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A note on singular integrals

by

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Abstract. The purpose of the paper is to further investigate relationships between various conditions on singular kernels K which imply continuity of the corresponding operator.

1. In the study of the existence and properties of singular integrals

$$\int_{R^n} f(y)K(x-y)\,dy$$

various hypotheses about the kernel K can be made, in addition to the basic properties that $K(x)$ is homogeneous of degree $-n$ (n the dimension of the space) and that the mean value of K over the surface

$$(\Sigma) \qquad |x| = 1$$

of the unit sphere is 0.

One of the earlierst assumptions used was (see e.g., [2]) that the kernel K satisfies the Dini condition on Σ, that is to say that the modulus of continuity $\omega(t)$ of K on Σ be such that

$$(1.1) \qquad \int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

This implicitly presupposes the continuity of K on Σ. If this holds then the transformation

$$\tilde{f}(x) = Tf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|\geqslant\varepsilon} f(y)K(x-y)\,dy = \text{P.V.} \int_{R^n} f(y)K(x-y)\,dy$$

is of type (p, p) for $1 < p < \infty$, and of weak type $(1, 1)$ (see [2]).

It may also be noted that condition (1.1) was merely used to show that

$$(1.2) \qquad \int_{|x|\geqslant 2|y|} |K(x-y)-K(x)|\,dx \leqslant C \quad (y \neq 0)$$

from which the properties of T just stated were derived (see also [1], [5]).

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In view of the importance of singular integrals any weakening or significant modification of assumptions about the kernel K may be of interest. For example, the theorem just stated about the operation $\tilde{f} = Tf$ holds if the modulus of continuity $\omega(t)$ is replaced by the integral modulus of continuity $\omega_1(t)$ (see below), that is, if

$$\int_0^1 \frac{\omega_1(t)}{t}\, dt < \infty \tag{1.3}$$

because, as was shown in [4], (1.3) implies (1.2). In that paper, it was also shown that (1.3) implies

$$\int_\Sigma |K(x)| \log^+ |K(x)|\, d\sigma_x, \tag{1.4}$$

which had been previously known to guarantee that T is of type (p, p), $1 < p < \infty$ (see [3]).

2. In this paper we want to establish some additional relations between the conditions (1.2), (1.3) and (1.4). As we said, (1.3) implies both (1.2) and (1.4). Here we shall show that, conversely, (1.2) implies (1.3) and (1.4). We recall the definition of $\omega_1(t)$ (see [4]). Let ϱ be a proper rotation of $\boldsymbol{R}^n$ about the origin and let

$$|\varrho| = \sup_{|x|=1} |x - \varrho x|.$$

Then

$$\omega_1(t) = \sup_{|\varrho| \leqslant t} \int_\Sigma |K(\varrho x) - K(x)|\, d\sigma_x,$$

where $d\sigma_x$ denotes the surface area element of $\Sigma = \{|x| = 1\}$. We shall also consider two more moduli of continuity of the kernel $K(x)$, namely

$$\omega_2(t) = \omega_2(t, a, b, \bar{y}) = \int_{a \leqslant |x| \leqslant b} |K(x - t\bar{y}) - K(x)|\, dx, \quad |\bar{y}| = 1, \tag{2.1}$$

$$\omega_3(t) = \omega_3(t, a, b) = \sup_{|y| \leqslant t} \int_{a \leqslant |x| \leqslant b} |K(x - y) - K(x)|\, dx, \tag{2.2}$$

where $0 < a < b$ and $|\bar{y}| = 1$.

Let $\alpha > 1$. Setting

$$I_\alpha(y) = \int_{|x| \geqslant \alpha|y|} |K(x - y) - K(x)|\, dx, \quad J_\alpha(K) = \sup_y I_\alpha(y) \tag{2.3}$$

(notice that $I_\alpha(y)$ is a homogeneous function of degree zero of y), our main result can be formulated as follows:

THEOREM. *Let $K(x)$ be positively homogeneous of degree $-n$ and locally integrable in $|x| \neq 0$. Then, if $1 < \alpha < \beta$,*

(i) $I_\beta(y) \leqslant I_\alpha(y) \leqslant 2 \dfrac{\beta - 1}{\alpha - 1} I_\beta(y)$;

(ii) $\dfrac{a}{b - a} \displaystyle\int_0^{a/\alpha} \frac{\omega_2(t)}{t}\, dt \leqslant I_\alpha(y) \leqslant \frac{b}{b - a} \int_0^{b/\alpha} \frac{\omega_2(t)}{t}\, dt$, *where a and b are as in (2.1) and $\bar{y} = y/|y|$;*

(iii) $\dfrac{1}{c} \displaystyle\int_0^{\delta(a/\alpha)} \frac{\omega_3(t)}{t}\, dt \leqslant J_\alpha(K) \leqslant c \int_0^{b/\alpha} \frac{\omega_3(t)}{t}\, dt$, $\delta > 0$, *where δ depends only on the dimension n, and c depends on n, a and b;*

(iv) $\omega_1(t) \leqslant c\omega_3(t)$, $0 < t \leqslant 2$, *where c depends on a, b and n; and finally, if*

$$\lambda = J_\alpha(K) + \int_{a \leqslant |x| \leqslant b} |K(x)|\, dx < \infty,$$

then

(v) $\displaystyle\int_{a \leqslant |x| \leqslant b} \frac{|K(x)|}{\lambda} \log\left(1 + \frac{|K(x)|}{\lambda}\right) dx \leqslant c$ *where c depends on a, b, α and n.*

3. We begin proving (i). Let $\beta - 1 \leqslant 2(\alpha - 1)$. Then

$$\int_{|x| \geqslant \alpha|y|} |K(x - y) - K(x)|\, dx \tag{3.1}$$
$$\leqslant \int_{|x| \geqslant \alpha|y|} |K(x - y) - K(x - y/2)|\, dx + \int_{|x| > \alpha|y|} |K(x - y/2) - K(x)|\, dx.$$

Setting $\bar{x} = x - y/2$ in the first integral on the right above and observing that $|\bar{x} + y/2| \geqslant \alpha|y|$ implies $|\bar{x}| \geqslant (2\alpha - 1)\left|\frac{y}{2}\right| \geqslant \beta\left|\frac{y}{2}\right|$, we see that this integral is majorized by

$$\int_{|\bar{x}| \geqslant \beta\left|\frac{y}{2}\right|} |K(\bar{x} - y/2) - K(\bar{x})|\, d\bar{x}.$$

Now, because $\beta \leqslant 2\alpha$, this also majorizes the second integral on the right of (3.1). Consequently,

$$\int_{|x| \geqslant \alpha|y|} |K(x - y) - K(x)|\, dx \leqslant 2 \int_{|x| \geqslant \beta\frac{|y|}{2}} |K(x - y/2) - K(x)|\, dx$$

and

$$I_\alpha(y) \leqslant 2I_\beta(y/2) = 2I_\beta(y)$$

which implies (i) for $\beta-1\leqslant 2(\alpha-1)$. In particular, we have

$$I_\alpha(y)\leqslant 2I_\beta(y),\qquad \beta-1=2(\alpha-1)$$

and from this we obtain

$$I_\alpha(y)\leqslant 2^kI_\beta(y),\qquad 2^{k-1}(\alpha-1)\leqslant\beta-1\leqslant 2^k(\alpha-1),\qquad k=1,2,\ldots$$

whence (i) follows in the general case.

To prove (ii) we set $x=t\bar{x}$, $y=s\bar{y}$, $|\bar{x}|=|\bar{y}|=1$ in the integral defining $I_\alpha(y)$ in (2.3) and obtain

$$I_\alpha(y)=\int\limits_{|x|\geqslant\alpha|y|}|K(x-y)-K(x)|\,dx=\int\limits_{\Sigma}d\sigma_{\bar{x}}\int\limits_{\alpha s}^{\infty}|K(t\bar{x}-s\bar{y})-K(t\bar{x})|\,t^n\frac{dt}{t},$$

where Σ denotes the unit sphere $|x|=1$ and $d\sigma_{\bar{x}}$ the surface area element. We replace now t by the variable $\bar{t}=\tau\frac{s}{t}$, where τ is a constant for the moment, and, using the homogeneity of $K(x)$, we find that

$$I_\alpha(y)=\int\limits_0^{\tau/\alpha}\frac{d\bar{t}}{\bar{t}}\tau^n\int\limits_{\Sigma}|K(\tau\bar{x}-\bar{t}\bar{y})-K(\tau\bar{x})|\,d\sigma_{\bar{x}}.$$

If we integrate this equation with respect to τ over the interval (a,b), $0<a<b$, and write t for $\bar{t}$, we obtain

$$\text{(3.2)}\qquad I_\alpha(y)\geqslant\frac{1}{b-a}\int\limits_0^{a/\alpha}\frac{dt}{t}\int\limits_a^b\tau^{n-1}a\,d\tau\int\limits_{\Sigma}|K(\tau\bar{x}-t\bar{y})-K(\tau\bar{x})|\,d\sigma_{\bar{x}}$$

$$=\frac{a}{b-a}\int\limits_0^{a/\alpha}\frac{dt}{t}\int\limits_{a\leqslant|x|\leqslant b}|K(x-t\bar{y})-K(x)|\,dx.$$

According to (2.1), this is the first inequality in (ii). Clearly, the second inequality in (ii) can be obtained by a similar argument which we leave to the reader.

The proof of (iii) is more elaborate and depends on the following lemma.

LEMMA 1. *Let A_λ be the annulus $\{x|\ \lambda\leqslant|x|\leqslant 2\lambda\}$ and E a subset of A_λ such that $|E|\geqslant|A_\lambda|c$, where $|E|$ and $|A_\lambda|$ denote the measures of E and A_λ respectively and $c>\frac{1}{2}$. Then the set*

$$E+E=\{x|\ x=x_1+x_2,\ x_1\in E,\ x_2\in E\}$$

contains a sphere $|x|\leqslant\delta\lambda$, δ being a positive number which depends only on c.

It evidently suffices to prove the lemma in the case when $\lambda=1$. Let $|y|\leqslant\delta$ and consider the sets E and $y-E=\{y-x|\ x\in E\}$. They are both contained in the annulus $\bar{A}=\{x|\ 1-\delta\leqslant|x|\leqslant 2+\delta\}$. But $|\bar{A}|=\dfrac{(2+\delta)^n-(1-\delta)^n}{2^n-1}|A_1|$, so that, if δ is chosen so small that $|\bar{A}|<2c|A_1|$, we will have

$$|\bar{A}|<2c|A_1|\leqslant 2|E|=|E|+|y-E|,$$

that is, $|\bar{A}|<|E|+|y-E|$. Because the sets E and $y-E$ are contained in $\bar{A}$, this implies that their intersection is non-empty or, equivalently, that $y\in E+E$. Consequently, every point of $\{|y|\leqslant\delta\}$ is contained in $E+E$, as we wished to show.

Returning to (iii), integrating (3.2) with respect to $\bar{y}$ over the unit sphere $\Sigma=\{|x|=1\}$ we obtain

$$\text{(3.3)}\qquad |\Sigma|J_\alpha(K)\geqslant\frac{a}{b-a}\int\limits_0^{a/\alpha}\frac{dt}{t}\int\limits_{a\leqslant|x|\leqslant b}|K(x-t\bar{y})-K(x)|\,dx\,d\sigma_{\bar{y}}$$

$$=\frac{a}{b-a}\int\limits_{|y|\leqslant a/\alpha}|y|^{-n}\int\limits_{a\leqslant|x|\leqslant b}|K(x-y)-K(x)|\,dx\,dy,$$

where $|\Sigma|$ denotes the surface area of Σ.

Let now $0<a_1<a<b<b_1$, $a_1=a/2$, $b_1=b+a/2$, and set

$$\theta(y)=\int\limits_{a\leqslant|x|\leqslant b}|K(x-y)-K(x)|\,dx,$$

$$\theta_1(y)=\int\limits_{a_1\leqslant|x|\leqslant b_1}|K(x-y)-K(x)|\,dx.$$

Then, if $|y_1|\leqslant a/2$, $|y_2|\leqslant a/2$, we have

$$\text{(3.4)}\qquad \theta(y_1+y_2)\leqslant\int\limits_{a\leqslant|x|\leqslant b}|K(x-y_1-y_2)-K(x-y_1)|\,dx+$$

$$+\int\limits_{a\leqslant|x|\leqslant b}|K(x-y_1)-K(x)|\,dx\leqslant\theta_1(y_1)+\theta_1(y_2).$$

If A_λ denotes the annulus $\lambda\leqslant|x|\leqslant 2\lambda$, and

$$a_\lambda=\frac{1}{|A_\lambda|}\int\limits_{A_\lambda}\theta_1(y)\,dy,\qquad E_\lambda=\{y|\ y\in A_\lambda,\ \theta_1(y)\leqslant 4a_\lambda\},$$

then $\theta_1(y)>4a_\lambda$ on $A_\lambda-E_\lambda$, which clearly implies that $|A_\lambda-E_\lambda|<\frac{1}{4}|A_\lambda|$ and, consequently, $|E_\lambda|>\frac{3}{4}|A_\lambda|$. Now, if $y=y_1+y_2$, $y_1,y_2\in E_\lambda$, then according to (3.4) we have

$$\theta(y)\leqslant\theta_1(y_1)+\theta_1(y_2)<8a_\lambda.$$

But Lemma 1 asserts that $E_\lambda+E_\lambda$ contains the sphere $|y|\leqslant\delta\lambda$ and we find that the preceding inequality holds for $y\leqslant\delta\lambda$. Recalling the definition of $\omega_3(t)$, this shows that $\omega_3(t)<8a_\lambda$ for $t\leqslant\delta\lambda$. Thus we have

$$\int_{\delta\frac{\lambda}{2}}^{\delta\lambda}\omega_3(t)\frac{dt}{t}<8a_\lambda\log 2=8\log 2\frac{1}{|A_\lambda|}\int_{A_\lambda}\theta_1(y)\,dy\leqslant c\int_{A_\lambda}\theta_1(y)\,|y|^{-n}dy.$$

Setting $\lambda=2^{-h}a_1/a=2^{-h}a/2a$, $h=1,2,\ldots$, and adding the corresponding inequalities we obtain

$$\int_0^{\delta\frac{a}{4a}}\omega_3(t)\frac{dt}{t}\leqslant c\int_{|y|\leqslant a_1/a}\theta_1(y)\,|y|^{-n}dy,$$

which combined with (3.3) yields

$$\int_0^{\delta\frac{a}{4a}}\omega_3(t)\frac{dt}{t}\leqslant cJ_a(K),$$

where c depends on a and b. Thus the first half of (iii) is established. The second half follows from the second inequality in (ii) by observing that $\omega_2(t)\leqslant\omega_3(t)$.

We pass now to the proof of (iv). Our argument depends on Lemma 6 in [4], which is also valid in the following slightly different situation.

LEMMA 2. *There exist positive constants c,η depending only on the dimension n such that if*

$$A=\{x|\ a\leqslant|x|\leqslant b\},\qquad a>2\delta_0,$$
$$A'=\{x|\ a-\delta_0\leqslant|x|\leqslant b+\delta_0\},$$
$$A''=\{x|\ a-2\delta_0\leqslant|x|\leqslant b+2\delta_0\},$$

and $h(x)=\big(h_1(x),\ldots,h_n(x)\big)$ is a C^1 vector-valued function satisfying

(a) $|h(x)|\leqslant\delta\leqslant\delta_0$, $\left|\dfrac{\partial h_i}{\partial x_j}(x)\right|\leqslant\eta$ *for all x in A',*

then

$$\int_A\big|f\big(x+h(x)\big)-f(x)\big|\,dx\leqslant c\sup_{|y|\leqslant\delta}\int_{A'}|f(x+y)-f(x)|\,dx \tag{3.5}$$

for every function f integrable in A''.

To prove the lemma we argue as follows. Choosing η sufficiently small, the matrices with entries $\delta_{ij}+\partial h_i/\partial x_j$, which is the functional matrix of the change of variables $z=x+h(x)$, will have a determinant of absolute value larger than $1/2$. Consequently, for any function $g(x)$ we shall have

$$\int_{A'}\big|g\big(x+h(x)\big)\big|\,dx\leqslant 2\int_{A''}|g(z)|\,dz, \tag{3.6}$$

as is readily seen by changing variables in the integral on the left. Let now $\varphi\geqslant 0$ be a function in C_0^∞ with support in $|x|\leqslant 1$ and such that $\int\varphi\,dx=1$, and let

$$\varphi_\delta(x)=\delta^{-n}\varphi\left(\frac{x}{\delta}\right),\qquad f_\delta(x)=f*\varphi_\delta.$$

Let us also denote the supremum on the right-hand side of (3.5) by $\omega(\delta)$. Then, since $\varphi_\delta(y)$ has support in $|y|\leqslant\delta$ and $\int\varphi_\delta dx=1$, we have

$$\int_{A'}|f(x)-f_\delta(x)|\,dx=\int_{A'}\left|\int[f(x)-f(x-y)]\varphi_\delta(y)\,dy\right|dx$$
$$\leqslant\int\varphi_\delta(y)\int_{A'}|f(x)-f(x-y)|\,dx\,dy\leqslant\omega(\delta),$$

that is,

$$\int_{A'}|f(x)-f_\delta(x)|\,dx\leqslant\omega(\delta), \tag{3.7}$$

which combined with (3.6) gives

$$\int_A\big|f\big(x+h(x)\big)-f_\delta\big(x+h(x)\big)\big|\,dx\leqslant 2\omega(\delta). \tag{3.8}$$

On the other hand, because

$$\int\frac{\partial}{\partial y_j}\varphi_\delta(y)\,dy=0\quad\text{and}\quad\int\left|\frac{\partial}{\partial y_j}\varphi_\delta(y)\right|dy\leqslant c\delta^{-1},$$

we have

$$\int_{A'}\left|\frac{\partial f_\delta}{\partial x_j}(x)\right|dx=\int_{A'}\left|\int f(x-y)\frac{\partial}{\partial y_j}\varphi_\delta(y)\,dy\right|dx$$
$$=\int_{A'}\left|\int[f(x-y)-f(x)]\frac{\partial}{\partial y_j}\varphi_\delta(y)\,dy\right|dx\leqslant c\delta^{-1}\omega(\delta).$$

From this and (3.6) which is also valid with $th(x)$, $0\leqslant t\leqslant 1$, replacing $h(x)$, we obtain

$$\int_A\big|f_\delta\big(x+h(x)\big)-f_\delta(x)\big|\,dx=\int_A\left|\int_0^1\sum_{j=1}^n\frac{\partial f_\delta}{\partial x_j}\big(x+th(x)\big)h_j(x)\,dt\right|dx$$
$$\leqslant\delta\sum_j\sup_t\int_A\left|\frac{\partial f_\delta}{\partial x_j}\big(x+th(x)\big)\right|dx$$
$$\leqslant 2\delta\sum_j\int_{A'}\left|\frac{\partial f_\delta}{\partial x_j}(x)\right|dx\leqslant 2nc\omega(\delta),$$

that is,

$$\int_A |f_\delta(x+h(x))-f_\delta(x)|\,dx \leqslant 2nc\,\omega(\delta),$$

and this combined with (3.7) and (3.8) gives the desired result.

Returning to the proof of (iv), let A be the annulus $\{x|\ a+\delta_0 \leqslant |x| \leqslant b-\delta_0\}$, where $\delta_0 < a$, $2\delta_0 < b-a$. Then, if ϱ denotes a rotation of $\boldsymbol{R}^n$ about the origin, we have

$$\int_\Sigma |K(\varrho x)-K(x)|\,d\sigma_x = \frac{1}{\log\dfrac{b-\delta_0}{a+\delta_0}} \int_A |K(\varrho x)-K(x)|\,dx.$$

This is readily seen if one takes into account the fact that $K(x)$ is homogeneous of degree $-n$. Setting $h(x) = \varrho(x)-x$, or $x+h(x) = \varrho(x)$, and using the preceding lemma we find that

$$(3.9)\quad \sup_{|\varrho|\leqslant\delta}\int_\Sigma |K(\varrho x)-K(x)|\,d\sigma_x \leqslant \frac{c}{\log\dfrac{b-\delta_0}{a+\delta_0}}\sup_{|y|\leqslant\delta}\int_{a\leqslant|x|\leqslant b}|K(x+y)-K(x)|\,dx,$$

provided that δ is sufficiently small, say $\delta\leqslant\varepsilon$, where ε depends only on the dimension n. But according to the definitions of ω_1 and ω_3, this inequality is the same as

$$\omega_1(t)\leqslant c\,\omega_3(t),\qquad t\leqslant\varepsilon.$$

In order to extend this inequality to the interval $\varepsilon\leqslant t\leqslant 2$, we observe that the group of proper rotations is compact and connected and, consequently, there exists a finite collection of rotations $\varrho_1, \varrho_2, \ldots, \varrho_k$ such that for every ϱ there exists an element ϱ_{j_1} of this collection with the property that

$$|\varrho x-\varrho_{j_1}x|\leqslant\varepsilon,$$

for all x with $|x| = 1$. Furthermore, there exist $\varrho_{j_1}, \varrho_{j_2}, \ldots, \varrho_{j_l} = I$, where I is the identity rotation, such that

$$|\varrho_{j_i}x-\varrho_{j_{i+1}}x|\leqslant\varepsilon,\qquad |x| = 1,\qquad i = 1, 2, \ldots, l-1.$$

In other words, we have

$$|\varrho_{j_1}^{-1}\varrho|\leqslant\varepsilon,\qquad |\varrho_{j_{i+1}}^{-1}\varrho_{j_i}|\leqslant\varepsilon,\qquad |\varrho_l| < \varepsilon.$$

Now

$$\int_\Sigma |K(\varrho x)-K(x)|\,d\sigma_x \leqslant \int_\Sigma |K(\varrho x)-K(\varrho_{j_1}x)|\,d\sigma_x + \sum_{i=1}^{l-1}\int_\Sigma |K(\varrho_{j_i}x)-K(\varrho_{j_{i+1}}x)|\,d\sigma_x + \int_\Sigma |K(\varrho_l x)-K(x)|\,d\sigma_x$$
$$= \int_\Sigma |K(\varrho_{j_1}^{-1}\varrho x)-K(x)|\,d\sigma_x + \sum_{i=1}^{l-1}\int_\Sigma |K(\varrho_{j_{i+1}}^{-1}\varrho_{j_i}x)-K(x)|\,d\sigma_x + \int_\Sigma |K(\varrho_l x)-K(x)|\,d\sigma_x,$$

and since all rotations in this last expression have modulus less than or equal to ε, we find that

$$\int_\Sigma |K(\varrho x)-K(x)|\,d\sigma_x \leqslant (l+1)\,\omega_1(\varepsilon)\leqslant (k+1)\,\omega_1(\varepsilon)\,d\sigma_x$$

which implies that for $t\geqslant\varepsilon$

$$\omega_1(t)\leqslant (k+1)\,\omega_1(\varepsilon)\leqslant c(k+1)\,\omega_3(t).$$

Now k evidently depends only on the dimension n. Thus (iv) is valid for all t.

Now there only remains to prove (v). Clearly it will suffice to prove this inequality for the positive part K^+ of K (without loss of generality we may assume that K is real). Evidently we have

$$|K^+(x-y)-K^+(x)|\leqslant |K(x-y)-K(x)|,$$

and, on account of (3.3),

$$(3.10)\quad \frac{a}{b-a}\int_{|y|\leqslant a/a}|y|^{-n}\int_{a\leqslant|x|\leqslant b}|K^+(x-y)-K^+(x)|\,dx\leqslant |\Sigma|\,J_a(K).$$

Consider now the maximal function of K^+:

$$\overline{K}(x) = \sup_{t\leqslant a/a}\frac{1}{t^n}\int_{|y|\leqslant t}K^+(x-y)\,dy.$$

Then

$$\overline{K}(x)\leqslant \sup_{t\leqslant a/a}\frac{1}{t^n}\int_{|y|\leqslant t}|K^+(x-y)-K^+(x)|\,dy + \Omega K^+(x),$$

where Ω is the measure of $\{|x|\leqslant 1\}$, and

$$\overline{K}(x)\leqslant \int_{|y|\leqslant a/a}\frac{|K^+(x-y)-K^+(x)|}{|y|^n}\,dy + \Omega K^+(x).$$

Integrating with respect to x and using (3.10) we find that

$$\int_{a\leqslant|x|\leqslant b}\overline{K}(x)\,dx\leqslant |\Sigma|\,\frac{b-a}{a}J_a + \Omega\int_{a\leqslant|x|\leqslant b}K^+(x)\,dx.$$

But a theorem of E. M. Stein (see [6]) asserts that if the maximal function $\bar{K}$ of K^+ is integrable, then so is $K^+\log(1+K^+)$, and since the same argument applies to the negative K^- of K, we conclude that

$$\int_{a\leqslant|x|\leqslant b} |K(x)|\log[1+|K(x)|]\,dx < \infty$$

for every function $K(x)$ which is homogeneous of degree $-n$, is locally integrable in $|x|>0$ and for which $J_a(K)<\infty$ for some a, $a>1$. But this implies the inequality in (v). To prove this implication consider the convex function $\Phi(t) = t\log(1+t)$, $t\geqslant 0$, and the space L_Φ of functions F in $a\leqslant|x|\leqslant b$ with the property that $\Phi(|F|)$ is integrable, and define a norm in L_Φ by (see [7], Chapter IV, Section 10)

$$\int_{a\leqslant|x|\leqslant b} \Phi\left(\frac{|F|}{\|F\|_\Phi}\right)dx = 1.$$

On the other hand, consider also the space B of functions $K(x)$ which are homogeneous of degree $-n$ and for which $J_a(K)<\infty$, $a>1$, with the norm

$$\|K\|_B = J_a(K) + \int_{a\leqslant|x|\leqslant b} |K(x)|\,dx.$$

As is readily verified, B is a Banach space and its embedding in $L^1\{a\leqslant|x|\leqslant b\}$ is continuous.

Now, what we have shown above is that $B\subset L_\Phi$. Consequently, we have

$$B\subset L_\Phi\subset L^1\{a\leqslant|x|\leqslant b\}.$$

But the embedding of B in $L^1\{a\leqslant|x|\leqslant b\}$ is continuous and, as is readily seen, so is that of L_Φ. Thus, according to the closed graph theorem, the embedding of B in L_Φ is also continuous, that is, there exists a constant c such that

$$\|K\|_\Phi \leqslant c\Big(J_a(K) + \int_{a\leqslant|x|\leqslant b} |K|\,dx\Big). \tag{3.11}$$

Clearly, we may assume that $c\geqslant 1$. Now, as is readily verified, the function $\Phi(t)/t^2$ is a decreasing function of t and therefore, since $c\geqslant 1$, we have

$$\frac{\Phi(t/c)}{(t/c)^2}\geqslant\frac{\Phi(t)}{t^2},$$

that is,

$$\Phi(t/c)\geqslant\frac{1}{c^2}\Phi(t),\qquad c\geqslant 1.$$

Thus, setting

$$\lambda = J_a(K) + \int_{a\leqslant|x|\leqslant b} |K(x)|\,dx,$$

(3.11) becomes $\|K\|_\Phi\leqslant c\lambda$, and we obtain

$$1 = \int_{a\leqslant|x|\leqslant b}\Phi\left(\frac{|K|}{\|K\|_\Phi}\right)dx \geqslant \int_{a\leqslant|x|\leqslant b}\Phi\left(\frac{|K|}{c\lambda}\right)dx \geqslant \frac{1}{c^2}\int_{a\leqslant|x|\leqslant b}\Phi\left(\frac{K}{\lambda}\right)dx,$$

that is

$$\int_{a\leqslant|x|\leqslant b}\Phi\left(\frac{K}{\lambda}\right)dx\leqslant c^2,$$

which is the inequality (v). This concludes the proof of our theorem.

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