STUDIA MATHEMATICA, T. LXV. (1979)

Group valued, α-additive set functions

by

J. A. CRENSHAW and R. B. KIRK (Carbondale, Ill.)

Abstract. Let m be a set function on an algebra $\mathscr{F}(\mathscr{W})$ of sets generated by a paving $\mathscr{W}$ into an Abelian topological group G which is complete and Hausdorff. If m is $\mathscr{W}$ regular, then α-additivity on $\mathscr{W}$, for α an infinite cardinal, is sufficient for m to have an extension to the algebra $\mathscr{F}(\mathscr{W}_\alpha)$ generated by the paving $\mathscr{W}_\alpha$ consisting of sets which are the intersection of β sets from $\mathscr{W}$ for $\beta \leqslant \alpha$.

Let G be a complete, Abelian Hausdorff group, and let m denote a G-valued, finitely-additive, s-bounded set function defined on an algebra $\mathscr{F}$ of sets. The problems of extension and decomposition of such set functions have received some attention in the last several years. In [4] Traynor has given a Carathéodory type extension for m when it is countably-additive and has also given a decomposition of m into a countably-additive and a purely finitely-additive part. In [1] Drewnowski has done the same for set functions satisfying more general additivity. Independently in [2], the present authors have given an extension and decomposition theory for real-valued, α-additive set functions satisfying a certain regularity condition.

In this paper we wish to emphasize the role of regularity. Given s-bounded set functions, we assume regularity with respect to an algebra generating paving $\mathscr{W}$. Then we are able to use a Daniell–Bourbaki process to extend the set function when we assume additivity only on the paving $\mathscr{W}$. Moreover, we show that the extended set function is also regular with respect to the larger paving $\mathscr{W}_\alpha$.

In the first section, we establish some notational conventions and prove several key lemmas. Section 2 culminates with Theorem 2.5 which is the main result on extension. For completeness in Section 3 we include a Hewitt–Yosida type decomposition theorem.

1. Notations and basic facts. Throughout the paper G will denote a complete, Abelian, Hausdorff topological group. The symbol $\mathscr{U}$ will denote a neighborhood base at 0 in G consisting of closed symmetric sets. The letter X will denote a fixed set; and $\mathscr{W}$ will denote a family of subsets of X, containing $\emptyset$, and closed with respect to finite unions and intersections. We will assume also that $X \in \mathscr{W}$ (although this condition can be relaxed somewhat for much of what follows). The algebra of subsets of X generated by $\mathscr{W}$ will be denoted by $\mathscr{F}(\mathscr{W})$. The complement of a subset A of X will be denoted by A^c.

A finitely-additive set function m from $\mathscr{F}(\mathscr{W})$ into G is called *strongly-bounded* (*s-bounded*) if for every increasing sequence $\{A_n\} \subset \mathscr{F}(\mathscr{W})$, $\lim m(A_n)$ exists. (Note that m is s-bounded if and only if $\lim m(A_n)$ exists for every decreasing sequence $\{A_n\} \subset \mathscr{F}(\mathscr{W})$.) The set function m is *$\mathscr{W}$-regular* if for all $U \in \mathscr{U}$ and all $A \in \mathscr{F}(\mathscr{W})$, there is a $W \in \mathscr{W}$ with $W \subset A$ such that $m(B) \in U$ whenever $B \in \mathscr{F}(\mathscr{W})$ and $B \subset A - W$.

Throughout the paper, a will stand for a fixed infinite cardinal number. A finitely-additive set function from $\mathscr{F}(\mathscr{W})$ into G is *a-additive* if for every downward directed set $\{W_i\colon i \in I\} \subset \mathscr{W}$ with $\operatorname{card}(I) \leqslant a$ and with $\bigcap \{W_i\colon i \in I\} = \emptyset$, it follows that $\lim m(W_i)$ exists and is 0. (Recall that $\{W_i\colon i \in I\}$ is downward directed if for all $i, j \in I$, there is $k \in I$ with $W_k \subset W_i \cap W_j$.)

Since much of the discussion below involves limits over directed sets of various types, we will develop some notation to streamline these arguments. The Greek letter Δ will be used to denote the collection of all directed families of subsets of X of cardinal at most a. We will use ϱ, σ, τ, ω, etc. to denote elements of Δ, and we will write $\varrho\uparrow$ $(\varrho\downarrow)$ if $\varrho \in \Delta$ is directed upward (downward). If $\varrho = \{R_i\colon i \in I\} \in \Delta$ with $\varrho\uparrow$ $(\varrho\downarrow)$, then for $i, j \in I$, we will write $i \leqslant j$ if $R_i \subset R_j$ $(R_j \subset R_i)$. For $\varrho = \{R_i\} \in \Delta$ and $\sigma = \{S_j\} \in \Delta$ with $\varrho\uparrow$ $(\varrho\downarrow)$ and $\sigma\uparrow$ $(\sigma\downarrow)$, we will write $\varrho \leqslant \sigma$ if for every j, there is an i with $S_j \subset R_i$ $(R_i \subset S_j)$. Let $\varrho \vee \sigma = \{R_i \cap S_j\}$ and $\varrho \wedge \sigma = \{R_i \cup S_j\}$. If $\varrho\uparrow$ $(\varrho\downarrow)$ and $\sigma\uparrow$ $(\sigma\downarrow)$, then $\varrho \vee \sigma$, $\varrho \wedge \sigma \in \Delta$ with $\varrho \vee \sigma\uparrow$ $(\varrho \vee \sigma\downarrow)$, $\varrho \wedge \sigma\uparrow$ $(\varrho \wedge \sigma\downarrow)$ and with $\varrho \wedge \sigma \leqslant \varrho$, $\sigma \leqslant \varrho \vee \sigma$ $(\varrho \vee \sigma \leqslant \varrho$, $\sigma \leqslant \varrho \wedge \sigma)$. For $A \subset X$ and $\varrho = \{R_i\} \in \Delta$, let $\varrho \cap A = \{R_i \cap A\}$. Then $\varrho \cap A \in \Delta$ and $\varrho \cap A\uparrow$ $(\varrho \cap A\downarrow)$ if $\varrho\uparrow$ $(\varrho\downarrow)$. Finally, if $\{g_i\colon i \in I\}$ is a net in G, then when the limit of this net exists, it will be denoted by $\lim\limits_{I} g_i$.

The following useful fact is due to Sion in [3].

LEMMA 1.1. *Let m be a finitely-additive set function from $\mathscr{F}(\mathscr{W})$ into G and let $\varrho = \{R_i\} \in \Delta$ with $\varrho \subset \mathscr{F}(\mathscr{W})$ and $\varrho\uparrow$ (or $\varrho\downarrow$). If $\{m(R_{i_n})\}$ is Cauchy in G for every non-decreasing (non-increasing) sequence $\{R_{i_n}\}$ in $\{R_i\}$, then $\{m(R_i)\}$ is Cauchy in G.*

Proof. Assume $\{m(R_i)\}$ is not Cauchy. Then there is a $U \in \mathscr{U}$ such that for every i, there are $j \geqslant i$ and $k \geqslant i$ with $m(R_j) - m(R_k) \notin U$. By induction it is possible to choose a non-decreasing sequence $\{R_{i_n}\}$ with $m(R_{i_{n+1}}) - m(R_{i_n}) \notin U$ for all n. But then $\{m(R_{i_n})\}$ is not Cauchy. ∎

The following result is another key to the extension theory. (Compare Lemma 1.2 in [1] and Lemma 2.3 in [4].)

LEMMA 1.2. *Let m be a finitely-additive, s-bounded set function from $\mathscr{F}(\mathscr{W})$ into G, and let $\varrho = \{R_i\} \in \Delta$ with $\varrho \subset \mathscr{F}(\mathscr{W})$ and $\varrho\uparrow$ (or $\varrho\downarrow$). Then $\lim\limits_{\varrho} m(R_i \cap B)$ exists uniformly over $B \in \mathscr{F}(\mathscr{W})$. (That is, for every $U \in \mathscr{U}$, there is an i such that $m(B \cap R_j) - m(B \cap R_k) \in U$ for every $B \in \mathscr{F}(\mathscr{W})$ whenever $i \leqslant j, k$.)*

Proof. Since

$$\begin{aligned} m(B \cap R_j) - m(B \cap R_k) &= m\big(B \cap (R_j - R_k)\big) - m\big(B \cap (R_k - R_j)\big) \\ &= m\big([B \cap (R_j \cap R_k \cap R_i^c)^c] \cap (R_j - R_i)\big) - m\big([B \cap (R_j \cap R_k \cap R_i^c)^c] \cap (R_k - R_i)\big), \end{aligned}$$

it is sufficient to prove the following: For every $U \in \mathscr{U}$, there is an i such that $m\big(B \cap (R_j - R_i)\big) \in U$ for all $B \in \mathscr{F}(\mathscr{W})$ whenever $i \leqslant j$. If this is false, then there is a $U \in \mathscr{U}$ such that for every i, there is $j \geqslant i$ and a $B_i \in \mathscr{F}(\mathscr{W})$ with $m\big(B_i \cap (R_j - R_i)\big) \notin U$. By induction we can find sequences $\{R_{i_n}\}\uparrow$ and $\{B_n\} \subset \mathscr{F}(\mathscr{W})$ with $m\big(B_n \cap (R_{i_{n+1}} - R_{i_n})\big) \notin U$. Define

$$C_n = \bigcup_{k=1}^{n} [B_k \cap (R_{i_{k+1}} - R_{i_k})].$$

Then $\{C_n\}\uparrow$ and $m(C_{n+1}) - m(C_n) \notin U$ for all n contrary to the assumption that m is s-bounded. ∎

Let $A \subset X$ and let A^* denote the family of all $\omega \in \Delta$ with $\omega\uparrow$, $\omega \subset \{W^c\colon W \in \mathscr{W}\}$ and $A \subset \bigcup \omega$. (Of course, $\omega \in \Delta$ also means $\operatorname{card}(\omega) \leqslant a$.) Since $\mathscr{W}$ is closed under finite unions, it is easy to check that A^* is a directed set with respect to the relation $\leqslant$ on Δ (defined above).

If m is a finitely-additive, s-bounded set function from $\mathscr{F}(\mathscr{W})$ into G, then by Lemma 1.1, $\lim\limits_{\varrho} m(R_i)$ exists for each $\varrho = \{R_i\} \in \Delta$. We will agree to denote this limit by $\mu(\varrho)$. Hence, for each $A \subset X$, $\{\mu(\omega)\colon \omega \in A^*\}$ is a net in G.

LEMMA 1.3. *Let m be a finitely-additive, s-bounded set function from $\mathscr{F}(\mathscr{W})$ into G. If $A \subset X$, then $\lim\limits_{A^*} \mu(\omega) = m^*(A)$ exists.*

Proof. Assume that this limit does not exist for some $A \in \mathscr{F}(\mathscr{W})$. Then there is a $U \in \mathscr{U}$ such that for all $\omega \in A^*$, there are $\sigma \in A^*$ with $\omega \leqslant \sigma$ and $\mu(\omega) - \mu(\sigma) \notin 3U$. By induction we can find a sequence $\{\omega_n\} \subset A^*$ with $\omega_n \leqslant \omega_{n+1}$ and $\mu(\omega_{n+1}) - \mu(\omega_n) \notin 3U$. Let $\omega_n = \{W_{ni}^c\}$. Choose a sequence $\{U_n\} \subset \mathscr{U}$ with $U_n + U_n \subset U_{n-1} \subset U_0 = U$ for all $n = 1, 2, \ldots$ (This is possible in any topological group by the continuity of addition.) By Lemma 1.2, for each n, we may choose an i_n such that $m\big(B \cap (W_{ni}^c - W_{ni_n}^c)\big) \in U_n$ whenever $i \geqslant i_n$ and $B \in \mathscr{F}(\mathscr{W})$. Since U is closed, we have that

$$\mu(\omega_n) - m(W_{ni_n}^c) = \lim_{\omega_n} m(W_{ni}^c - W_{ni_n}^c) \in U_n. \tag{1}$$

Define $B_n = \bigcap_{k=1}^{n} W^c_{ki_k}$ so that,

$$W^c_{ni_n} - B_n = \bigcup_{k=1}^{n-1} \Big[\Big(\bigcap_{j=k+1}^{n} W^c_{ji_j}\Big) - W^c_{ki_k}\Big]. \tag{2}$$

Since $\omega_k \leqslant \omega_{k+1}$, we may choose an $W^c_{ki} \in \omega_k$ with $W^c_{ki_k} \subset W^c_{ki}$ and $W^c_{k+1,i_{k+1}} \subset W^c_{ki}$. It is now easy to check that there is an $F_k \in \mathscr{F}(\mathscr{W})$ such that

$$\Big(\bigcap_{j=k+1}^{n} W^c_{ji_j}\Big) - W^c_{ki_k} = F_k \cap (W^c_{ki} - W^c_{ki_k}).$$

Next using (2) and the definition of i_k, we obtain

$$m(W^c_{ni_n}) - m(B_n) \in \sum_{k=1}^{n-1} U_k. \tag{3}$$

Since $\{B_n\}\downarrow$, the s-boundedness of m guarantees an n_0 such that $m(B_n - B_m) \in U$ for $n_0 \leqslant m \leqslant n$. Combining this fact with (1) and (3), we find:

$$\begin{aligned}\mu(\omega_{n_0+1}) - \mu(\omega_{n_0}) &= [\mu(\omega_{n_0+1}) - m(W^c_{n_0+1,i_{n_0+1}})] + \\ &\quad + [m(W^c_{n_0+1,i_{n_0+1}}) - m(B_{n_0+1})] + [m(B_{n_0+1}) - m(B_{n_0})] + \\ &\quad + [m(B_{n_0}) - m(W^c_{n_0 i_{n_0}})] + [m(W^c_{n_0 i_{n_0}}) - \mu(\omega_{n_0})] \\ &\in U^c_{n_0+1} + \sum_{k=1}^{n_0} U_k + U + \sum_{k=1}^{n_0-1} U_k + U_{n_0} \subset 3U.\end{aligned}$$

Hence $\mu(\omega_{n_0+1}) - \mu(\omega_{n_0}) \in 3U$ contrary to assumption. ■

2. The α-outer measure. Let $M(\mathscr{W})$ denote the set of all finitely-additive, s-bounded, $\mathscr{W}$-regular set functions from $\mathscr{F}(\mathscr{W})$ into G. The function m^*, defined on the collection of all subsets of X into G by $m^*(A) = \lim_{A^*} \mu(\omega)$ for each $A \subset X$, is the *α-outer measure* associated with m. A set $A \subset X$ is *m^*-measurable* if for each set $B \subset X$, $m^*(B) = m^*(A \cap B) + m^*(A^c \cap B)$. The collection of all m^*-measurable sets is denoted by $\mathscr{F}(m^*)$. Finally, let $\mathscr{W}_a$ denote the family of all subsets of X of the form $\cap\{W_i\colon i \in I\}$ where $\omega = \{W_i\colon i \in I\} \in \Delta$ satisfies $\omega \subset \mathscr{W}$ and $\omega\downarrow$. It is clear that $X \in \mathscr{W}_a$ and that $\mathscr{W}_a$ is closed under finite unions and intersections. Furthermore, since $\mathscr{W} \subset \mathscr{W}_a$, it follows that $\mathscr{F}(\mathscr{W}) \subset \mathscr{F}(\mathscr{W}_a)$. We then have the following

THEOREM 2.1. *Let $m \in M(\mathscr{W})$ with α-outer measure m^*. Then $\mathscr{F}(m^*)$ is an algebra of subsets of X with $\mathscr{F}(\mathscr{W}_a) \subset \mathscr{F}(m^*)$.*

Proof. It is obvious that $X \in \mathscr{F}(m^*)$ and that $A \in \mathscr{F}(m^*)$ if and only if $A^c \in \mathscr{F}(m^*)$. Hence to show that $\mathscr{F}(m^*)$ is an algebra, it is sufficient to show that $A \cap B \in \mathscr{F}(m^*)$ whenever $A, B \in \mathscr{F}(m^*)$. But if $R \subset X$ and $A, B \in \mathscr{F}(m^*)$ we have:

$$\begin{aligned} m^*(R) &= m^*(R \cap A^c) + m^*(R \cap A) \\ &= m^*(R \cap A^c) + m^*(R \cap A \cap B^c) + m^*(R \cap A \cap B) \\ &= m^*\big(R \cap (A \cap B)^c \cap A^c\big) + m^*\big(R \cap (A \cap B)^c \cap A\big) + m^*(R \cap A \cap B) \\ &= m^*\big(R \cap (A \cap B)^c\big) + m^*(R \cap A \cap B). \end{aligned}$$

Hence $A \cap B \in \mathscr{F}(m^*)$ as claimed.

We will now show that $\mathscr{W} \subset \mathscr{F}(m^*)$. (Hence $\mathscr{F}(\mathscr{W}) \subset \mathscr{F}(m^*)$.) Fix $A \subset X$, $U \in \mathscr{U}$ and $W \in \mathscr{W}$. By the $\mathscr{W}$-regularity of m, there is $T \in \mathscr{W}$ with $T \subset W^c$ and $m(C) \in U$ whenever $C \subset T^c \cap W^c$. Let

$$\Gamma = \{\varrho \vee [(\sigma \cap W^c) \wedge (\tau \cap T^c)]\colon\ \varrho \in A^*,\ \sigma \in (A \cap W^c)^* \text{ and } \tau \in (A \cap W)^*\}.$$

Then Γ is a cofinal, directed subset of A^*. (Hence, $m^*(A) = \lim_{\Gamma} \mu(\gamma)$.) Similarly,

$$\Lambda = \{\varrho \vee (\sigma \cap W^c)\colon\ \varrho \in A^* \text{ and } \sigma \in (A \cap W^c)^*\}$$

and

$$\Phi = \{\varrho \vee (\tau \cap T^c)\colon\ \varrho \in (A^*) \text{ and } \tau \in (A \cap W)^*\}$$

are cofinal, directed subsets of $(A \cap W^c)^*$ and $(A \cap W)^*$, respectively. If $\varrho = \{R^c_i\}$, $\sigma = \{S^c_j\}$ and $\tau = \{T^c_k\}$, then $\gamma = \{R^c_i \cap [(S^c_j \cap W^c) \cup (T^c_k \cap T^c)]\}$, $\lambda = \{R^c_i \cap S^c_j \cap W^c\}$ and $\varphi = \{R^c_i \cap T^c_k \cap T^c\}$ are in Γ, Λ and Φ, respectively. We may thus choose i, j and k such that:

$$m^*(A) - m\big(R^c_i \cap [(S^c_j \cap W^c) \cup (T^c_k \cap T^c)]\big) \in U, \tag{1}$$

$$m^*(A \cap W^c) - m(R^c_i \cap S^c_j \cap W^c) \in U, \tag{2}$$

$$m^*(A \cap W) - m(R^c_i \cap T^c_k \cap T^c) \in U. \tag{3}$$

Also we have that,

$$\begin{aligned} &m\big(R^c_i \cap [(S^c_j \cap W^c) \cup (T^c_k \cap T^c)]\big) \\ &\quad = m(R^c_i \cap S^c_j \cap W^c) + m(R^c_i \cap T^c_k \cap T^c) - m(R^c_i \cap S^c_j \cap W^c \cap T^c_k \cap T^c). \end{aligned}$$

Hence by (1), (2), (3) and the fact that $R^c_i \cap S^c_j \cap W^c \cap T^c_k \cap T^c \subset W^c \cap T^c$, we obtain:

$$m^*(A) - m^*(A \cap W^c) - m^*(A \cap W) \in 4U. \tag{4}$$

Since $U \in \mathscr{U}$ was arbitrary, it follows from (4) that $m^*(A) = m^*(A \cap W^c) + m^*(A \cap W)$. Thus $\mathscr{W} \subset \mathscr{F}(m^*)$ as claimed.

In order to show that $\mathscr{W}_a \subset \mathscr{F}(m^*)$, fix $W \in \mathscr{W}_a$ and $U \in \mathscr{U}$. Let $\omega = \{W_r\} \in \Delta$ with $\omega \subset \mathscr{W}$, $\omega\downarrow$ and $W = \bigcap\{W_r\}$. By Lemma 1.2 we may choose r_0 such that $m(B \cap (W_r - W_{r_0})) \in U$ for all $B \in \mathscr{F}(\mathscr{W})$ if $r \geq r_0$. Since m is $\mathscr{W}$-regular, there is $T \in \mathscr{W}$ with $T \subset W^c_{r_0}$ such that $m(B) \in U$ whenever $B \in \mathscr{F}(\mathscr{W})$ and $B \subset W^c_{r_0} \cap T^c$. If $\beta = \{W^c_r\}$, let $\Gamma = \{(\varrho \wedge \beta) \cap T^c$: $\varrho \in (A \cap W^c \cap W_{r_0})^*\}$. Then Γ is a cofinal, directed subset of $(A \cap W^c \cap W_{r_0})^*$. Hence, if $\varrho = \{R^c_i\}$, we may choose i and r such that:

(5) $$m^*(A \cap W^c \cap W_{r_0}) - m(R^c_i \cap W^c_r \cap T^c) \in U,$$

(6) $$W^c_{r_0} \subset W^c_r.$$

Since

$$m(R^c_i \cap W^c_r \cap T^c) = m(R^c_i \cap W^c_{r_0} \cap T^c) + m\big(R^c_i \cap T^c \cap (W^c_r - W^c_{r_0})\big),$$

it follows that:

(7) $$m(R^c_i \cap W^c_r \cap T^c) \in 2U.$$

From (5) and (7), we obtain:

(8) $$m^*(A \cap W^c \cap W_{r_0}) \in 3U.$$

Since $W_{r_0} \in \mathscr{F}(m^*)$ (as shown above), and since $W \subset W_{r_0}$, we have:

$$\begin{aligned} m^*(A \cap W^c) &+ m^*(A \cap W) - m^*(A) \\ &= m^*(A \cap W^c \cap W_{r_0}) + m^*(A \cap W^c_{r_0}) + m^*(A \cap W) - m^*(A) \\ &= m^*(A \cap W^c \cap W_{r_0}) + m^*(A \cap W) - m^*(A \cap W_{r_0}). \end{aligned}$$

Hence from (8), we obtain,

(9) $$m^*(A \cap W^c) + m^*(A \cap W) - m^*(A) \in m^*(A \cap W) - m^*(A \cap W_{r_0}) + 3U.$$

Next let

$$\Lambda = \{(\tau \cap T^c) \vee (\beta \wedge \sigma)\colon\ \tau \in (A \cap W_{r_0})^* \text{ and } \sigma \in (A \cap W)^*\}$$

and

$$\Phi = \{(\tau \cap T^c) \vee \sigma\colon\ \tau \in (A \cap W_{r_0})^* \text{ and } \sigma \in (A \cap W)^*\}.$$

Then Λ and Φ are cofinal, directed subsets of $(A \cap W_{r_0})^*$ and $(A \cap W)^*$, respectively. Hence if $\tau = \{T^c_k\}$ and $\sigma = \{S^c_j\}$, we may choose k, j and r such that:

(10) $$m^*(A \cap W_{r_0}) - m\big(T^c_k \cap T^c \cap (W^c_r \cup S^c_j)\big) \in U,$$

(11) $$m^*(A \cap W) - m(T^c_k \cap T^c \cap S^c_j) \in U,$$

(12) $$W_r \subset W_{r_0}.$$

Since

$$\begin{aligned} m\big(T^c_k \cap T^c \cap (W^c_r \cup S^c_j)\big) &- m(T^c_k \cap T^c \cap S^c_j) \\ &= m(T^c_k \cap T^c \cap W^c_r) - m(T^c_k \cap T^c \cap W^c_r \cap S^c_j) \\ &= m(T^c_k \cap S_j \cap T^c \cap W^c_r) \\ &= m\big(T^c_k \cap S_j \cap T^c \cap (W_{r_0} - W_r)\big) + m(T^c_k \cap S_j \cap T^c \cap W^c_{r_0}), \end{aligned}$$

we see that:

(13) $$m\big(T^c_k \cap T^c \cap (W^c_r \cup S^c_j)\big) - m(T^c_k \cap T^c \cap S^c_j) \in 2U.$$

Combining (10), (11) and (13) it follows that:

(14) $$m^*(A \cap W_{r_0}) - m^*(A \cap W) \in 4U.$$

Finally from (9) and (14) we have:

(15) $$m^*(A \cap W) + m^*(A \cap W^c) - m^*(A) \in 7U.$$

Since $U \in \mathscr{U}$ was arbitrary, we have $m^*(A) = m^*(A \cap W) + m^*(A \cap W^c)$. Hence $W \in \mathscr{F}(m^*)$, and the proof is complete. ■

If $m \in M(\mathscr{W})$, $\overline{m}$ and m_a will denote the restrictions of the α-outer measure m^* to $\mathscr{F}(\mathscr{W}_a)$ and $\mathscr{F}(\mathscr{W})$, respectively. We then have the following

LEMMA 2.2. *Let $W \in \mathscr{W}_a$ and take $\omega = \{W_r\} \in \Delta$, $\omega \subset \mathscr{W}$ with $\omega\downarrow$ and $W = \bigcap\{W_r\}$. Then*

$$\overline{m}(A \cap W^c) = \lim \overline{m}(A \cap W^c_r) \quad \textit{for all } A \in \mathscr{F}(\mathscr{W}_a).$$

Proof. Fix $U \in \mathscr{U}$ arbitrarily. Since $\beta = \{W^c_r\}\uparrow$, by Lemma 1.2 there is an r_0 such that $m(B) \in U$ whenever $B \in \mathscr{F}(\mathscr{W})$ with $B \subset W^c_s - W^c_r$ and $s \geq r \geq r_0$. Fix $r_1 \geq r_0$. Since m is $\mathscr{W}$-regular, there is $T \subset W^c_{r_1}$ with $T \in \mathscr{W}$ such that $m(B) \in U$ whenever $B \in \mathscr{F}(\mathscr{W})$ and $B \subset W^c_{r_1} \cap T^c$. Let $\Gamma = \{\varrho \vee \beta \vee (\sigma \cup T^c)\colon\ \varrho \in (A \cap W^c)^*$ and $\sigma \in (A \cap W^c_{r_1})^*\}$ and $\Lambda = \{\varrho \vee \vee (\sigma \cap W^c_{r_1})\colon\ \varrho \in (A \cap W^c)^*$ and $\sigma \in (A \cap W^c_{r_1})^*\}$. Then Γ and Λ are cofinal, directed subsets of $(A \cap W^c)^*$ and $(A \cap W^c_{r_1})^*$. Hence if $\varrho = \{R^c_i\}$ and $\sigma = \{S^c_j\}$, we may choose i, j and r such that:

(1) $$\overline{m}(A \cap W^c) - m\big(R_i \cap W^c_r \cap (S^c_j \cup T^c)\big) \in U,$$

(2) $$\overline{m}(A \cap W^c_{r_1}) - m(R^c_i \cap S^c_j \cap W^c_{r_1}) \in U,$$

(3) $$W^c_{r_1} \subset W^c_r.$$

Since $m(C \cup D) = m(C) + m(D) - m(C \cap D)$ for $C, D \in \mathscr{F}(\mathscr{W})$, we have,

$$\begin{aligned} m\big(R^c_i \cap W^c_r \cap (S^c_j \cup T^c)\big) &- m(R^c_i \cap S^c_j \cap W^c_{r_1}) \\ &= m\big(R^c_i \cap S^c_j \cap (W^c_r - W^c_{r_1})\big) + m(R^c_i \cap W^c_r \cap T^c) - m(R^c_i \cap S^c_j \cap W^c_r \cap T^c) \\ &= m\big(R^c_i \cap S^c_j \cap (W^c_r - W^c_{r_1})\big) + m(R^c_i \cap S_j \cap W^c_r \cap T^c). \end{aligned}$$

Since $r \geq r_1 \geq r_0$ by (3), we obtain from this that:

(4) $$m\big(R^c_i \cap W^c_r \cap (S^c_j \cup T^c)\big) - m(R^c_i \cap S^c_j \cap W^c_{r_1}) \in m(R^c_i \cap S_j \cap W^c_r \cap T^c) + U.$$

Since

$$m(R_i^c\cap S_j\cap W_r^c\cap T^c)$$
$$= m(R_i^c\cap S_j\cap W_r^c\cap T^c\cap W_{r_1}^c)+m\big(R_i^c\cap S_j\cap T^c\cap (W_r^c-W_{r_1}^c)\big)$$

and since $r_0\leqslant r$, we obtain:

$$m(R_i^c\cap S_j\cap W_r^c\cap T^c)\in 2U. \tag{5}$$

Combining (1), (2), (4) and (5), we have:

$$\overline{m}(A\cap W^c)-\overline{m}(A\cap W_{r_1}^c)\in 5U. \tag{7}$$

Since $U\in\mathscr{U}$ and $r_1\geqslant r_0$ were arbitrary, the proof is complete. ■

LEMMA 2.3. *Let $A, B\in\mathscr{F}(\mathscr{W})$ with $A\subset B$ and let $U\in\mathscr{U}$. Assume that $m(C)\in U$ whenever $C\in\mathscr{F}(\mathscr{W})$ and $C\subset B-A$. Then $\overline{m}(D)\in U$ whenever $D\in\mathscr{F}(\mathscr{W}_a)$ and $D\subset B-A$.*

Proof. Fix $V\in\mathscr{U}$ and fix $D\in\mathscr{F}(\mathscr{W}_a)$. Since m is $\mathscr{W}$-regular, we may take $W, T\in\mathscr{W}$ with $W\subset A$ and $T\subset B^c$ such that $m(C)\in V$ whenever $C\in\mathscr{F}(\mathscr{W})$ and either $C\subset A\cap W^c$ or $C\subset B^c\cap T^c$. Let $\Gamma=\{(\varrho\cap T^c)\vee(\sigma\cup W^c)\colon \varrho\in(B\cap D)^*$ and $\sigma\in(A\cap D)^*\}$ and $\Lambda=\{(\varrho\cap T^c)\vee\sigma\colon \varrho\in(B\cap D)^*$ and $\sigma\in(A\cap D)^*\}$. Then Γ and Λ are cofinal, directed subsets of $(B\cap D)^*$ and $(A\cap D)^*$, respectively. Hence, if $\varrho=\{R_i^c\}$ and $\sigma=\{S_j^c\}$, we may choose i and j such that

$$\overline{m}(B\cap D)-m\big(R_i^c\cap T^c\cap(S_j^c\cup W^c)\big)\in V, \tag{1}$$

$$\overline{m}(A\cap D)-m(R_i^c\cap T^c\cap S_j^c)\in V. \tag{2}$$

We have

$$m\big(R_i^c\cap T^c\cap(S_j^c\cup W^c)\big)-m(R_i^c\cap T^c\cap S_j^c)$$
$$= m(R_i^c\cap T^c\cap W^c)-m(R_i^c\cap T^c\cap S_j^c\cap W^c)$$
$$= m(R_i^c\cap S_j\cap T^c\cap W^c)$$
$$= m(R_i^c\cap S_j\cap T^c\cap A\cap W^c)+m(R_i^c\cap S_j\cap A^c\cap W^c\cap T^c\cap B^c)+$$
$$+m\big(R_i^c\cap S_j\cap W^c\cap T^c\cap(B-A)\big).$$

Using the fact that $m(C)\in U$ if $C\subset B-A$ and that $m(C)\in V$ if $C\subset A\cap W^c$ or $C\subset B^c\cap T^c$, we obtain from this last calculation:

$$m\big(R_i^c\cap T^c\cap(S_j^c\cup W^c)\big)-m(R_i^c\cap T^c\cap S_j^c)\in 2V+U. \tag{3}$$

Combining (1), (2) and (3), we obtain:

$$\overline{m}(B\cap D)-\overline{m}(A\cap D)\in 4V+U. \tag{4}$$

Since $V\in\mathscr{U}$ was arbitrary and since U is closed, it follows that $\overline{m}(B\cap D)-\overline{m}(A\cap D)\in U$.

LEMMA 2.4. *Let $\{W_r\colon r\in I\}\subset\mathscr{W}_a$ be a downward directed set of cardinal at most a and let $W=\bigcap\{W_r\}$. If $U\in\mathscr{U}$, then there is an r_0 such that if $r\geqslant r_0$, then*

$$\overline{m}(A\cap W_r)-\overline{m}(A\cap W)\in U \quad \text{for all } A\in\mathscr{F}(\mathscr{W}_a).$$

Proof. From Theorem 2.1, it is immediate that $\overline{m}$ is finitely-additive on $\mathscr{F}(\mathscr{W}_a)$. Hence, it is enough to show $\overline{m}(A\cap W_r^c)-\overline{m}(A\cap W^c)\in U$ for $r\geqslant r_0$ and $A\in\mathscr{F}(\mathscr{W}_a)$. For each $r\in I$ choose a downward directed set $\{W_{ri}\colon i\in I_r\}\subset\mathscr{W}$ of cardinal at most a such that $W_r=\bigcap\{W_{ri}\colon i\in I_r\}$. Then the family ϱ consisting of all sets of the form $W_{r_1i_1}^c\cup\ldots\cup W_{r_ni_n}^c$ (where n is a natural number and where $r_j\in I$ and $i_j\in I_{r_j}$ for $j=1,\ldots,n$) belongs to $(W^c)^*$. Now fix $U\in\mathscr{U}$ arbitrarily. If $\varrho=\{R_k^c\}$, then by Lemma 1.2 there is a k_0 such that whenever $k_0\leqslant k_1\leqslant k_2$, the following holds:

$$m\big(C\cap(R_{k_2}^c-R_{k_1}^c)\big)\in U, \quad \text{for all } C\in\mathscr{F}(\mathscr{W}). \tag{1}$$

Applying Lemma 2.3, we have that (1) gives for all $k_0\leqslant k_1\leqslant k_2$:

$$\overline{m}(B\cap R_{k_2}^c)-\overline{m}(B\cap R_{k_1}^c)\in U, \quad \text{for all } B\in\mathscr{F}(\mathscr{W}_a). \tag{2}$$

Choose r_0 such that $R_{k_0}^c\subset W_{r_0}^c$, and take both $r_1\geqslant r_0$ and $A\in\mathscr{F}(\mathscr{W}_a)$ arbitrarily. Since $\varrho=\{R_k^c\}\uparrow W^c$, by Lemma 2.2 there is a $k_1\geqslant k_0$ such that:

$$\overline{m}(A\cap W^c)-\overline{m}(A\cap R_{k_1}^c)\in U. \tag{3}$$

Also since $\{R_k^c\cap W_{r_1i}^c\}\uparrow W_{r_1}^c$ and $\{R_{k_0}^c\cap W_{r_1i}^c\}\uparrow R_{k_0}^c$, by Lemma 2.2 there are k_2 and i_1 with $k_2\geqslant k_1$ such that:

$$\overline{m}(A\cap W_{r_1}^c)-\overline{m}(A\cap R_{k_2}^c\cap W_{r_1i_1}^c)\in U, \tag{4}$$

$$\overline{m}(A\cap R_{k_0}^c)-\overline{m}(A\cap R_{k_0}^c\cap W_{r_1i_1}^c)\in U. \tag{5}$$

Since $k_2\geqslant k_1\geqslant k_0$, we have from (2) that

$$\overline{m}(A\cap R_{k_2}^c\cap W_{r_1i_1}^c)-\overline{m}(A\cap R_{k_0}^c\cap W_{r_1i_1}^c)\in U, \tag{6}$$

and

$$\overline{m}(A\cap R_{k_1}^c)-\overline{m}(A\cap R_{k_0}^c)\in U. \tag{7}$$

Combining (3), (4), (5), (6) and (7) we obtain:

$$\overline{m}(A\cap W^c)-\overline{m}(A\cap W_{r_1}^c)\in 5U. \tag{8}$$

Since $U\in\mathscr{U}$, $r_1\geqslant r_0$ and $A\in\mathscr{F}(\mathscr{W}_a)$ were all arbitrarily chosen, the proof is complete.

THEOREM 2.5. *Let $m\in M(\mathscr{W})$. Then $\overline{m}\in M(\mathscr{W}_a)$ and $\overline{m}$ is a-additive.*

Proof. It is immediate from Theorem 2.1 that $\overline{m}$ is finitely-additive and from Lemma 2.4 that $\overline{m}$ is a-additive on $\mathscr{F}(\mathscr{W}_a)$. We must show that $\overline{m}$ is $\mathscr{W}_a$-regular and s-bounded.

Step 1. *$\overline{m}$ is $\mathscr{W}_a$-regular.* As is shown in [2], 1.4, each $A \in \mathscr{F}(\mathscr{W}_a)$ has a representation $A = \bigcup_{s=1}^{n} (W_s^c - V_s^c)$ where n is a natural number, $W_s, V_s \in \mathscr{W}_a$ and $(W_s - V_s) \cap (W_t - V_t) = \emptyset$ for $s \neq t$. In view of this and the finite-additivity of $\overline{m}$, it suffices to check $\mathscr{W}_a$-regularity on sets of the form W^c with $W \in \mathscr{W}_a$. Hence fix $W \in \mathscr{W}_a$ and $U \in \mathscr{U}$. Take $V \in \mathscr{U}$ with $V + V \subset U$. Let $\{W_r\} \subset \mathscr{W}$ be a downward directed family of cardinal at most a with $W = \bigcap \{W_r\}$. By Lemma 2.4 there is an r_0 such that for all $r \geqslant r_0$ and all $B \in \mathscr{F}(\mathscr{W}_a)$,

$$\overline{m}(B \cap W^c) - \overline{m}(B \cap W_r^c) \in V. \tag{1}$$

Since m is $\mathscr{W}$-regular, there is a $T \in \mathscr{W}$ with $T \subset W_{r_0}^c$ and $m(B) \in V$ whenever $B \in \mathscr{F}(\mathscr{W})$ and $B \subset W_{r_0}^c \cap T^c$. Let $A \in \mathscr{F}(\mathscr{W}_a)$ be arbitrary. Since $\Gamma = \{\varrho \cap T^c \cap W_{r_0}^c : \varrho \in (A \cap T^c \cap W_{r_0}^c)^*\}$ is a cofinal, directed subset of $(A \cap T^c \cap W_{r_0}^c)^*$ (if $\varrho = \{R_i^c\}$) the fact that V is closed gives

$$\overline{m}(A \cap T^c \cap W_{r_0}^c) = \lim_{\Gamma} \lim_{\gamma} m(R_i^c \cap T^c \cap W_{r_0}^c) \in V. \tag{2}$$

Combining (1) and (2) (with $B = A \cap T^c$ in (1)), we have that

$$\overline{m}(A \cap W^c \cap T^c) = \overline{m}\big(A \cap T^c \cap (W^c - W_{r_0}^c)\big) + \overline{m}(A \cap T^c \cap W_{r_0}^c) \in 2V \subset U. \tag{3}$$

Since $A \in \mathscr{F}(\mathscr{W}_a)$ was arbitrary, it follows from (3) that $\overline{m}(B) \in U$ whenever $B \in \mathscr{F}(\mathscr{W}_a)$ and $B \subset W^c \cap T^c$. Thus $\overline{m}$ is $\mathscr{W}_a$-regular as claimed.

Step 2. *$\overline{m}$ is s-bounded.* Let $\{A_n\}$ be a sequence in $\mathscr{F}(\mathscr{W}_a)$ with $\{A_n\}\downarrow$. Let $U \in \mathscr{U}$ be arbitrary. For $n = 1, 2, \ldots$ choose $U_n \in \mathscr{U}$ with $U_n + U_n \subset U_{n-1} \subset U_0 = U$. Since $\overline{m}$ is $\mathscr{W}_a$-regular, there is $T_n \in \mathscr{W}_a$ with $T_n \subset A_n$ and with $\overline{m}(B) \in U_n$ whenever $B \in \mathscr{F}(\mathscr{W}_a)$ and $B \subset A_n - T_n$. Define $W_n = \bigcap_{k=1}^{n} T_k$. Then

$$\begin{aligned}\overline{m}(A_n - T_n) &= \overline{m}\Big((A_n - T_n) \cup \bigcup_{k=1}^{n-1} \Big[\Big(A_n \cap \bigcap_{j=k+1}^{n} T_j\Big) - T_k\Big]\Big)\\ &= \overline{m}(A_n - T_n) + \sum_{k=1}^{n-1} \overline{m}\Big(\Big(A_n \cap \bigcap_{j=k+1}^{n} T_j\Big) - T_k\Big),\end{aligned}$$

so that

$$\overline{m}(A_n - W_n) \in \sum_{k=1}^{n} U_k \subset U. \tag{4}$$

Since a is an infinite cardinal, $\mathscr{W}_a$ is closed under countable intersections so that $W = \bigcap W_n \in \mathscr{W}_a$. Since $\{W_n\}\downarrow$, it follows from Lemma 2.4 that $\{\overline{m}(W_n)\}$ is a Cauchy sequence in G. Hence there is an n_0 such that

$$\overline{m}(W_n) - \overline{m}(W_m) \in U \quad \text{if} \quad n_0 \leqslant n \leqslant m. \tag{5}$$

From (4) and (5), we obtain for $n_0 \leqslant n \leqslant m$,

$$\begin{aligned}\overline{m}(A_n) - \overline{m}(A_m) = [\overline{m}(A_n) - \overline{m}(W_n)] + [\overline{m}(W_n) - \overline{m}(W_m)] +\\ + [\overline{m}(W_m) - \overline{m}(A_m)],\end{aligned}$$

so that

$$\overline{m}(A_n) - \overline{m}(A_m) \in 3U \quad \text{if} \quad n_0 \leqslant n, m. \tag{6}$$

Since $U \in \mathscr{U}$ was arbitrary, it follows that $\{\overline{m}(A_n)\}$ is a Cauchy sequence in G. The proof is complete. ∎

COROLLARY 2.6. *Let $m \in M(\mathscr{W})$. Then $m_a \in M(\mathscr{W})$ and m_a is a-additive. Furthermore, $m = m_a$ if and only if m is a-additive.*

Proof. Since m_a is the restriction of $\overline{m}$ to $\mathscr{F}(\mathscr{W})$, it is immediate from Theorem 2.5 that m_a is finitely-additive, s-bounded and a-additive. Note that in Step 1 of the proof of Theorem 2.5, we found for an arbitrary $U \in \mathscr{U}$ and $W \in \mathscr{W}_a$ a $T \in \mathscr{W}$ with $T \subset W^c$ and $\overline{m}(B) \in U$ whenever $B \in \mathscr{F}(\mathscr{W}_a)$ and $B \subset W^c \cap T^c$. This fact together with the representation of $A \in \mathscr{F}(\mathscr{W})$ as $A = \bigcup_{j=1}^{n} (W_j - V_j)$ (where $W_j, V_j \in \mathscr{W}$ and $(W_j - V_j) \cap \cap (W_i - V_k) = \emptyset$ for $i \neq j$) yields the $\mathscr{W}$-regularity of $\overline{m}$.

If $m = m_a$, then m is obviously a-additive. Hence assume that m is a-additive. Then for $W \in \mathscr{W}$, we have

$$m_a(W^c) = \lim_{(W^c)^*} \lim_{|\omega|} m(W_i^c \cap W^c) = m(W^c).$$

Since, in particular, $m_a(X) = m(X)$, we have $m_a(W) = m(W)$ for all $W \in \mathscr{W}$. Since m_a and m are both $\mathscr{W}$-regular, it follows that $m = m_a$. ∎

COROLLARY 2.7. *Let $m \in M(\mathscr{W})$. If $v \in M(\mathscr{W}_a)$ is a-additive and if m_a is the restriction of v to $\mathscr{F}(\mathscr{W})$, then $v = \overline{m}$.*

Proof. Let $W \in \mathscr{W}_a$ and let $\{W_r\} \subset \mathscr{W}$ be a downward directed set of cardinal at most a with $W = \bigcap \{W_r\}$. Then $v(W) = \lim v(W_r) = \lim m_a(W_r) = \overline{m}(W)$. The $\mathscr{W}_a$-regularity of v and $\overline{m}$ now gives $v = \overline{m}$. ∎

3. A Hewitt–Yosida type decomposition. Let $m \in M(\mathscr{W})$. Then m is a-singular if for each $U \in \mathscr{U}$ and for each a-additive $m_0 \in M(\mathscr{W})$, there is an $A \in \mathscr{F}(\mathscr{W})$ such that $m(A \cap B) \in \mathscr{U}$ and $m_0(A^c \cap B) \in \mathscr{U}$ for all $B \in \mathscr{F}(\mathscr{W})$. (We remark that the proof of the converse of the following theorem is modeled on that given by Traynor in [4].)

THEOREM 3.1. *Let $m \in M(\mathscr{W})$. Then m is a-singular if and only if $m_a = 0$.*

Proof. Let m be a-singular. If $U \in \mathscr{U}$, there is an $A \in \mathscr{F}(\mathscr{W})$ with $m(A \cap B) \in \mathscr{U}$ and $m_a(A^c \cap B) \in U$ for all $B \in \mathscr{F}(\mathscr{W})$. Using Lemma 2.3

it follows that $m_a(A\cap B)\in\mathscr{U}$ for all $B\in\mathscr{F}(\mathscr{W})$. But then for $B\in\mathscr{F}(\mathscr{W})$, we have

$$m_a(B) = m_a(A\cap B)+m_a(A^c\cap B)\in 2U.$$

Since $U\in\mathscr{U}$ was arbitrary, it follows that $m_a(B) = 0$ for all $B\in\mathscr{F}(\mathscr{W})$.

Now assume that $m_a = 0$ and assume that m is not a-singular. This means that there is an a-additive $m_0\in M(\mathscr{W})$ and a $U\in\mathscr{U}$ such that for every $A\in\mathscr{F}(\mathscr{W})$, if $m_0(B\cap A)\in U$ for every $B\in\mathscr{F}(\mathscr{W})$, then there is a $B\in\mathscr{F}(\mathscr{W})$ with $m(B\cap A^c)\notin U$. Let $\{U_n\}$ be a sequence in $\mathscr{U}$ with $U_n+ +U_n\subset U_{n-1}\subset U_0 = U$ for $n = 1, 2, \ldots$ Let $A_0 = \emptyset$ and assume that $A_0, A_1, \ldots, A_n$ have been defined such that:

$$A_i\cap A_j = \emptyset \quad \text{for } i\neq j, \tag{1}$$

$$m(A_i)\notin U_1 \quad \text{for } i = 1, \ldots, n, \tag{2}$$

$$m_0(B\cap A_i)\in U_i \quad \text{for all } B\in\mathscr{F}(\mathscr{W}) \text{ and all } i = 1, \ldots, n. \tag{3}$$

Define $C = \left(\bigcup_{i=0}^{n} A_i\right)^c$. Then

$$m_0(B\cap C^c) = \sum_{i=0}^{n} m(B\cap A_i)\in\sum_{i=1}^{n} U_i\subset U \quad \text{for all } B\in\mathscr{F}(\mathscr{W})$$

by (1) and (3). Hence there is $B_0\in\mathscr{F}(\mathscr{W})$ with $B_0\subset C$ with $m(B_0)\notin U$. Since $m_a(B_0) = 0$, there is $\omega = \{W_r^c\}\in(B_0)^*$ such that $\lim_{\varrho} m(R_i^c)\in U_3$ for all $\varrho = \{R_i^c\}\in(B_0)^*$ with $\varrho\geqslant\omega$. Since m is $\mathscr{W}$-regular, there is $T\in\mathscr{W}$ with $T\subset B_0^c$ such that $m(B\cap B_0^c\cap T^c)\in U_2$ for all $B\in\mathscr{F}(\mathscr{W})$. Since m_0 is a-additive, $m_0 = (m_0)_a$ by Corollary 2.6. Let $\varrho = \{W_r^c\cap T^c\}$. Then $\varrho\in(B_0)^*$ and $\varrho\geqslant\omega$. By Lemma 2.4 and the above remarks, we may take r_0 such that

$$m(W_{r_0}^c\cap T^c)\in U_2, \tag{7}$$

$$m_0(B\cap W_{r_0}\cap B_0)\in U_{n+1} \quad \text{for all } B\in\mathscr{F}(\mathscr{W}). \tag{8}$$

Then we have that

$$m(B_0\cap W_{r_0})\notin U_1. \tag{9}$$

(Otherwise,

$$\begin{aligned} m(B_0) &= m(B_0\cap W_{r_0})+m(B_0\cap W_{r_0}^c)\\ &= m(B_0\cap W_{r_0})+m(T^c\cap W_{r_0}^c)-m(T^c\cap W_{r_0}^c\cap B_0^c)\\ &\in U_1+U_2+U_2\subset U, \end{aligned}$$

contrary to the fact that $m(B_0)\notin U$.) Thus, from (8) and (9), the sequence $A_0, A_1, \ldots, A_{n+1}$ satisfies (1), (2) and (3). Hence by induction there is an infinite sequence $\{A_n\}\subset\mathscr{F}(\mathscr{W})$ of pairwise disjoint sets with $m(A_n)\notin U_1$ for $n = 1, 2, \ldots$ This contradicts the fact that m is s-bounded. The proof is complete. ■

THEOREM 3.2. *Let $m\in M(\mathscr{W})$. Then there are unique elements $m_1, m_2\in M(\mathscr{W})$ such that m_1 is a-additive, m_2 is a-singular and $m = m_1+m_2$.*

Proof. Let $m_1 = m_a$ and $m_2 = m-m_a$. Then m_1 is an a-additive element of $M(\mathscr{W})$ by Corollary 2.6. Since $m, m_a\in M(\mathscr{W})$, it is immediate that $m_2\in M(\mathscr{W})$. From the definition of m^*, it is immediate that $(m_2)_a = (m-m_a)_a = m_a-(m_a)_a = m_a-m_a = 0$. (That $(m_a)_a = m$ is immediate from Lemma 2.4 and Corollary 2.6.) Since $(m_2)_a = 0$, m_2 is a-singular by Theorem 3.1. Finally, if $m = m_1'+m_2'$ with $m_1', m_2'\in M(\mathscr{W})$, m_1' a-additive and m_2' a-singular, then $m_1 = m_a = (m_1'+m_2')_a = (m_1')_a+(m_2')_a = m_1'$ by Lemma 2.4, Corollary 2.6 and Theorem 3.1. Since $m_1' = m_1$, we also have $m_2' = m_2$. ■

References

[1] L. Drewnowski, *Decompositions of set functions*, Studia Math. 48 (1973), pp. 23–48.

[2] R. B. Kirk, and J. A. Crenshaw, *Extension of real-valued a-additive set functions*, Studia Math. 58 (1976), pp. 129–139.

[3] M. Sion, *Outer measures with values in a topological group*, Proc. London Math. Soc. 19 (1969), pp. 89–106.

[4] Tim Traynor, *A general Hewitt-Yosida decomposition*, Canad. J. Math. 26 (1972), pp. 1164–1169.

Received June 25, 1976
Revised version April, 1977 (1174)