**A local version of the generalized retract theorem
of Ważewski with applications in the theory of partial
differential equations of first order**

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Abstract. In paper [3] the author considered a generalization of the classical retract theorem of Ważewski. This generalization had been stated for generalized semi-dynamical systems. Here we give another version of the generalized retract theorem with respect to local semi-dynamical systems and we give some applications of it in the theory of partial differential equations of the first order. The results of the paper concerning partial differential equations formally generalize some previous results of the author. In particular, now we need not consider a common domain of existence of solutions of corresponding partial differential equations.

Introduction. In the author's paper [3] there was considered a generalization of the classical retract theorem of Ważewski (cf. [7]–[9]). This generalization was formulated for generalized semi-dynamical systems and involved the notion of a local ordered abelian semi-group. Here we give another version of the generalized retract theorem with regard to local semi-dynamical systems in the sense of Sell [5] and Bhatia and Hajek [1]. Such an approach had been already presented in paper [4]; here we recall it and — since we wish to make the paper reasonably self-contained — we give also the proof, omitting some details. However, the principal purpose of the present paper is to give some applications for partial differential equations of first order. The results of the present paper concerning partial differential equations formally generalize some results of [3]. Moreover, the formulations of our present theorems are simpler than those presented in [3]. In particular, now we need not any conditions concerning estimation constants occurring in [3] and we need not consider a common domain of existence of a family of solutions.

Note that one can extend the general Theorem A (see Section 1.2 below) to local generalized semi-dynamical systems in which R_* can be replaced by an abelian ordered semi-group T such as in [3].

1. GENERAL RESULTS: A GENERALIZED RETRACT THEOREM

1.1. Let X be a topological space, $R_* = [0, \infty)$ the topological semi-group of non-negative numbers, A a subset of $R_* \times X$ such that

$$(0.1) \quad \{0\} \times X \subset A$$

and let π be a mapping from A into X . We put

$$(1.1) \quad I_x = \{t \in R_* : (t, x) \in A\}, \quad \omega_x = \sup I_x \leq \infty$$

(we put $\omega_x = \infty$ if $\sup I_x$ does not exist).

DEFINITION 1 (see Bhatia and Hajek [1], Sell [5]). We say that (X, R_*, A, π) is a *local semi-dynamical system* if and only if

(I) I_x is an interval and the mapping

$$X \ni x \mapsto \omega_x \in R_* \cup \{\infty\}$$

is lower semi-continuous in the sense that for every point $x \in X$:

(a) if $\omega_x < \infty$, then for every $\eta > 0$ there exists a neighbourhood U of x such that

$$y \in U \Rightarrow \omega_y > \omega_x - \eta;$$

(b) if $\omega_x = \infty$, then for every $c \in R_*$ there exists a neighbourhood U of x such that

$$y \in U \Rightarrow \omega_y > c;$$

(II) π is continuous (in A there is the natural topology induced by the Cartesian product $R_* \times X$);

(III) $\pi(0, x) = x$ for every $x \in X$;

(IV) if $t \in I_x$ and $s \in I_{\pi(t, x)}$, then $s + t \in I_x$;

(V) $\pi(s, \pi(t, x)) = \pi(s + t, x)$ for every $t \in I_x$, $s \in I_{\pi(t, x)}$.

We put for every $x \in X$:

$$(1.2) \quad \pi(x) = \{\pi(t, x) : t \in I_x\},$$

$$(1.3) \quad \pi^{-1}(x) = \{y \in X : \text{there exists a number } t \in I_y \text{ such that } \pi(t, y) = x\}$$

and for $x \in X$ and $z \in \pi(x)$:

$$(1.4) \quad t(x, z) = \inf \{s : \pi(s, x) = z\},$$

$$[x, z] = \{\pi(t, x) : t \in I_x \cap [0, t(x, z)]\},$$

$$[x, z) = \{\pi(t, x) : t \in I_x \cap [0, t(x, z))\},$$

$$(1.5) \quad (x, z] = \{\pi(t, x) : t \in I_x \cap (0, t(x, z)]\},$$

$$(x, z) = \{\pi(t, x) : t \in I_x \cap (0, t(x, z))\}.$$

Let M be an open and non-empty subset of the space X ; we denote by ∂M and $\bar{M}$ the boundary and the closure of M , respectively. We put

$$(1.6) \quad S^0(\partial M) = \{x \in \partial M : \text{there is } t \in I_x, t > 0 \text{ such that} \\ (x, \pi(t, x)] \neq \emptyset \text{ and } (x, \pi(t, x)] \cap \bar{M} = \emptyset\},$$

$$(1.7) \quad S(M) = \{x \in \partial M : \text{there is } y \in M \text{ such that } [y, x) \subset M \text{ and} \\ x = \pi(t(y, x), y)\},$$

$$(1.8) \quad S^*(M) = S^0(\partial M) \cap S(M).$$

Remark 1. The points of $S(M)$ have been called by T. Ważewski (cf. [7]–[9]) *egress points*, the points of $S^*(M)$ *strict egress points*. The same notation had been used in [4], but in [2] and [3] there were used the symbols M_e and M_{se} . The concept of egress and strict egress points had been introduced by T. Ważewski.

If $x \in \bar{M}$, then (see the notation in [4]) we denote by

$$(1.9) \quad c(x; M)$$

the point $z \in \partial M$ such that:

(i) in the case of $x \in M$

$$z = \pi(t(x, z), x) \quad \text{and} \quad [x, z) \subset M,$$

(ii) in the case of $x \in \partial M$

$$z = x.$$

Remark 2. Such points (called “consequent” for x) have been previously considered by Ważewski, following Poincaré.

Remark 3. It can happen that for some $x \in M$ the point $c(x; M)$ does not exist; but if $\pi(x) \cap (X \setminus \bar{M}) \neq \emptyset$, then $c(x; M)$ exists and is well defined as the unique point $z \in \partial M$ satisfying

$$(1.10) \quad z = \pi(\inf \{t \in I_x : \pi(t, x) \notin \bar{M}\}, x).$$

1.2. THEOREM A. Let $(X, R_*, A; \pi)$ be a local semi-dynamical system, let $M \subset X$ be open and non-empty. Suppose that $D \subset S(M) \cap S^0(\partial M)$ and $N \subset M \cap D$ are such that

$$N \neq \emptyset, \quad D \neq \emptyset$$

and that

- (α) there are no retractions $N \rightarrow N \cap D$,
- (β) the set $N \cap D$ is a retract of the set D ,
- (γ) $S^*(M) = S(M)$.

Then there exists a point $x \in M \cap N$ such that $\pi(x) \subset M$ or there exists a point $x \in M \cap N$ for which $c(x; M) \in S(M) \setminus D$.

COROLLARY. *If all assumptions of Theorem A are satisfied and, moreover, if $S(M) = S^*(M) \subset D$, then there is a point $x \in M \cap N$ such that $\pi(x) \subset M$.*

The above Theorem A was presented in [4]; the essential idea of the proof is the same as that of the proof of the main results of [3]. We now outline the proof omitting the details. Suppose that the assertion of Theorem A is false. Hence for every point $x \in M \cap N$ we have

$$\pi(x) \cap (X \setminus \bar{M}) \neq \emptyset \quad \text{and} \quad c(x; M) \in D.$$

Consider the mapping

$$\sigma: (M \cap N) \cup \partial M \ni x \mapsto t(x, c(x; M)) \in R_+.$$

We have obviously

$$\pi(\sigma(x), x) = c(x; M).$$

PROPOSITION 1. *The mapping σ restricted to the set*

$$(M \cap N) \cup S^0(\partial M)$$

is continuous.

Proof of Proposition 1. 1° Let $x \in M \cap N$ be fixed, let $\varepsilon > 0$ be given. We shall show that there exists a neighbourhood Q of x such that

$$y \in Q \Rightarrow \{\sigma(y) \text{ is well defined and } \sigma(y) \in (\sigma(x) - \varepsilon, \sigma(x) + \varepsilon)\},$$

which means that σ is continuous at the point x .

We put

$$J_{x,\varepsilon} \stackrel{\text{df}}{=} (\sigma(x) - \varepsilon, \sigma(x) + \varepsilon) \cap I_x.$$

Of course,

$$\sigma(x) < \sup J_{x,\varepsilon} \leq \omega_x.$$

We can find a neighbourhood U of x such that

$$(1.11) \quad y \in U \Rightarrow \frac{1}{2}(\sigma(x) + \sup J_{x,\varepsilon}) < \omega_y$$

(since π fulfils condition (I)). Assume that

$$U \subset M \cap N$$

(this is possible without loss of generality). Let us put

$$z = \pi(\sigma(x), x) = c(x; M).$$

We have

$$z \in \partial M \cap S(M) = \partial M \cap S^*(M) \subset S^0(\partial M).$$

Thus there is $t \in I_x$, $t > 0$, such that

$$(1.12) \quad (z, \pi(t, z)] \neq \emptyset \quad \text{and} \quad (z, \pi(t, z)] \cap \bar{M} = \emptyset.$$

It is easy to see that, replacing in (1.12) the number t by any number $s \in (0, t)$, we obtain also true relations. Let us consider a number $s \in (0, t)$ such that

$$(1.13) \quad \sigma(x) + s < \frac{1}{2}(\sigma(x) + \sup J_{x,\varepsilon}).$$

Now, putting $w = \pi(s, z)$, we obtain

$$w = \pi(\sigma(x) + s, x) \in X \setminus \bar{M}.$$

In view of the continuity of the mapping π we can choose a neighbourhood W of x such that

$$(1.14) \quad y \in W \cap U \Rightarrow \sigma(x) + \varepsilon < \omega_y \quad (\text{see (1.11)}) \quad \text{and} \\ \pi(\sigma(x) + s, y) \subset X \setminus \bar{M}.$$

The first condition asserted in (1.14) says that $\sigma(y)$ is well defined, the second means that

$$\sigma(y) < \sigma(x) + s < \sigma(x) + \varepsilon.$$

Hence, in the case where

$$\sigma(x) - \varepsilon < 0,$$

(1.14) gives the following implications:

$$(1.15) \quad y \in W \cap U \Rightarrow \sigma(y) \in [0, \sigma(x) + \varepsilon] \Rightarrow \sigma(y) \in (\sigma(x) - \varepsilon, \sigma(x) + \varepsilon).$$

If

$$\sigma(x) - \varepsilon \geq 0,$$

we consider the interval

$$[0, \sigma(x) - \varepsilon].$$

We have

$$\pi(s, x) \in M \quad \text{for } s \in [0, \sigma(x) - \varepsilon].$$

Since π is continuous and $[0, \sigma(x) - \varepsilon]$ is compact, we have the following

PROPOSITION 2. *There is a neighbourhood Z of the point x (it can be assumed that $Z \subset M \cap N$) such that*

$$(1.16) \quad y \in Z \Rightarrow \pi(s, y) \in M \quad \text{for } s \in [0, \sigma(x) - \varepsilon].$$

Furthermore, we have the following simple implication:

$$(1.17) \quad \text{if } y \in M, \pi(y) \cap (X \setminus \bar{M}) \neq \emptyset \text{ and } [y, \pi(\sigma(x) - \varepsilon, y)) \subset M, \\ \text{then } \sigma(y) > \sigma(x) - \varepsilon.$$

Hence, in virtue of (1.14) and (1.16), we obtain

$$(1.18) \quad y \in W \cap U \cap Z \Rightarrow \sigma(y) \in (\sigma(x) - \varepsilon, \sigma(x) + \varepsilon).$$

The continuity of the mapping π at the point x is thus proved.

2° If $x \in \partial M$, the reasoning is simpler because $\sigma(x) = 0$. We omit the details.

Thus we have proved Proposition 1.

In order to finish the proof of Theorem A let us consider the mapping:

$$(1.19) \quad p: N' \ni x \mapsto c(x; M) \in \partial M,$$

where

$$N' = \{x \in N: \pi(x) \cap (X \setminus \bar{M}) \neq \emptyset\}.$$

From the assumption that our theorem is false, it follows that $N = N'$. Moreover, in virtue of Proposition 1, the mapping (1.19) is continuous, since

$$c(x; M) = \pi(\sigma(x), x) = (\pi \circ (\sigma, \text{Identity in } X))(x) \quad \text{for } x \in N' = N.$$

Let now

$$r: D \rightarrow N \cap D$$

be a retraction (see (β)); consider

$$R = r \circ p.$$

The mapping R is a retraction of the set $N' = N$ onto the set $N \cap D$. However, the existence of such a retraction is impossible according to assumption (α) of the theorem. This contradiction completes the proof of Theorem A.

2. PRELIMINARIES FOR APPLICATIONS

2.1. Suppose that there are given positive constants

$$a, \alpha, K, L \quad (\text{we admit also } a = \infty; \text{ in this case } [0, a) = R_*)$$

and continuous functions

$$s^0, h, N_0, M_0: [0, a) \rightarrow R_*$$

such that

$$(2.1) \quad N_0(0) = h(0) = K, \quad s^0(0) = \alpha.$$

We shall say that *Hypothesis A₁* is satisfied if and only if:

$$(2.2) \quad K \cdot h(x) \cdot M_0(x) \leq L \cdot N_0^2(x) \quad \text{for every } x \in [0, a).$$

We introduce the following sets:

$$(2.3) \quad Q = \{(x, y) \in R^2: 0 \leq x < a, |y| \leq s^0(x)\}$$

and for $x \in [0, a)$

$$(2.4) \quad Q_x = Q \cap ([0, x] \times R) \quad (= \{(t, y): 0 \leq t \leq x, |y| \leq s^0(x)\}),$$

$$(2.5) \quad Q_x^0 = Q \cap ([0, x) \times R),$$

$$(2.6) \quad Q_{(x)} = \{(x, y): |y| \leq s^0(x)\}.$$

Further, we define two classes of functions:

$$(2.7) \quad C_0(a, K) = \text{the family of all functions: } [-a, a] \rightarrow [-K, K] \\ \text{fulfilling the Lipschitz condition with the constant } K,$$

$$(2.8) \quad C_1(a, K, L) = \{\varphi \in C_0(a, K): \varphi \text{ is of class } C^1 \text{ and } \varphi' \text{ fulfils the} \\ \text{Lipschitz condition with the constant } L\}.$$

Let us put

$$(2.9) \quad s_1(x) \stackrel{\text{df}}{=} a \cdot \frac{h(x)}{N_0(x)} \quad \text{and} \quad s(x) \stackrel{\text{df}}{=} \min(s^0(x), s_1(x)), \quad x \in [0, a)$$

(there and elsewhere, by a function of class C^1 in a closed interval we mean a function which is of class C^1 in the interior of this interval and has the right-hand and left-hand derivatives at the endpoints equal to the one-sided limits of the derivative considered in the interior).

2.2. We have the following two lemmas:

LEMMA 1. If $x \in [0, a)$ is fixed, $u: [-s(x), s(x)] \rightarrow [-h(x), h(x)]$ fulfils the Lipschitz condition with a constant $N_0(x)$, then the function

$$(2.10) \quad w^{u,x}: [-a, a] \ni y \mapsto w^{u,x}(y) \stackrel{\text{df}}{=} \frac{K}{h(x)} \cdot u\left(\frac{s(x)}{a} \cdot y\right) \in R$$

belongs to the class $C_0(a, K)$.

LEMMA 2. Assume Hypothesis A_1 . If $x \in [0, a)$ is fixed and

$$u: [-s(x), s(x)] \rightarrow [-h(x), h(x)]$$

is a C^1 -function such that

$$|u'(y)| \leq N_0(x) \quad \text{and} \quad |u'(y) - u'(z)| \leq M_0(x)|y - z|,$$

then the function $w^{u,x}$ defined by formula (2.10) belongs to the class $C_1(a, K, L)$.

Proof of Lemma 1.

$$\begin{aligned}
 |w^{u,x}(y)| &\leq K \cdot (h(x))^{-1} \left| u \left(\frac{s(x)}{a} \cdot y \right) \right| \leq K \cdot (h(x))^{-1} \cdot h(x) = K, \\
 |w^{u,x}(y) - w^{u,x}(z)| &\leq K \cdot (h(x))^{-1} \left| u \left(\frac{s(x)}{a} y \right) - u \left(\frac{s(x)}{a} z \right) \right| \\
 &\leq K \cdot (h(x))^{-1} \cdot N_0(x) \cdot s_1(x) \cdot a^{-1} |y - z| = K |y - z|.
 \end{aligned}$$

Proof of Lemma 2. It is obvious that $w^{u,x} \in C_0(a, K)$ (see Lemma 1). It is sufficient to prove that the derivative of $w^{u,x}$ fulfils the Lipschitz condition with the constant L . We have:

$$\begin{aligned}
 |(w^{u,x})'(y) - (w^{u,x})'(z)| &= \frac{K \cdot s(x)}{h(x) a} \left| u' \left(\frac{s(x)}{a} y \right) - u' \left(\frac{s(x)}{a} z \right) \right| \\
 &\leq \frac{K \cdot s(x)}{h(x) a} \cdot \frac{s(x)}{a} \cdot M_0(x) |y - z| \\
 &\leq \frac{K}{h(x)} \cdot (s_1(x))^2 \cdot \frac{1}{a^2} \cdot M_0(x) |y - z| \\
 &= K \cdot (h(x))^{-1} \cdot \frac{h(x)}{N_0(x)} \cdot \frac{h(x)}{N_0(x)} \cdot M_0(x) \cdot |y - z| \leq L \cdot |y - z|.
 \end{aligned}$$

3.2. It is clear that $C_0(a, K)$ and $C_1(a, K, L)$ are compact in the space $C_0(a, R)$ of all continuous functions from $[-a, a]$ into R , with the usual topology of the uniform convergence. The set $C_1(a, K, L)$ is also compact in the space $C_1(a, R)$ of all C^1 -functions from $[-a, a]$ into R , with the usual topology of the uniform convergence of functions together with their first derivatives. In order to prove these assertions we can apply the classical Arzelà–Ascoli theorem.

Moreover, $C_0(a, K)$ and $C_1(a, K, L)$ are convex in $C_0(a, R)$; the second set is obviously convex also in the space $C_1(a, R)$.

We put

$$(2.11) \quad C_0^*(a, K) = \{\varphi \in C_0(a, K) : \text{there is } y \in [-a, a] \text{ such that } |\varphi(y)| = K\},$$

$$(2.12) \quad C_1^*(a, K, L) = C_1(a, K, L) \cap C_0(a, K) \quad (= \{\varphi \in C_1(a, K, L) : \text{there is } y \in [-a, a] \text{ such that } |\varphi(y)| = K\}).$$

LEMMA 3. *There are no retractions $C_0(a, K) \rightarrow C_0^*(a, K)$ and there are no retractions $C_1(a, K, L) \rightarrow C_1^*(a, K, L)$; the second assertion holds true with respect to the topological space $C_0(a, K)$ as well as with respect to the topological space $C_1(a, R)$.*

Proof. We repeat here the classical reasoning. Suppose that there is a retraction

$$r: C_0(a, K) \rightarrow C_0^*(a, K)$$

and consider the mapping

$$\varrho = -r.$$

The mapping ϱ is a continuous mapping from $C_0(a, K)$ into itself. In view of the compactness and the convexity of $C_0(a, K)$, there is a fixed point $\varphi = \varrho(\varphi)$ of the mapping ϱ (by the Schauder fixed point theorem). This fixed point, however, belongs to the set $C_0^*(a, K)$, since $r(\varphi) \in C_0^*(a, K)$ and $C_0^*(a, K)$ is invariant with respect to the mapping: $z \mapsto -z$. Hence we have $\varphi = \varrho(\varphi) = -r(\varphi) = -\varphi$ and so the function φ must be identically equal to zero. But the zero function does not belong to the set $C_0^*(a, K)$. This contradiction proves that there are no retractions from $C_0(a, K)$ onto $C_0^*(a, K)$.

The proof for $C_1(a, K, L)$ is quite similar. In order to consider both cases of C^0 - and C^1 -topologies simultaneously, we observe that under the conditions assumed for $C_1(a, K, L)$ every mapping $C_1(a, K, L) \rightarrow C_1(a, K, L)$ is continuous in the C^0 -topology if and only if it is continuous in the C^1 -topology.

3. FAMILIES OF TYPE $\mathcal{S}^0$ AND $\mathcal{S}^1$

Fundamental convention. Throughout Sections 3 and 4 we assume that constants a, α, K, L and functions $s^0, h, N_0: [0, a] \rightarrow R_*$ are fixed and satisfy condition (2.1). We use the notation and definitions of sets and functions from Section 2. Whenever, we assume that a function $M_0: [0, a] \rightarrow R^*$ is given, we suppose that Hypothesis A_1 is satisfied.

3.1. Let S be a family of real continuous functions defined in some subset of $R_* \times R$.

DEFINITION 2. We say that the family S is of type $\mathcal{S}^0$, or shortly, that $S \in \mathcal{S}^0$, if and only if:

(C₁) For every $\varphi \in S$ there is a number ω_φ belonging to the interval $[0, a]$ such that φ is defined and continuous in the set

$$(3.1) \quad Q^\varphi \stackrel{\text{df}}{=} Q_{\omega_\varphi}^{\varphi} \quad (= \{(t, y): 0 \leq t < \omega_\varphi, |y| \leq s^0(x)\}) \quad (\text{see (2.5)});$$

(C₂) For every $\varphi \in S$ the function

$$(3.2) \quad \varphi(0, \cdot): [-\alpha, \alpha] \ni y \mapsto \varphi(0, y) \in R$$

belongs to the class $C_0(a, K)$ (one can say shortly — accepting for simplicity a natural identification — that

$$\varphi|_{Q_0} \in C_0(a, K) \quad \text{for every } \varphi \in S);$$

- (C₂) If $\varphi, \psi \in S$ and $\varphi|_{Q_0} = \psi|_{Q_0}$, then $\varphi = \psi$;
 (C₄) (i) If $\varphi \in S$ and $\omega_\varphi = \infty$ (this is possible, of course, only in the case $a = \infty$), then for every positive number σ there exists a positive number λ such that

$$(3.3) \quad (\psi \in S, \|\psi|_{Q_0} - \varphi|_{Q_0}\| < \lambda) \Rightarrow \omega_\psi > \sigma,$$

- (ii) If $\varphi \in S$ and $\omega_\varphi < \infty$, then for every positive number ε there exists a positive number λ such that

$$(3.4) \quad (\psi \in S, \|\psi|_{Q_0} - \varphi|_{Q_0}\| < \lambda) \Rightarrow \omega_\psi > \omega_\varphi - \varepsilon,$$

where $\|\cdot\|$ denotes the usual norm:

$$\|f\| = \sup \{|f(x)| : x \in \text{domain of definition of } f\}$$

(this condition can be expressed shortly as the lower semi-continuity of the mapping

$$(3.5) \quad C_0(a, K) \ni \varphi(0, \cdot) \mapsto \omega_\varphi \in \bar{R}_* = R_* \cup \{\infty\});$$

- (C₅) For every $\varphi \in S$, every $x \in [0, \omega_\varphi)$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(3.6) \quad (\psi \in S, \|\psi|_{Q_0} - \varphi|_{Q_0}\| < \delta) \Rightarrow (\|\psi|_{Q_x} - \varphi|_{Q_x}\| < \varepsilon \\ \text{for every } x \in [0, x] \cap [0, \omega_\varphi))$$

(this condition means intuitively that the mapping

$$(3.7) \quad C_0(a, K) \ni \varphi(0, \cdot) \mapsto \varphi \in S$$

is continuous);

- (C₆) For every $\varphi \in S$, every $x \in [0, \omega_\varphi]$, every $y, z \in [-s^0(x), s^0(x)]$ the following Lipschitz inequality

$$(3.8) \quad |\varphi(x, y) - \varphi(x, z)| \leq N_0(x) |y - z|$$

holds true;

- (C₇) For every $u \in C_0(a, K)$ there exists a function $\varphi \in S$ such that $\varphi(0, \cdot) = u$.

Remark 4. Conditions (C₃) and (C₇) imply that for every function u belonging to the set $C_0(a, K)$ there exists exactly one function φ belonging to S such that $\varphi(0, \cdot) = u$.

3.2. Let S be a family of real continuous functions defined in some subset of $R_* \times R$, continuously differentiable with respect to the second variable.

DEFINITION 3. We say that S is of type $\mathcal{S}^1$ (shortly: $S \in \mathcal{S}^1$) if and only if:

- (C'₁) For every $\varphi \in S$ there is a number ω_φ belonging to the closed interval $[0, a]$ such that φ is defined and continuous in the set Q^φ given by (3.1) and for every $x \in [0, \omega_\varphi)$ the function $\varphi(x, \cdot): [-s^0(x), s^0(x)] \rightarrow R$ belongs to the class C^1 ;
- (C'₂) For every $\varphi \in S$ the function $\varphi(0, \cdot)$ belongs to the class $C_1(a, K, L)$;
- (C'₃) $\Leftrightarrow$ (C₃) (see Definition 2);
- (C'₄) $\Leftrightarrow$ (C₄) (see Definition 2; here in place of the lower semi-continuity of the mapping (3.5) we have the lower semi-continuity of the mapping
- (3.9) $C_1(a, K, L) \ni \varphi(0, \cdot) \mapsto \omega_\varphi \in \bar{R}$;
- (C'₅) $\Leftrightarrow$ (C₅) (see Definition 2; here we have the continuity of the mapping
- (3.10) $C_1(a, K, L) \ni \varphi(0, \cdot) \mapsto \varphi \in S$;
- (C'₆) $\Leftrightarrow$ (C₆) (see Definition 2);
- (C'₇) For every $u \in C_1(a, K, L)$ there exists a function $\varphi \in S$ such that $\varphi(0, \cdot) = u$;
- (C'₈) For every $\varphi \in S$ and every $x \in [0, \omega_\varphi)$ the following estimation

$$(3.11) \quad \left| \frac{\partial}{\partial y} \varphi(x, y) \right| \leq N_0(x)$$

holds true and the partial derivative $\frac{\partial}{\partial y} \varphi(x, \cdot)$ fulfils in the interval $[-s^0(x), s^0(x)]$ the Lipschitz condition with respect to y with the constant $M_0(x)$.

Remark 5. In conditions (C'₄) and (C'₅) we consider the C^0 -norms, not the C^1 -norms.

4. APPLICATIONS OF THE RETRACT THEOREM FOR FAMILIES OF TYPE $\mathcal{S}^0$ AND $\mathcal{S}^1$

4.1. Let S be a family of functions such that $S \in \mathcal{S}^0$ or $S \in \mathcal{S}^1$.

DEFINITION 4. We say that the family S fulfils *condition (W)* if and only if the following implication holds true:

If

(j) $\varphi \in S, x \in [0, \omega_\varphi)$,

(jj) $|\varphi|_{Q_x}(t, y) < h(t)$ for every $t \in [0, x), |y| \leq s(x)$,

(jjj) $|\varphi(x, y)| = h(x)$ for some $y \in [-s(x), s(x)]$, then there exists a positive number t such that

(k) $t + x < \omega_\varphi$,

(kk) for every $\tau \in [x, x + t]$ there exists a point $y \in [-s(\tau), s(\tau)]$ such that

$$(4.1) \quad |\varphi(\tau, y)| > h(\tau).$$

Remark 5. It is not difficult to show that if (W) holds true, then, under the assumption that (j)–(jjj) are satisfied, there is $t > 0$ fulfilling (k) and there exists a continuous function $y: [x, x+t] \rightarrow R$ such that

$$|y(\tau)| \leq s(\tau) \quad \text{and} \quad |\varphi(\tau, y(\tau))| > h(\tau) \quad \text{for every } \tau \in (x, x+t].$$

This is, however, not necessary for us and therefore we do not prove this property.

4.2. Let the constants a, α, K, L and the functions s^0, h and N_0 be fixed as in Section 2 (cf. the convention introduced in Section 3); we recall that a can be ∞ .

Let S be a family of type $\mathcal{S}^0$.

THEOREM 1. *If S fulfils condition (W), then there is a function φ belonging to S such that*

$$(4.2) \quad |\varphi(x, y)| < h(x)$$

for every $(x, y) \in Q^\circ$.

Remark 6. The assertion of this theorem is equivalent to the following statement: there exists a function u belonging to the set $C_0(\alpha, K)$ such that the function $\varphi \in S$ uniquely determined by the equality $\varphi(0, y) = u(y)$ for $y \in [-\alpha, \alpha]$, fulfils inequality (4.2). This observation is formally trivial, but is important for further applications in the theory of partial differential equations of first order; the functions from $C_0(\alpha, K)$ (or $C_1(\alpha, K, L)$) play the role of initial data determining solutions with a certain class S .

Proof of Theorem 1. Let us consider the set $\mathcal{X}$ (called in the sequel the space) of some pairs (t, φ) , defined by the formula:

$$(4.3) \quad (t, \varphi) \in \mathcal{X} \stackrel{\text{df}}{\iff} (\text{there exists } \tilde{\varphi} \in S \text{ such that } t \in [0, \omega_{\tilde{\varphi}}) \text{ and } \varphi = \tilde{\varphi}|_{Q_t}).$$

Remark 7. In virtue of the assumptions on the class S , the function $\tilde{\varphi}$ is uniquely determined by φ .

In the space $\mathcal{X}$ we introduce a topology by defining the fundamental neighbourhood bases. If (x, φ) is a point of $\mathcal{X}$, then by

$$V_{\varepsilon, \eta} = V_{\varepsilon, \eta}(x, \varphi)$$

we denote the set defined by the formula:

$$(4.4) \quad (y, \psi) \in V_{\varepsilon, \eta} \stackrel{\text{df}}{\iff} [(y, \psi) \in \mathcal{X} \text{ and } y \in (x - \varepsilon, x + \varepsilon) \cap [0, \omega_{\tilde{\varphi}}), \|\tilde{\psi}|_{Q_\xi} - \tilde{\varphi}|_{Q_\xi}\| < \eta \text{ for every } \xi \in [0, x] \cap [0, y], \text{ where } \tilde{\varphi} \text{ and } \tilde{\psi} \text{ are the functions belonging to } S \text{ and such that, respectively, } \tilde{\varphi}|_{Q_x} = \varphi \text{ and } \tilde{\psi}|_{Q_y} = \psi].$$

The family

$$\{\{V_{\varepsilon, \eta}(x, \varphi)\}_{\varepsilon > 0, \eta > 0}\}_{(x, \varphi) \in \mathcal{X}}$$

induces in $\mathcal{X}$ a topology in the usual way; we omit the details.

Let us consider a set $\mathcal{M} \subset \mathcal{X}$ defined as follows:

$$(4.5) \quad \mathcal{M} = \{(x, \varphi) \in \mathcal{X} : |\varphi(t, y)| < h(t) \text{ for } t \in [0, x], |y| \leq s(t)\}.$$

PROPOSITION 3. *The set $\mathcal{M}$ is open.*

PROPOSITION 4. *The boundary $\partial\mathcal{M}$ of the set $\mathcal{M}$ is equal to the set*

$$(4.6) \quad \mathcal{D} = \{(x, \varphi) \in \overline{\mathcal{M}} : |\varphi(t, y)| = h(y) \text{ for some } t \in [0, x], y \in [-s(t), s(t)]\}.$$

Easy proofs of these two propositions will be omitted.

Now we define a certain subset $\mathcal{A}$ of the set $R_* \times \mathcal{X}$:

$$(4.7) \quad (t, (x, \varphi)) \in \mathcal{A} \stackrel{\text{df}}{\Leftrightarrow} [x+t \in [0, \omega_{\tilde{\varphi}}], \text{ where } \tilde{\varphi} \in S \text{ is such that } \tilde{\varphi}|_{Q_x} = \varphi].$$

Finally, we define a mapping $\pi^0: \mathcal{A} \rightarrow \mathcal{X}$ as follows:

$$(4.8) \quad \pi^0(t, (x, \varphi)) = (x+t, \tilde{\varphi}|_{Q_{x+t}}).$$

We have the following obvious

PROPOSITION 5. *The quadruple $(\mathcal{X}, R_*, \mathcal{A}, \pi^0)$ is a local semi-dynamical system.*

PROPOSITION 6. *If $(x, \varphi) \in \mathcal{D}$, then there exists a positive number t such that $\pi^0(r, (x, \varphi)) \in \mathcal{X} \setminus \overline{\mathcal{M}}$ for $r \in [0, t)$.*

Proposition 6 is an easy consequence of the assumption that condition (W) is fulfilled. From the last proposition it follows easily that:

PROPOSITION 7. $S^*(\mathcal{M}) = S(\mathcal{M})$.

Now we put

$$(4.9) \quad \mathcal{N} = \mathcal{D} \cup \mathcal{M};$$

we prove that

$$(4.10) \quad \text{there are no retractions } \mathcal{N} \rightarrow \mathcal{D} \cap \mathcal{N} = \mathcal{D}$$

and we observe that (trivially):

$$(4.11) \quad \mathcal{N} \cap \mathcal{D} \text{ is a retract of } \mathcal{D} = \mathcal{N} \cap \mathcal{D}.$$

In order to prove (4.10) we use Lemmas 1 and 3. Indeed, suppose that there is a retraction

$$(4.12) \quad \Phi: \mathcal{N} \rightarrow \mathcal{D},$$

and consider the mapping

$$(4.13) \quad i: C_0(a, K) \ni u \mapsto (0, \varphi|_{Q_0}) \in \mathcal{N},$$

where $\varphi \in S$ is such that $\varphi(0, \cdot) = u$.

This mapping is obviously continuous; it can be identified with an inclusion. Further, we define the mapping

$$(4.14) \quad j: \mathcal{D} \ni (x, \varphi) \mapsto w^{x, \varphi(x, \cdot)} \in C_0^*(\alpha, K)$$

(see Lemma 1 in Section 2.2). Finally, we consider the composition

$$(4.15) \quad \Psi = j \circ \Phi \circ i: C_0(\alpha, K) \rightarrow C_0^*(\alpha, K),$$

which is, of course, continuous. Moreover, for $u \in C_0^*(\alpha, K)$ we have $i(u) \in \mathcal{D}$ and so

$$\Psi(u) = j(\Phi(i(u))) = j(\Phi((0, \varphi|_{Q_0})))$$

for $\varphi \in S$ such that $\varphi(0, \cdot) = u$; since $(0, \varphi|_{Q_0})$ belongs to $\mathcal{D}$, we have:

$$\Psi(u) = j(\Phi((0, \varphi|_{Q_0}))) = j((0, \varphi|_{Q_0})) = w^{0, \varphi(0, \cdot)} = w^{0, u} = u.$$

Hence Ψ is a retraction $C_0(\alpha, K) \rightarrow C_0^*(\alpha, K)$. But the existence of such a retraction is impossible in virtue of Lemma 3. This finishes the proof of (4.10).

Now we can apply Theorem A from Section 1.2. For $X = \mathcal{X}$, $M = \mathcal{M}$, $N = \mathcal{N}$ (and $A = \mathcal{A}$) all the assumptions of Theorem A are satisfied with respect to the local semi-dynamical system $(\mathcal{X}, R_*, \mathcal{A}, \pi^0)$; hence the assertion of this theorem holds true. This means that the assertion of our Theorem 1 holds true. The proof is complete.

4.3. Suppose that α, α, K, L and the functions h, s^0, N_0 and M_0 are as in Section 2.1.

Let S be a family of type $\mathcal{S}^1$.

THEOREM 2. *If S fulfils condition (W) and if Hypothesis A_1 is satisfied, then there is a function $\varphi \in S$ such that*

$$|\varphi(x, y)| < h(x) \quad \text{for every } (x, y) \in Q^0.$$

Proof. We consider the system $(\mathcal{X}, R_*, \mathcal{A}, \pi^0)$ defined in Section 4.2; here of course, S being of type $\mathcal{S}^1$, the elements of $\mathcal{X}$ are of stronger regularity; but we consider the same topology and the same mapping π^0 as in the preceding section. The same is also the reasoning showing that the set $\mathcal{M}$ defined by (4.5) is open and that condition (W) implies the equality $S(\mathcal{M}) = S^*(\mathcal{M})$. The only change which we have to do in the proof of Theorem 1 in order to obtain the proof of Theorem 2 is the following: in place of Lemma 1 and the first part of the assertion of Lemma 3, we must use Lemmas 1, 2 and the second part of the assertion of Lemma 3. In that way we obtain (4.10) because of the fact that there are no retractions $C_1(\alpha, K, L) \rightarrow C_1^*(\alpha, K, L)$. All other details are exactly the same as in Section 4.2. We finish the proof by using Theorem A from Section 1.2.

**5. AN APPLICATION OF THEOREM 2 IN THE THEORY
OF PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER**

5.1. Suppose that the constants a, α, K, L and continuous functions h and s^0 from $[0, a]$ into $R_+ = (0, \infty)$ are given and are such as in Section 2.1. Let f be a continuous real function defined on an open subset Ω of R^4 and such that

$$(5.1) \quad [0, a] \times [-\alpha, \alpha] \times J \times R \subset \Omega,$$

where

$$J = [-\sup \{h(x): x \in [0, a]\}, \sup \{h(x): x \in [0, a]\}]$$

if $\sup \{h(x): x \in [0, a]\} < \infty$ and

$$J = R$$

if $\sup \{h(x): x \in [0, a]\} = \infty$.

HYPOTHESIS A₂. Assume that for every $\lambda \in C_1(\alpha, K, L)$ there is exactly one solution $z[\lambda]$ of the Cauchy problem

$$(5.2) \quad z_x = f(x, y, z, z_y),$$

$$(5.3) \quad z(0, y) = \lambda(y),$$

defined and continuously differentiable in the set

$$(5.4) \quad Q^{(\lambda)} \stackrel{\text{df}}{=} \{(x, y) \in R^2: x \in [0, \alpha_\lambda], |y| \leq s^0(x)\}$$

with some positive number α_λ (uniquely determined by λ), we admit the possibility $\alpha_\lambda = \infty$. Moreover, every solution $z[\lambda]$ of (5.2)–(5.3) is such that

$$(5.5) \quad \left| \frac{\partial}{\partial y} z[\lambda](\xi, \eta) \right| \leq N_0(\xi) \quad \text{for } (\xi, \eta) \in Q^{(\lambda)}$$

and

$$(5.6) \quad \left| \frac{\partial}{\partial y} z[\lambda](\xi, \eta) - \frac{\partial}{\partial y} z[\lambda](\xi, \eta^0) \right| \leq M_0(\xi) |\eta - \eta^0|$$

for $(\xi, \eta), (\xi, \eta^0)$ belonging to $Q^{(\lambda)}$, where $N_0, M_0: [0, a] \rightarrow R_+$ are some continuous functions, common for all λ (and then common for the solutions $z[\lambda]$).

Furthermore, we assume that solutions z of (5.2)–(5.3) depend on the initial data λ continuously, which means that for every $\lambda \in C_1(\alpha, K, L)$ and every positive number ε , there exists a positive number δ such that

$$(5.7) \quad \text{if } \hat{\lambda} \in C_1(\alpha, K, L) \text{ and } \|\lambda - \hat{\lambda}\| < \delta, \text{ then}$$

$$|z[\lambda](\xi, \eta) - z[\hat{\lambda}](\xi, \eta)| < \varepsilon \quad \text{for every } (\xi, \eta) \in Q^{(\lambda)} \cap Q^{(\hat{\lambda})}.$$

Finally, we assume that the mapping

$$(5.8) \quad \lambda \mapsto a_\lambda$$

is lower semi-continuous in the sense that: for every $\lambda \in C_1(\alpha, K, L)$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[\hat{\lambda} \in C_1(\alpha, K, L), \|\lambda - \hat{\lambda}\| < \delta] \Rightarrow a_\lambda - \varepsilon < a_{\hat{\lambda}},$$

or — in the case $a_\lambda = \infty$ — for every $\beta \in R$ there is $\delta > 0$ such that

$$[\hat{\lambda} \in C_1(\alpha, K, L), \|\lambda - \hat{\lambda}\| < \delta] \Rightarrow a_{\hat{\lambda}} > \beta.$$

Remark 8. Using the notation from Section 2.1 we can write the set $Q^{(\lambda)} \cap Q^{(\hat{\lambda})}$ as $Q_{\min(a_\lambda, a_{\hat{\lambda}})}^0$.

HYPOTHESIS A_3 . If $z[\lambda]$ is a solution of (5.2)–(5.3) for some function λ belonging to $C_1(\alpha, K, L)$, if x is a point belonging to the interval $[0, a_\lambda]$ and if

$$\begin{aligned} |z[\lambda](\xi, \eta)| &< h(\xi) && \text{for } (\xi, \eta) \in Q_x \text{ with } Q_x \text{ given by (2.4),} \\ |z[\lambda](x, \eta)| &= h(x) && \text{for some } \eta \in [-s(x), s(x)], \end{aligned}$$

where $s(x)$ is given by (2.9),

then there exists a positive number t such that for every $\xi \in [x, x+t]$ there is $\eta \in [-s(\xi), s(\xi)]$ for which the inequality

$$|z[\lambda](\xi, \eta)| > h(\xi)$$

holds true.

5.2. Before we formulate our theorem, we give a short discussion of Hypotheses A_2 and A_3 . It is known that under suitable assumptions on the regularity of the function f the conditions assumed in Hypothesis A_2 are fulfilled. In particular, if $f = f(x, y, z, q)$ is continuous and has first partial derivatives f_y, f_z, f_q which are continuous functions of (x, y, z, q) with absolute values bounded by a constant M_1 , and, moreover, if these partial derivatives fulfil the Lipschitz condition with respect to (y, z, q) with the constant M_1 , then for each sufficiently regular and bounded initial function λ , there is exactly one solution z of the initial problem (5.2)–(5.3) defined in the set

$$\hat{Q} = \{(x, y) \in R: 0 \leq x < \delta, |y| \leq \mu - Mx\},$$

where δ, μ and M are some positive constants; moreover, there are: a constant K_1 and a function $N(\cdot)$ such that for every solution of (5.2)–(5.3) corresponding to an initial (sufficiently regular) function λ , the following conditions

$$z_y(x, y) \leq K_1$$

and

$$|z_y(x, y) - z_y(x, y^0)| \leq N(x)|y - y^0|$$

hold true (see T. Ważewski [10]; cf. also J. Szarski [6]).

Note that in this case the interval $[0, \delta)$ is common for all solutions, which means that the mapping (5.8) is trivially lower semi-continuous, even continuous; it is just the constant mapping.

Now the following question seems to be natural: what reasonable conditions are sufficient for Hypothesis A_3 to be satisfied? The answer is given implicitly in [3]. We shall recall it, giving some details and — first of all — making an obvious correction to inequalities (6.17) and (6.18) in [3]: in (6.17) and (6.18) (p. 52 in [3]) in the place of $d - cx$ and $-d + cx$ we have to write $-c$ and c , respectively.

The sense of this correction will be made clear after a short discussion which we give below.

Suppose that s and h are differentiable. Assume that

$$(5.9) \quad f(x, y, h(x), q) > h'(x), \quad f(x, y, -h(x), q) < -h'(x)$$

for (x, y, q) with $0 \leq x < a$, $|y| < s(x)$, $|q| \leq N_0(x)$, and that

$$(5.10) \quad \begin{aligned} f(x, s(x), h(x), q) &> -q \cdot s'(x) + h'(x) && \text{for } q \in [0, N_0(x)], \\ f(x, -s(x), h(x), q) &> q \cdot s'(x) + h'(x) && \text{for } q \in [-N_0(x), 0], \\ f(x, s(x), -h(x), q) &< -s'(x) \cdot q - h'(x) && \text{for } q \in [0, N_0(x)], \\ f(x, -s(x), -h(x), q) &< s'(x) \cdot q - h'(x) && \text{for } q \in [-N_0(x), 0] \end{aligned}$$

for $(x, y) \in R^2$ with $0 \leq x < a$ and $|y| \leq s(x)$.

If these conditions are satisfied, then Hypothesis A_3 is fulfilled. The proof is very easy. It is sufficient to observe that the above inequalities mean that in the corresponding sets, or along suitable curves $y = \varphi(x)$, the functions $z(x, \varphi(x)) - h(x)$ and $z(x, \varphi(x)) + h(x)$ are strictly monotone. In the case which has been discussed in [3] (inequalities (6.17) and (6.18)) the function h was linear; thus it is clear that its derivative $c = \text{const}$.

5.3. Suppose that the constants a, α, K, L are as in Section 2.1, as well as the functions s^0, h, N_0 and M_0 ; in particular, we assume (2.1).

THEOREM 3. *Let the function f be as in Section 5.1. Suppose that Hypotheses A_1, A_2 and A_3 are satisfied; we assume that the functions N_0 and M_0 considered in Hypothesis A_2 (see (5.5) and (5.6)) and in Hypothesis A_1 (see (2.2)) are the same. Under these assumptions there exists a function λ belonging to the class $C_1(a, K, L)$ such that*

$$|z[\lambda](x, y)| < h(x) \quad \text{for } (x, y) \in R^2, 0 \leq x < a_\lambda, |y| \leq s(x)$$

(where $s(\cdot)$ is the function defined by (2.9)).

In order to prove this theorem we apply directly Theorem 2. An easy proof of the fact that all the assumptions of Theorem 2 are satisfied

will be omitted; just note that condition (W) (see Definition 4 in Section 4.1) follows directly from Hypothesis A_3 .

The above theorem formally generalizes the main results from the second part of paper [3].

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