrac{1}{\pi}\dfrac{\operatorname{sgn}x}{|x|^{1+i\gamma}}=k_\gamma(x)$ does not approximate $\dfrac{1}{\pi}\dfrac{\operatorname{sgn}x}{|x|^{1}}$ sufficiently well. In this section we examine two other approximating methods.

I. *Non-tangential approximation.* Our first approach is to replace $1+i\gamma$ by $\beta+i\gamma$ and study convergence as $\beta+i\gamma\to 1$ in different ways. Since $\beta=1$ was not quite successful we will stay away from the line $\beta=1$. More exactly, let $\beta=1+\delta$ where $\delta>0$ and define

$$k_{\delta+i\gamma}(x)=\frac{1}{x}\,|x|^{(\delta+i\gamma)\operatorname{sgn}(1-|x|)}=\begin{cases}\dfrac{1}{x}\,|x|^{\delta+i\gamma}, & 0<|x|\leqslant 1,\\[2mm] \dfrac{1}{x}\,\dfrac{1}{|x|^{\delta+i\gamma}}, & |x|>1.\end{cases}$$

THEOREM 4.1. *If $f\in L(\boldsymbol{R})$, then at almost every point of $\boldsymbol{R}$ we have*

(i) $f_{\delta+i\gamma}(x)=\lim\limits_{\varepsilon\to0}\dfrac{-1}{\pi}\displaystyle\int_\varepsilon^\infty [f(x+t)-f(x-t)]k_{\delta+i\gamma}(t)\,dt$ *exists for each $\delta>0$ and*

(ii) $f_{\delta+i\gamma}(x)\to\tilde f(x)$ *as $\delta+i\gamma\to 0$ non-tangentially, i.e., $\delta\searrow 0$ and $\dfrac{|\gamma|}{\delta}$ remains bounded.*

Proof. It suffices to prove the theorem at a point x where the Hilbert transform $\tilde f(x)$ exists ([14], p. 217). Fix such an x and let

$$S(t)=\frac{-1}{\pi}\int_1^t\frac{f(x+s)-f(x-s)}{s}\,ds,\quad s_1=\lim_{\varepsilon\to0}-S(\varepsilon),\quad\text{and}\quad s_2=\lim_{t\to\infty}S(t)$$

so that $\tilde f(x)=s_1+s_2$.

Let $c(\delta+i\gamma,t)=-\dfrac{d}{dt}\big(tk_{\delta+i\gamma}(t)\big)$ and integrate the expression in (i) by parts to get

$$S(t)\big(tk_{\delta+i\gamma}(t)\big)\Big|_\varepsilon^\infty+\int_\varepsilon^\infty S(y)c(\delta+i\gamma,y)\,dy.$$

As $\varepsilon\to0$ the integrated term tends to 0, so that

$$f_{\delta+i\gamma}(x)=\lim_{\varepsilon\to0}\int_\varepsilon^\infty S(y)c(\delta+i\gamma,y)\,dy$$

if that limit should exist. Write $c(\delta+i\gamma,y)=c(\delta+i\gamma,y)\chi_{[0,1]}(y)+{}$ $+c(\delta+i\gamma,y)\chi_{(1,\infty)}(y)=c_1+c_2$, where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a,b). It suffices to show that as $\delta+i\gamma\to0$ non-tangentially

$$\lim_{\varepsilon\to0}\int_\varepsilon^1 S(y)c_1(\delta+i\gamma,y)\,dy\to s_1\quad\text{and}\quad\int_1^\infty S(y)c_2(\delta+i\gamma,y)\,dy\to s_2.$$

Both these facts are simple summability results which require verifying only that the kernels c_1 and c_2 satisfy the usual regularity conditions:

$$\int_0^1|c_1(\delta+i\gamma,y)|\,dy\leqslant C,\qquad\int_1^\infty|c_2(\delta+i\gamma,y)|\,dy\leqslant C$$

where C is independent of $\delta+i\gamma$,

$$\int_\varepsilon^1|c_1(\delta+i\gamma,y)|\,dy\to0,\qquad\int_1^{1/\varepsilon}|c_2(\delta+i\gamma,y)|\,dy\to0$$

as $\delta+i\gamma\to0$ non-tangentially for every positive $\varepsilon<1$, and

$$\int_0^1c_1(\delta+i\gamma,y)\,dy=-1,\qquad\int_1^\infty c_2(\delta+i\gamma,y)\,dy=1$$

when $\delta>0$. (Compare [10], p. 50.)

Remarks. 1. The only deep point in the above proof is the almost everywhere existence of the Hilbert transform. Theorem 4.1 is a regularity result; the corresponding Tauberian theorem would be a proof of the almost everywhere existence of the Hilbert transform of an L^1 function.

2. The kernel $k_{\delta+i\gamma}$ can be replaced by simpler kernels. First we can let $\gamma=0$, thus replacing non-tangential convergence by radial convergence. The result, a special case of Theorem 4.1, is that a.e. the Hilbert transform is given by

$$\lim_{\delta\to 0+}(\text{p.v. }\{f*x^{-1}|x|^{\delta\operatorname{sgn}(1-|x|)}\})$$

where p.v. $(f*g)$ denotes the convolution of f and g in the principal value sense. Second, we can use $|x|^{\delta}$ to damp out the singularity at 0 and not worry about the singularity at infinity. Thus if

$$l_{\delta+i\gamma}(x)=\begin{cases}\frac{1}{x}|x|^{\delta+i\gamma}, & 0<|x|\leqslant 1,\\ \frac{1}{x}|x|^{i\gamma}, & 1<|x|,\end{cases}$$

we have as before that the Hilbert transform is given a.e. as the non-tangential limit of p.v. $(f*l_{\delta+i\gamma})$. Again the special "radial" case of $\gamma\equiv 0$ may be of the most interest.

3. Instead of modifying the kernel we might damp the multiplier instead, replacing $-i\operatorname{sgn}(\xi)$ by $\hat{k}_{\delta+i\gamma}(\xi)=-i\operatorname{sgn}(\xi)\cdot|\xi|^{(\delta+i\gamma)(1-|\xi|)}$, letting $f_{\delta+i\gamma}(x)=\lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}\hat{k}_{\delta+i\gamma}(\xi)\hat{f}(\xi)e^{-ix\xi}d\xi$, and approximating the Hilbert transform $\tilde{f}(x)$ by the non-tangential limit of $f_{\delta+i\gamma}(x)$. The regularity theorem is still just as easy except that this time we need the a.e. existence of the Hilbert transform and a real line version of the Carleson–Hunt theorem on the convergence of Fourier series ([5], [11], [18], vol. 2, p. 242).

4. In the periodic case everything is simpler and we may either convolve the function with $\frac{1}{x}|x|^{\delta+i\gamma}$ or multiply the Fourier coefficients termwise by $|n|^{-(\delta+i\gamma)}$ to obtain similar regularity results.

II. *Averaging*. For simplicity we will restrict ourselves to the Fourier series setting. (The statements and proofs for $\boldsymbol{R}$ are similar.)

In Section 2 of this paper it was shown that if the multiplier $-i\operatorname{sgn}(n)\cdot|n|^{i\gamma}$ is applied to the series $\sum a_ne^{inx}$, the result may not converge as $\gamma\to 0$, even at points where the conjugate series does converge. In an attempt to salvage something from this distressing behavior, and following a suggestion of Professor Zygmund, we tried "smoother" versions of the multiplier.

For simplicity (and, due to Plessner's theorem ([18], vol. 2, p. 216), without loss of generality) we will only look at one-sided series, replacing $-i\operatorname{sgn}n|n|^{i\gamma}$ with $n^{i\gamma}$, etc.

The first version was obtained by replacing $n^{i\gamma}$ by an average

$$\frac{1}{\gamma}\int_0^{\gamma}n^{i\beta}d\beta=\frac{n^{i\gamma}-1}{i\gamma\log n},$$

and taking the limit as $\gamma\to 0$. It is clear that this multiplier is better than the original. Indeed any super-lacunary series (i.e., of the form $\sum a_ke^{in_kx}$ with $\frac{\log n_{k+1}}{\log n_k}\geqslant c>1$), and consequently the counterexample of Theorem 2.2, converges when this multiplier is applied to it. However, we were unable to prove convergence in general. In particular, the multiplier $\frac{n^{i\gamma}-1}{i\gamma\log n}$ does not induce a regular summation method.

However, if we smooth the multiplier again, by taking a second average, the resulting multiplier yields a summation method. More precisely we consider the multiplier

$$m_n(\gamma)=\frac{1}{\gamma}\int_0^{\gamma}\frac{n^{i\beta}-1}{i\beta\log n}d\beta=\frac{1}{i\gamma\log n}\int_0^{\gamma\log n}\frac{e^{iu}-1}{u}du.$$

THEOREM 4.2. *Assume that the series $\sum_{n=1}^{\infty}a_ne^{inx}$ converges, then the series $f_\gamma(x)=\sum_{n=1}^{\infty}a_nm_n(\gamma)e^{inx}$ converges for each fixed γ, and the $\lim_{\gamma\to 0}f_\gamma(x)$ exists.*

Proof. After a summation by parts,

$$f_\gamma(x)=\sum_{n=1}^{\infty}s_n(x)[m_n(\gamma)-m_{n+1}(\gamma)]=\sum_{n=1}^{\infty}s_n(x)[\Delta_n(\gamma)],$$

where

$$s_n(x)=\sum_{k=1}^{n}a_ke^{ikx}.$$

We must show that $\Delta_n(\gamma)$ satisfies the requirements for a regular method. The three conditions are

$$\sum_{n=1}^{\infty}|\Delta_n(\gamma)|<c<\infty\quad\text{for all }\gamma;\tag{4.1}$$

$$\lim_{\gamma\to 0}\Delta_n(\gamma)=0\quad\text{for every }n;\tag{4.2}$$

$$\lim_{\gamma\to 0}\sum_{n=1}^{\infty}\Delta_n(\gamma)=1.\tag{4.3}$$

The second and third conditions are easily checked. The first seems to require some delicate estimates.

We need to study

$$(4.4)\qquad \Delta_n(\gamma) = \frac{1}{\gamma}\frac{1}{\log n}\int_0^{\gamma\log n}\frac{e^{iu}-1}{iu}\,du - \frac{1}{\gamma\log(n+1)}\int_0^{\gamma\log(n+1)}\frac{e^{iu}-1}{iu}\,du$$

$$= \frac{1}{\gamma}\int_0^{\gamma\log n}\frac{e^{iu}-1}{iu}\left[\frac{1}{\log n}-\frac{1}{\log(n+1)}\right]du - $$

$$- \frac{1}{\gamma\log(n+1)}\int_{\gamma\log n}^{\gamma\log(n+1)}\frac{e^{iu}-1}{iu}\,du.$$

For small values of n ($\gamma\log n < 1$) we will use the estimate

$$\frac{e^{iu}-1}{iu} = 1+O(u).$$

Combining terms and using the above estimate we get

$$\Delta_n(\gamma) = \frac{1}{\gamma}\int_0^{\gamma\log n}\big(1+O(u)\big)\frac{\log\left(1+\frac{1}{n}\right)}{\log n\log(n+1)}\,du -$$

$$- \frac{1}{\gamma\log(n+1)}\int_{\gamma\log n}^{\gamma\log(n+1)}\big(1+O(u)\big)\,du$$

$$= \frac{\log\left(1+\frac{1}{n}\right)}{\log(n+1)} + \frac{\log\left(1+\frac{1}{n}\right)}{\log(n+1)}O(\gamma\log n) -$$

$$- \frac{\gamma\log(n+1)-\gamma\log n}{\gamma\log(n+1)}\big[1+O\big(\gamma\log(n+1)\big)\big]$$

$$= \frac{\log\left(1+\frac{1}{n}\right)}{\log(n+1)}O(\gamma\log n) - \frac{\log\left(1+\frac{1}{n}\right)}{\log(n+1)}O\big(\gamma\log(n+1)\big)$$

$$= \frac{\log\left(1+\frac{1}{n}\right)O(\gamma\log n)}{\log(n+1)} = O\left(\frac{\gamma}{n}\right).$$

For large values of n ($\gamma\log n \geqslant 1$), we use the fact that

$$\left|\int_0^x \frac{e^{iu}-1}{iu}\,du\right| \leqslant c\log x + c,$$

which follows from the fact that $\left|\dfrac{e^{iu}-1}{iu}\right| = 1+O(u)$ near the origin and is bounded by $2/u$ away from the origin. We use this estimate on the first integral in (4.4) and the estimate

$$\left|\frac{e^{iu}-1}{iu}\right| \leqslant \frac{2}{u}$$

on the second. We get

$$|\Delta_n(\gamma)|$$

$$\leqslant \frac{1}{\gamma}\left[\frac{1}{\log n}-\frac{1}{\log(n+1)}\right][c\log(\gamma\log n)+c] + \frac{\gamma\log(n+1)-\gamma\log n}{\gamma\log(n+1)}\,\frac{2}{\gamma\log n}$$

$$\leqslant \frac{c\log\left(1+\frac{1}{n}\right)\log(\gamma\log n)}{\gamma\log n\log(n+1)} \leqslant \frac{c\log(\gamma\log n)}{\gamma n\log^2 n}.$$

Combining the two estimates and setting $N = [e^{1/\gamma}]$, we get

$$\sum_{n=1}^{\infty}|\Delta_n(\gamma)| \leqslant \sum_{n=1}^{N}|\Delta_n(\gamma)| + \sum_{n=N}^{\infty}|\Delta_n(\gamma)|$$

$$\leqslant c\sum_{n=1}^{N}\frac{\gamma}{n} + c\sum_{n=N}^{\infty}\frac{\log(\gamma\log n)}{\gamma n\log^2 n}$$

$$\leqslant c\int_1^N \frac{\gamma}{x}\,dx + c\int_N^{\infty}\frac{\log(\gamma\log x)}{\gamma x\log^2 x}\,dx$$

$$\leqslant c\gamma\log N + c\int_{\gamma\log N}^{\infty}\frac{\log u}{u^2}\,du$$

$$\leqslant c\gamma\log e^{1/\gamma} + c\int_1^{\infty}\frac{\log u}{u^2}\,du$$

$$= c < \infty.$$

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Received December 21, 1976 (1237)

STUDIA MATHEMATICA, T. LXV. (1979)

On the sum of two Brownian paths

by

R. KAUFMAN (Bloomington, Ind.)

Abstract. We study a mapping property of the random function of two variables given by $X(s)+X(t)$. This process has a complicated dependence structure, and a combinatorial estimation of product measures is used, in place of martingales. The property is suggested by the Peano curve and the known modulus of continuity of Brownian motion.

Let X denote Brownian motion on the half-line $t \geqslant 0$, let $Z(s, t) = X(s)+X(t)$ on the quadrant $s \geqslant 0$, $t \geqslant 0$, and let F be a compact set in this quadrant.

THEOREM. *If the Hausdorff dimension of F exceeds $1/2$, then for almost all paths X, $Z(F)$ has an interior point.*

Before entering upon the proof, we point out how the present theorem differs from previous results, including [3]. From the viewpoint of probability theory, we observe that the process Z has a complicated dependence structure, for example an identity $Z(a, b)+Z(c, d) = Z(a, d)+Z(b, c)$. We did not succeed in finding a proof based on martingale inequalities, but rely on a direct estimation of moments; the obstacle to a proof following [3] is precisely the presence of relations like the one cited. In the calculation of moments we require a new estimate for the product measures of certain sets in $F \times \ldots \times F$, which may be of interest for Gaussian processes in general. Finally, the process Z seems to be intractable by the method of calculating individual Fourier coefficients, e.g. [1]. If, for example $F = F_1 \times F_2$, wherein $\dim F_1 = \dim F_2 = 0$, then the sets $X(F_1)$ and $X(F_2)$ are subject to no workable restriction; indeed we can find F_1 and F_2 so that the additive groups generated by $X(F_1)$ and $X(F_2)$ have Hausdorff dimension 0 for almost all paths X.

1. In this paragraph we make a few preliminary reductions and write down the integrals whose estimation is the main burden of the proof. The quadrant $s \geqslant 0$, $t \geqslant 0$ is covered by its subsets $s = t$, $0 \leqslant s < t$, $0 \leqslant t < s$. Then F meets one of these sets in a subset of the same dimension as F itself, say F_0, F_1, F_2. In case $\dim F_0 = \dim F > 1/2$, we observe that $Z(s, t) = 2X(s)$ on F_0, and this possibility is easily included in the