Some properties of accessible sets in non-linear control systems

by STANISŁAW ŁOJASIEWICZ, Jr. (Kraków)

Abstract. In this paper we study uniform convexity of accessible sets in non-linear control systems. We give some sufficient conditions for the uniform convexity of accessible sets in two-dimensional control systems. We prove that these conditions can not be generalized for higher dimensional systems.

Introduction. The main problem in this paper is the convexity of the accessible sets in control systems with bounded control. For systems depending only on time the first theorem of this kind has been proved by Liapunov in 1940 [3], but in a different situation. For linear systems (in the space variable), the problem has been resolved by Olech in 1966 [5]. For non-linear systems we have the theorem of Blagodatskih. 1972 [1] with the concavity assumption (for the orientor field associated with the system) which is rather a strong one and is close to linearity of the system. Another theorem of Pliś, 1975 [8], has been proved under different assumptions: loosely speaking, the system has to be of class C^2 and the orientors should be uniformly convex.

In this paper the author tries to get rid of uniform convexity and to replace it by some assumption on the rotation movement of the orientor field. As in [8], we require uniform convexity of the accessible sets, and not its convexity. It seems that it is more effective and natural (for non-linear systems), the uniform convexity being a property more stable than convexity, in the following sense. If the orientor field is reduced to an ordinary vector field of class C^2 , then starting from a uniformly convex set we get the uniform convexity of the accessible sets in a certain neighbourhood of the initial moment. This is not true (in general) if we replace uniform convexity by a weaker assumption. The problem has been suggested by Professor A. Pliś. I thank for his valuable advice.

Basic notions and notation. At first let us introduce some well-known notions and recall their basic properties.

Let $\mathcal{E}^k$ be the k -dimensional euclidian space with the standard scalar product. Let u and v be two vectors in $\mathcal{E}^k$. We write:

uv (without a dot) the scalar product of u and v ,
 $|v| = (vv)^{1/2}$ the corresponding norm,
 $[u, v]$ and $]u, v[$ the closed (respectively open) interval with the end points u and v (if $u \neq v$).

We call an orientor any non-empty compact convex subset of $\mathcal{E}^k$. If the boundary of an orientor does not contain any open interval, this orientor is called *strictly convex*.

Let R be a non-negative constant. A non-empty subset of $\mathcal{E}^k$ is called *R-convex* if it is the intersection of closed balls of radius R . Each *R-convex* set is called *uniformly convex* (the most usual terminology). Obviously, every uniformly convex set is strictly convex. We will use the $+$ to denote algebraic sum of sets and we will not use the $\{ \}$ if a term of this sum reduces to a single point. Let A, B, C be orientors such that $A + B = C$; then C is strictly (uniformly) convex iff A and B are so. The intersection (not reduced to a single point) of a strictly (uniformly) convex set with an affine subspace of $\mathcal{E}^k$ is strictly (uniformly) convex relatively to this subspace, but not in $\mathcal{E}^k$ if this subspace is different from $\mathcal{E}^k$ (with $k \geq 2$). We denote by B_k the unit closed ball and we put $S_{k-1} = \partial B_k$ (the symbol ∂ will denote the boundary). If A is strictly convex and $n \in S_{k-1}$, we denote by $v(A, n)$ the unique point in ∂A such that n is an outward normal to the set A at the point $v(A, n)$. The function $v(A, \cdot)$ is well defined and continuous. The set A is *R-convex* iff the function $v(A, \cdot)$ satisfies the Lipschitz condition with the constant R . Let I be an open interval of the real line $\mathcal{E} = \mathcal{E}^1$. A set function F from $I \times \mathcal{E}^k$ into $\mathcal{E}^k$ is called an *orientor field* if its values are orientors in $\mathcal{E}^k$. A function x is called a *trajectory of F* if $\dot{x}(t) \in F(t, x(t))$ for almost every $t \in I'$, where I' is an open subinterval of I , and x is absolutely continuous on compact subintervals of I' .

Let A be a non-empty subset of $\mathcal{E}^k$ and t_0 a point of I . We denote by $E(F, t_0 \times A, t)$ (for $t \geq t_0$) and call an accessible set, the set of all points $x(t)$, where x is a trajectory of F such that $x(t_0) \in A$. We write $E(F, t_0 \times A) = \bigcup_{t \in I, t \geq t_0} \{t_0\} \times E(F, t_0 \times A, t)$ and call it (right) emission zone of $t_0 \times A$ with respect to F . For the basic properties of accessible sets, existence theorems and the connections between control systems and orientor fields, see Ważewski [9].

Known theorems on the uniform convexity of accessible sets. Let A and B be two strictly convex orientors in $\mathcal{E}^k$. Write:

$$r(A, B) = \max_{n \in S_{k-1}} |nv(A, n) - nv(B, n)|, \quad s(A, B) = \max_{n \in S_{k-1}} |v(A, n) - v(B, n)|.$$

The function r is the well-known Hausdorff metric (which can be defined on the space of all non-empty compact sets of $\mathcal{E}^k$) and s is the metric (on the space of all strictly convex orientors) first introduced by A. Plis in

his paper [7]. For R -convex sets these metrics are equivalent. If we will speak of the continuity of a set valued function, it will always be meant with respect to the Hausdorff metric r .

Let F be an orientor field from $\mathcal{E} \times \mathcal{E}^k$ into $\mathcal{E}^k$, with $k \geq 2$, and assume that:

- (1) $F(t, x)$ are R -convex for $(t, x) \in \mathcal{E} \times \mathcal{E}^k$,
- (2) there exists two constants: L and M such that, $s(F(t, x), F(t, y)) \leq M|x - y|$ for $t \in \mathcal{E}$, x and y in $\mathcal{E}^k$.
- (3) $r(F(t, px + (1-p)y), pF(t, x) + (1-p)F(t, y)) \leq Lp(1-p)|x - y|^2$ for $t \in \mathcal{E}$, $x \in \mathcal{E}^k$, $y \in \mathcal{E}^k$ and $p \in [0, 1]$.

Let A be a uniformly convex orientor in $\mathcal{E}^k$. We have the following theorem of Pliś [8].

THEOREM A. Assume that (1), (2) and (3) hold and F is continuous. Then for each initial moment t_0 there exists $\delta > 0$ such that the accessible sets $E(F, t_0 \times A, t)$ are uniformly convex for $t \in [t_0, t_0 + \delta]$.

The following theorem of the author prepared for publication, is of a similar nature.

Let (t_0, x_0) be a given point in $\mathcal{E} \times \mathcal{E}^k$. Let $T > 0$ be such number that $[0, T[$ is the interval of existence of the full solution of the following differential equation:

$$\dot{\gamma} = L + 3M\gamma + 2R\gamma^2, \quad \gamma(0) = 0.$$

THEOREM B. Assume that (1), (2), (3') hold, where (3') is condition (3) with the metric s instead of r and that F is measurable with respect to $t \in \mathcal{E}$ for each $x \in \mathcal{E}^k$, and is bounded by an integrable function on each compact set in $\mathcal{E} \times \mathcal{E}^k$. Then $E(F, (t_0, x_0), t)$ are uniformly convex for $t \in [t_0, t_0 + T[$.

The main theorems. Now, we are going to prove that if the orientor field is a "turning interval" in $\mathcal{E}^2$, then the accessible sets are uniformly convex. Let us remark that in this case the orientors are not strictly convex.

Denote by M_k the space of all square $k \times k$ matrices with real coefficients.

THEOREM 1. Let f and g be continuously differentiable functions from $\mathcal{E}$ into $\mathcal{E}^2$ and let A be a continuous function from $\mathcal{E}$ into M_2 .

Write

$$F(t, x) = A(t)x + [f(t), g(t)] \quad \text{for } (t, x) \in \mathcal{E} \times \mathcal{E}^2.$$

Let (t_0, x_0) be a given point in $\mathcal{E} \times \mathcal{E}^2$ and suppose that

$$\text{Span} \{g(t_0) - f(t_0), \dot{g}(t_0) - \dot{f}(t_0) - A(t_0)(g(t_0) - f(t_0))\} = \mathcal{E}^2.$$

Then the sets $E(F, (t_0, x_0), t)$ are uniformly convex for t close to t_0 .

Proof. Without loss of generality we can suppose $(t_0, x_0) = (0, 0)$. Let X be the fundamental matrix solution to the equation $\dot{x} = Ax$ (i.e.,

$\dot{X}(t) = A(t)X(t)$ and $X(0) = \text{Identity}$. Define:

$$2G(t) = X^{-1}(t)([f(t) - g(t), g(t) - f(t)]), \quad y = X^{-1}(t)x.$$

Then the inclusion $\dot{x} \in F(t, x)$ is equivalent to the inclusion

$$\dot{y} \in G(t) + \frac{1}{2}X^{-1}(t)(f(t) + g(t)).$$

Obviously, it is sufficient to prove our theorem for the orientor field G , our assertion being invariant under affine isomorphisms. Put $2h(t) = X^{-1}(t)(g(t) - f(t))$; then $G(t) = [-h(t), h(t)]$. Span $\{h(0), \dot{h}(0)\} = \mathcal{E}^2$ and h is continuously differentiable. Let $n_0 \in S_1$ be such that $n_0 h(0) = 0$.

Consider the equation

$$h(t)n = 0 \quad \text{for } n \in S_1.$$

Using twice (for n_0 and $-n_0$) the implicit function theorem, we get the existence of positive constants δ and M , such that the following two possibilities can occur:

1° $h(t)n \neq 0$ for $t \in [0, \delta]$ iff $(nh(0))(nh(\delta)) > 0$.

2° $h(t)n = 0$ for some $t = t(n) \in [0, \delta]$, then $t(n)$ is uniquely determined by n and the function t satisfies $|t(n) - t(m)| \leq M|n - m|$. Denote:

$$v(\cdot) = v(E(G, (0, 0), \delta), \cdot).$$

Now, using [5] we get from 1° and 2°

$$v(n) = \text{sgn}(nh(0)) \int_0^\delta h(s) ds$$

in the first case, and

$$v(n) = \text{sgn}(nh(0)) \int_0^{t(n)} h(s) ds + \text{sgn}(nh(\delta)) \int_{t(n)}^\delta h(s) ds$$

in the second case, for all $n \in S_1$.

It results that v is Lipschitz continuous and therefore $E(G, (0, 0), \delta)$ is uniformly convex. We may suppose that δ is arbitrarily small and so the proof of our theorem is complete.

Remark 1. The function v satisfies the Lipschitz condition with some constant R uniformly with respect to the parameter δ close to zero, and so we get the stronger assertion, namely, that the accessible set are R -convex for t close to t_0 .

Remark 2. A generalization of Theorem 1 for non-linear case with the main assumptions analogous to those of the Theorem 1 is being prepared for publication by A. Pliś and the author.

We are going to give an application (Remark 4) of Theorem 1, namely we will prove that if the orientor field is a "turning polygon" in $\mathcal{E}^2$, then the accessible set are uniformly convex.

Let f_i be functions from $\mathcal{E} \times \mathcal{E}^2$ into $\mathcal{E}^2$ (with $1 \leq i \leq j$, $i \in N$ and $j \geq 3$) and (t_0, x_0) a given point in $\mathcal{E} \times \mathcal{E}^2$. Assume that $f_i(t_0, x_0)$ are all the consecutive vertices of a convex polygon in $\mathcal{E}^2$ (i.e., $f_i(t_0, x_0)$ are all different extremal points of $\text{conv} \{f_i(t_0, x_0)\}_{1 \leq i \leq j}$). For convenience of notation put $f_{j+1} = f_1$. Define oriator field on $\mathcal{E} \times \mathcal{E}^2$ by

$$F(t, x) = \text{conv} \{f_i(t, x)\}_{1 \leq i \leq j}$$

and

$$F_i(t, x) = [f_i(t, x), f_{i+1}(t, x)].$$

Denote:

$$E = E(F, (t_0, x_0)), \quad E_i = E(F_i, (t_0, x_0)),$$

$$E(t) = E(F(t_0, x_0), t), \quad E_i(t) = E(F_i(t_0, x_0), t) \quad \text{for } t \geq t_0.$$

THEOREM 2. Assume that the functions f_i are continuous and satisfy the Lipschitz condition in x (with a constant K). Assume that $E_i(t)$ are R -convex on the interval $[t_0, t_1]$, $t_0 < t_1$. Then there exists a positive constant δ such that $E(t)$ are R -convex on the interval $[t_0, t_0 + \delta]$.

Proof. In the proof we will use (without explicitly referring to it) the compactness of accessible sets, the Lipschitz continuity of accessible sets (as a function in t) and the existence of the trajectory (defined on $\mathcal{E}$) passing through the given point, for the oriator fields F and F_i . All this is guaranteed by the continuity of F and F_i in (t, x) and the uniform Lipschitz condition in x for F and F_i .

Now we are going to present a needed variant of a known lemma on the approximation of trajectories of an oriator field by trajectories of class C^1 .

Let Ω be a non-empty convex compact set in $\mathcal{E}^2$ and let Ω_n be convex, compact sets such that $\Omega_n \subset \Omega$ and $\Omega_n \rightarrow \Omega$ as $n \rightarrow \infty$ (in the Hausdorff metric). Let H be an oriator field from $\mathcal{E} \times \mathcal{E}^2$ into $\mathcal{E}^2$.

LEMMA 1. Assume that H admits the representation:

$$H(t, x) = \{h(t, x, u): u \in \Omega\}$$

for

$$(t, x) \in E(H, (t_0, x_0)), \quad t \in [t_0, t_1]$$

and the function h is continuous and satisfies the uniform Lipschitz condition in x and u on $(E(H, (t_0, x_0)) \cap ([t_0, t_1] \times \mathcal{E}^2)) \times \Omega$.

Define

$$H_n(t, x) = \{h(t, x, u): u \in \Omega_n\}.$$

Then for each trajectory z of H satisfying $z(t_0) = x_0$ there exists a sequence $\{z_n\}$ of C^1 -trajectories of H_n converging uniformly to z on $[t_0, t_1]$, with $z_n(t_0) = x_0$.

Idea of the proof. From the measurable implicit function theorem ("Filippov Lemma", see [4]) we get the existence of a measurable control u with values in Ω such that $\dot{z}(t) = h(t, z(t), u(t))$ a.e. on $[t_0, t_1]$. Choose u_n as continuous controls with values in Ω_n such that

$$\int_{t_0}^{t_1} |u(t) - u_n(t)| dt$$

converges to zero as $n \rightarrow \infty$ (the possibility of such a choice results from the same possibility for real valued functions, using the convexity of Ω_n). Take z_n as the corresponding solutions (with $z_n(t_0) = x_0$); then z_n converge uniformly to z on $[t_0, t_1]$.

Remark. A similar theorem has been proved in [2]. Our proof is very simple contrary to that given in [2]. Lemma 1 is not a direct consequence of [2], because we have the regularity assumptions for the orienor field only on the emission zone, and not in a neighbourhood of the emission zone as in [2].

Denote by A_i the smallest closed cone (with the vertex at the origin) such that $F(t_0, x_0) \subset f_i(t_0, x_0) + A_i$. For positive ε denote by A_i^ε the closed cone with the same axis of symmetry as A_i , and with the angle enlarged by ε . Now fix $\varepsilon > 0$ such that $\text{conv } A_i^\varepsilon \neq \mathcal{E}^2$ for each i (i.e., the angles of $F(t_0, x_0)$ are not larger than $\pi - 2\varepsilon$). Let x_i be the solution of the equation $\dot{x} = f_i(t, x)$ with the initial condition $x_i(t_0) = x_0$.

LEMMA 2. *There exists $\delta_\varepsilon > 0$ such that the following inclusion holds for $t \in [t_0, t_0 + \delta_\varepsilon]$*

$$(1) \quad E(t) \subset \bigcap_i (x_i(t) + A_i^{2\varepsilon}).$$

Proof. It results from the continuity of the function f_i that there exists $\delta' > 0$ (with $\delta' \leq t_1 - t_0$) such that $f_i(t, x)$ are distinct vertices of $F(t, x)$ and $F(t, x) \subset f_i(t, x) + A_i^\varepsilon$ for $(t, x) \in E$ and $t \in [t_0, t_0 + \delta']$. Fix an index i . Define

$$G(t, x) = F(t, x + x_i(t)) - \dot{x}_i(t)$$

and

$$g(t, x) = f_i(t, x + x(t)) - \dot{x}_i(t).$$

Then $G(t, x) \subset g(t, x) + A_i^\varepsilon$ for $(t, x) \in E$ and $t \in [t_0, t_0 + \delta']$.

We are going to prove that there exists $\delta_i > 0$ such that $E(G, (t_0, 0), t) \subset A_i^{2\varepsilon}$ for $t \in [t_0, t_0 + \delta_i]$. From this it results that $E(t) \subset x_i(t) + A_i^{2\varepsilon}$ for $t \in [t_0, t_0 + \delta_i]$ and our lemma will be proved if we take $\delta_\varepsilon = \min_{1 \leq i \leq j} \delta_i$.

Therefore we must show that each trajectory z of the orienor field G

starting from the point $(t_0, 0)$ satisfies

$$(2) \quad z(t) \in A_i^{2s} \quad \text{for } t \in [t_0, t_0 + \delta_i].$$

Now choose the coordinate system in $\mathcal{E}^2$ such that the origin is at the vertex of A_i , the second coordinate axis is the symmetry axis of A_i and the set A_i has non-positive second coordinates. Define

$$H(t, x) = g(t, x) + (A_i^s \cap \{(u_1, u_2) \in \mathcal{E}^2: u_2 \geq -u_0\})$$

taking and fixing $u_0 > 0$ such that $G(t, x) \subset H(t, x)$ for $(t, x) \in E(G, (t_0, 0))$ and $t \in [t_0, t_0 + \delta']$.

It is sufficient to prove (2) for the trajectories of H . Denote $\Omega = A_i^s \cap \{(u_1, u_2) \in \mathcal{E}^2: u_2 \geq -u_0\}$ and $\Omega_n = (\Omega - (0, 1/n)) \cap \Omega$ for $n > 1/u_0$.

Define $h(t, x, u) = g(t, x) + u$ for $(t, x) \in E(G, (t_0, 0))$, $t \in [t_0, t_0 + \delta']$ and $u \in \Omega$. Using Lemma 1, we see that it suffices to prove (2) for the C^1 -trajectories of H_n , where

$$H_n(t, x) = \{h(t, x, u): u \in \Omega_n\}.$$

Let us remark that

$$A_i^s = \{(u_1, u_2): u_2 \leq -p|u_1|\}$$

and

$$A_i^{2s} = \{(u_1, u_2): u_2 \leq -q|u_1|\},$$

where p and q ($p > q \geq 0$) are the constants determined by A_i^s and A_i^{2s} . Fix $n > 1/u_0$. Take $M = K(1+p)^2$ and let z be a C^1 -trajectory of H_n starting from the point $(t_0, 0)$. We have the following

LEMMA 3. *The inequality*

$$(3) \quad z_2(t) \leq -(p - (t - t_0)M) |z_1(t)|$$

holds for $t \in [t_0, t_0 + \min\{\delta', pM^{-1}\}]$, where $z_1(t)$ and $z_2(t)$ denote the coordinates of $z(t)$.

Proof. Suppose that (3) does not hold. Denote by t' the infimum of points such that (3) is not satisfied. Then

$$(4) \quad \begin{aligned} z_2(t') &= -(p - (t' - t_0)M) |z_1(t')| \\ \text{and } \dot{z}_2(t') &\geq M |z_1(t')| - (p - (t' - t_0)M) |\dot{z}_1(t')|. \end{aligned}$$

The function g satisfies $|g(t, x)| \leq K|x|$ therefore

$$\dot{z}_2(t') \leq -p |\dot{z}_1(t')| - \frac{1}{n} + K(1+p) (|z_1(t)| + |z_2(t')|).$$

Using this and (4) we get

$$(t' - t_0)M |\dot{z}_1(t')| + \frac{1}{n} \leq -(t' - t_0)MK(1+p) |z_1(t')|$$

which contradicts the inequality $t' \geq t_0$ and so the proof of (3) is complete. Taking $\delta_i = \min\{\delta', (p-q)M^{-1}\}$ we obtain (2) by Lemma 3 and so Lemma 2 is proved.

Define $X(t) = \text{conv}_{1 \leq i \leq j} \{x_i(t)\}$. It results from the continuity of the functions f_i that there exists $\delta'' > 0$ such that $x_i(t)$ are all distinct vertices of $X(t)$ for $t \in [t_0, t_0 + \delta'']$. Put $\delta = \min\{\delta_s, \delta''\}$. Let $P_i(t)$ be a closed half-plane such that $P_i(t) \cap X(t) = [x_i(t), x_{i+1}(t)]$. Write

$$T_i(t) = P_i(t) \cap (x_i(t) + A_i^{2e}) \cap (x_{i+1}(t) + A_{i+1}^{2s}) \quad \text{for } t \in [t_0, t_0 + \delta].$$

Till the end of the proof of theorem we suppose that $t \in [t_0, t_0 + \delta]$.

LEMMA 4. *The following inclusion holds for $t \in [t_0, t_0 + \delta]$ and $1 \leq i \leq j$*

$$(5) \quad T_i(t) \cap E(t) \subset T_i(t) \cap E_i(t).$$

(In reality we have not only inclusion but equality.)

Proof. We are going to show that the set of points accessible from (t_0, x_0) by the field F , with use of C_1 -trajectories only, is dense in $E(t)$ and therefore in the proof of (5) we can restrict ourself to these trajectories (remark that the set $T_i(t) \cap E_i(t)$ is closed). Put $\Omega = F(t_0, x_0)$ and $f(t, x)$

$$= j^{-1} \sum_{i=1}^j f_i(t, x).$$

For $(t, x) \in E$ and each i we have a uniquely determined affine mapping $a_i(t, x)$ defined on the set

$$\text{conv}\{f_i(t_0, x_0), f_{i+1}(t_0, x_0), f(t_0, x_0)\}$$

such that

$$a_i(t, x)(f_i(t_0, x_0)) = f_i(t, x),$$

$$a_i(t, x)(f_{i+1}(t_0, x_0)) = f_{i+1}(t, x) \quad \text{and} \quad a_i(t, x)(f(t_0, x_0)) = f(t, x).$$

Define $h_i(t, x, u) = a_i(t, x)(u)$, $h = \bigcup_i h_i$ (the union of the mappings h_i).

The function h is well defined, continuous and satisfies the uniform Lipschitz condition in x and u on the set $(E \cap [t_0, t_0 + \delta]) \times \Omega$. Remark that $F(t, x) = \{h(t, x, u) : u \in \Omega\}$ for $(t, x) \in E$. Now, using Lemma 1 (with $\Omega_n = \Omega$ and $H = F$) we get our hypothesis.

Fix an index i . Choose a (new) coordinate system in $\mathcal{E}^2$ such that the interval $[f_i(t_0, x_0), f_{i+1}(t_0, x_0)]$ is contained in the first (horizontal) coordinate axis, the first coordinate of $f_i(t_0, x_0)$ is smaller than that of $f_{i+1}(t_0, x_0)$ and the set $F(t_0, x_0)$ has non-positive second coordinates. Let z be a C^1 -trajectory of the orientor field F starting from the point (t_0, x_0) . Denote by $z_1(t)$ and $z_2(t)$ its coordinates. For t such that $z(t) \in T_i(t)$ define

$$y(t) = z_2(t) - \max\{u : (z_1(t), u) \in E_i(t)\}.$$

If $y(t) \leq 0$, then $z(t) \in E_i(t)$, and therefore we are going to prove that $y(t) \leq 0$ for each t , which will imply our lemma. If $x(t) \in [x_i(t), x_{i+1}(t)]$, then obviously $y(t) \leq 0$. Suppose that $z(t') \in T_i(t') \setminus [x_i(t'), x_{i+1}(t')]$, then by the continuity of T_i and z , we have the same property in some neighbourhood of t' . Take $]a, \beta[$, the maximal interval (with $t_0 \leq a < t' < \beta \leq t_0 + \delta$) such that $z(t) \in T_i(t) \setminus [x_i(t), x_{i+1}(t)]$ for $t \in]a, \beta[$. Then, using Lemma 2 we get $z(a) \in [x_i(a), x_{i+1}(a)]$ and $y(a) \leq 0$. Let us remark that the inclination of the sides of the triangle $T_i(t)$ relatively to the horizontal axis is bounded (by $\tan \varepsilon$) which implies the same boundedness for $\partial E_i(t) \cap T_i(t)$ because $E_i(t)$ is convex and contains $[x_i(t), x_{i+1}(t)]$. Therefore the Lipschitz continuity of z_2 and E_i implies the Lipschitz continuity of y . Now, we are going to prove:

LEMMA 5. *The function y satisfies the following differential inequality*

$$(6) \quad \dot{y}(t) \leq \frac{2K}{\cos \varepsilon} |y(t)| \quad \text{for a.e. } t \in [\alpha, \beta].$$

Integrating this we will get that $y(t) \leq 0$ for $t \in [\alpha, \beta]$ and Lemma 4 will follow.

Proof. Fix $t \in]a, \beta[$. Choose a vector $v = (v_2, v_1)$, $v \in F_i(t, z_1(t), z_2(t) - y(t))$ such that $|v_1 - \dot{z}_1(t)|$ is minimal. Let $Q(x)$ be the closed half-plane containing $F(t, x)$ such that $F(t, x) \cap \partial Q(x) = F_i(t, x)$. Let us remark that for $x \in E(t)$:

$$F(t, x) \subset Q(x) \cap (f_i(t, x) + \{(u_1, u_2): u_2 \leq u_1 \tan \varepsilon\}) \cap (f_{i+1}(t, x) + \{(u_1, u_2): u_2 \leq -u_1 \tan \varepsilon\});$$

therefore the first coordinate of $f_i(t, x)$ is smaller than that of $f_{i+1}(t, x)$ and the inclination of $F_i(t, x)$ relatively to the horizontal axis is bounded by $\tan \varepsilon$.

Using this we get easily:

$$(7) \quad |\dot{z}_1(t) - v_1| \leq (\tan \varepsilon)^{-1} |\dot{z}_2(t) - v_2| + (\sin \varepsilon)^{-1} \max_{m=\{i, i+1\}} \{|f_m(t, x(t)) - f_m(t, z_1(t), z_2(t) - y(t))|\}$$

and for $\dot{z}_2(t) \geq v_2$ we get the inequalities

$$(8) \quad |\dot{z}_1(t) - v_1| \leq (\sin \varepsilon)^{-1} \max_{m=\{i, i+1\}} \{|f_m(t, x(t)) - f_m(t, z_1(t), z_2(t) - y(t))|\},$$

$$(9) \quad |\dot{z}_2(t) - v_2| \leq (\cos \varepsilon)^{-1} \max_{m=\{i, i+1\}} \{|f_m(t, z(t)) - f_m(t, z_1(t), z_2(t) - y(t))|\}.$$

The vector v is of the form:

$$pf_i(t, z_1(t), z_2(t) - y(t)) + (1-p)f_{i+1}(t, z_1(t), z_2(t) - y(t))$$

with $p \in [0, 1]$.

Denote by w the solution of the equation

$$\dot{x} = pf_i(s, x) + (1-p)f_{i+1}(s, x)$$

with the initial condition

$$(w_1(t), w_2(t)) = (z_1(t), z_2(t) - y(t)),$$

where $w_1(s)$ and $w_2(s)$ denote the coordinates of $w(s)$. Let us remark that $\dot{w}(t) = v$ and

$$w(t + \Delta t) \in E_i(t + \Delta t) \quad \text{for } \Delta t \geq 0.$$

From the boundedness (by $\tan \varepsilon$) of the inclination of $\partial E_i(t + \Delta t) \cap T_i(t + \Delta t)$ relatively to the horizontal axis we get for small $\Delta t \geq 0$

$$(10) \quad y(t + \Delta t) \leq z_2(t + \Delta t) - w_2(t + \Delta t) + |z_1(t + \Delta t) - w_1(t + \Delta t)| \tan \varepsilon.$$

Recall that the function f_i and f_{i+1} satisfy the Lipschitz condition in x (with the constant K). Therefore using (7), (8), (9), (10) and the first order development of the functions z and w at the point t , we get for small $\Delta t \geq 0$

$$y(t + \Delta t) \leq y(t) + 2K(\cos \varepsilon)^{-1}|y(t)|\Delta t + o(\Delta t)$$

which implies (6). So the proof of Lemmas 5 and 4 is complete.

Now we are going to finish the proof of our theorem. From (1) we get $\text{conv } E(t) \subset X(t) \cup \bigcup_i T_i(t)$. Let us remark that $X(t) \setminus \bigcup_i E_i(t) \subset \text{int } X(t)$ therefore it results easily from Lemma 4 that

$$(11) \quad \partial \text{conv } E(t) \subset \bigcup_i \partial E_i(t).$$

Using the R -convexity of $E_i(t)$ we get the R -convexity of $\text{conv } E(t)$ (remember that a strictly convex set A is R -convex iff the function $v(A, \cdot)$ satisfies the Lipschitz condition with the constant R). We are going to prove that $E(t) = \text{conv } E(t)$ which will finish the proof of our theorem. It is clear for $t = t_0$. Suppose that $a \in \text{conv } E(t')$ and $a \notin E(t')$, where $t' \in [t_0, t_0 + \delta]$. There exists a trajectory x of F such that $x(t') = a$. Define $d(t) = \text{dist}(x(t), \text{conv } E(t))$, the function d is continuous. Obviously $d(t_0) > 0$ and $d(t') \leq 0$. Define $t'' = \inf\{t > t_0 : d(t) \leq 0\}$. We have $d(t'') = 0$ and $d(t) > 0$ for $t \in [t_0, t'']$. It follows that $x(t'') \in \partial \text{conv } E(t'')$, but from (11) we get $x(t'') \in E(t'')$, which contradicts $a \notin E(t_1)$, and so the proof of Theorem 2 is complete.

Remark 3. If we assume (in Theorem 2) that $E_i(t)$ are only convex (strictly convex), we get that $E(t)$ are convex (respectively strictly convex).

Remark 4. Theorem 1 (Remark 1) gives sufficient conditions for $E_i(t)$ to be R -convex in the linear case, and for the non-linear case we can apply the theorem announced in Remark 2.

Now we are going to show that it is not possible to generalize Theorem 2 for systems in $\mathcal{E}^k$ with $k \geq 3$ because, even for linear systems, the assumption that $E_i(t)$ are uniformly convex (here $F_i(t, x)$ denote the $(k-1)$ -dimensional extremal faces of $F(t, x)$, see [6]) is never satisfied.

Namely, we are going to prove the following theorem.

Let A be a function from $\mathcal{E}$ into M_k , where $k \geq 3$, and let f_i (with $1 \leq i \leq q$, $i, q \in N$) be functions from $\mathcal{E}$ into $\mathcal{E}^k$. Define

$$F(t, x) = A(t)x + \operatorname{conv}_{1 \leq i \leq q} \{f_i(t)\}$$

for $(t, x) \in \mathcal{E} \times \mathcal{E}^k$.

THEOREM 3. Assume that elements of matrix function A are Lebesgue integrable on each finite interval in $\mathcal{E}$, that the functions f_i are absolutely continuous and that $\operatorname{int} F(t, x) = \emptyset$ for each (t, x) . Let (t_0, x_0) be a given point in $\mathcal{E} \times \mathcal{E}^k$. Assume that the points $f_i(t_0, x_0)$ are all distinct extremal point of the set $\operatorname{conv}_{1 \leq i \leq q} \{f_i(t_0, x_0)\}$ and that the dimension of this set is $k-1$. Then $E(F, (t_0, x_0), t)$ is not uniformly convex for $t > t_0$.

Proof. Without loss of generality we may suppose that $(t_0, x_0) = (0, 0)$. Let X be a fundamental matrix solution to the equation $\dot{x} = Ax$. Define

$$g_i(t) = X^{-1}(t)(f_i(t))$$

and

$$G(t) = \operatorname{conv}_{1 \leq i \leq q} \{g_i(t)\}.$$

As in the proof of Theorem 3, it is sufficient to prove our theorem for the orientor field G . It is also sufficient to prove that the accessible sets are not uniformly convex for small positive t . Taking $d > 0$ sufficiently small, we get that $g_i(t)$ are all distinct extremal points of $G(t)$ and the dimension of $G(t)$ is $k-1$ for $t \in [0, d]$. The orientor $G(t)$ for $t \in [0, d]$ determines uniquely the hyperplane $P(t)$ passing through the origin and the vector $a(t)$ perpendicular to $P(t)$ such that $G(t) \subset a(t) + P(t)$. Define

$$h_i(t) = g_i(t) - a(t)$$

and

$$H(t) = G(t) - a(t).$$

It is sufficient to prove our assertion for the orientor field H , because a translation of the orientor field induced a translation of accessible sets. There exists an absolutely continuous function n from $[0, d]$ into S_{k-1} such that $n(t) \perp P(t)$ ($\perp$ denotes perpendicularity). Let D be the set of all points in $]0, d[$ at which n and h_i (for all i) are simultaneously differentiable. Let I be the set of all pairs (i, j) such that $[h_i(0), h_j(0)]$ is a one dimensional extremal face of $H(0)$. There exists $d' \in]0, d[$ such that $[h_i(t), h_j(t)]$ are one dimensional extremal faces of $H(t)$ for $(i, j) \in I$ and

$t \in [0, d']$. The time interval considered $[0, d']$ may be chosen arbitrarily small and therefore it suffices to prove that $E(H, (0, 0), d')$ is not uniformly convex. Define for $(i, j) \in I$

$$D_{i,j} = \{t \in D \cap [0, d'] : (\dot{h}_i(t) - \dot{h}_j(t))n(t) \neq 0\}.$$

We are going to consider two following cases:

Case 1. $D_{i,j} = \emptyset$ for all $(i, j) \in I$. Let $t_0 \in D \cap [0, d']$. Then $\dot{h}_i(t_0)n(t_0) = \dot{h}_j(t_0)n(t_0)$ for $(i, j) \in I$, but we have also $H(t) \subset P(t)$, $0 \in P(t)$, $\dim H(t) = k-1$ and $n(t) \perp P(t)$ for t in some neighbourhood of t_0 , therefore $\dot{h}_i(t_0)n(t_0) = 0$ for $1 \leq i \leq q$. It results (from the above and $\dot{h}_i(t)n(t) = 0$ for $t \in D$) that $\dot{h}_i(t_0)\dot{n}(t_0) = 0$, hence $\dot{n}(t_0) \perp P(t_0)$ and so $\dot{n}(t_0) = 0$. It follows that $n(t) = \text{const}$ for $t \in [0, d']$. Now using [5], we get that $E(H, (0, 0), d')$ is not strictly convex, which proves our theorem in Case 1.

Case 2. There exists $(i', j') \in I$ such that $D_{i',j'} \neq \emptyset$.

We need to prove the following:

LEMMA. Let f be an absolutely continuous function from some neighbourhood of the interval $[0, d_1]$ into $\mathcal{E}^k$ ($k \geq 3$). Assume f is not vanishing and is differentiable at the origin. Define an orientor field L by the formula $L(t) = [0, f(t)]$. Let $n_0 \in S_{k-1}$ be a vector such that $n_0 \perp \text{Span}\{\dot{f}(0), \ddot{f}(0)\}$. Then there exists a sequence $\{n_p\}$ of vectors from S_{k-1} such that

$$(1) \quad |v(n_p) - v(n_0)| |n_p - n_0|^{-1} \rightarrow \infty \quad \text{as } p \rightarrow \infty,$$

where $v(\cdot) = v(E(L, (0, 0), d_1), \cdot)$ or $E(L, (0, 0), d_1)$ is not strictly convex.

Proof of the lemma. Suppose that $E(L, (0, 0), d_1)$ is strictly convex, then $E(L, (0, 0), t)$ is strictly convex for $t \in [0, d_1]$. Denote $f_0 = f(0)$, $\dot{f}_0 = \dot{f}(0)$ and write

$$A_0 = \{t \in [0, d_1] : n_0 f(t) > 0\}, \quad O(t) = f(t) - f_0 - t\dot{f}_0$$

for $t \geq 0$, and $a = \limsup_{\delta \rightarrow 0} \delta^{-1} m(A_0 \cap [0, \delta])$, where m denotes the Lebesgue measure.

We can choose $\{\delta_p\}$ and $\{\varepsilon_p\}$ two decreasing sequences of positives numbers converging to zero such that

$$\lim_{p \rightarrow \infty} \delta_p^{-1} m(A_0 \cap [0, \delta_p]) = a$$

and $|o(t)| < \varepsilon_p t$ for $t \in]0, \delta_p]$.

Let us define a sequence $\{n_p\}$ of vectors from S_{k-1} such that $n_p \in \text{Span}\{f_0, n_0\}$.

$$n_p n_0 = [1 - \varepsilon_p^2 \delta_p^2 (|f_0| - |\dot{f}_0| \delta_p)^{-2}]^{1/2}$$

and $n_p f_0 < 0$ if $a > 0$ and $n_p f_0 > 0$ if $a = 0$. $\{n_p\}$ is well defined for p sufficiently large.

Let x_p and x_0 be the unique extremal solutions of L corresponding to the controls u_p and u_0 determined by n_p and n_0 (see [5]).

We are going to prove that

$$(2) \quad |n_p - n_0|^{-2} \left| \int_0^{\delta_p} (n_p - n_0) (u_p(s) - u_0(s)) ds \right| \rightarrow \infty \quad \text{as } p \rightarrow \infty.$$

At first we consider the case $a > 0$.

From our construction we get $u_p(s) = 0$ for $s \in [0, \delta_p]$ and $u_0(s) = f(s)\chi_{A_0}(s)$ for $s \in [0, d_1]$ (by χ_A we denote the characteristic function of A). Write $A_p = A_0 \cap [0, \delta_p]$; then

$$\int_0^{\delta_p} (n_p - n_0) (u_p(s) - u_0(s)) ds = \int_{A_p} n_p f_0 + n_p \dot{f}_0 s + (n_p - n_0) o(s) ds.$$

Let us remark that:

$$(3) \quad \begin{aligned} |n_p f_0| &= \varepsilon_p \delta_p |f_0| (|f_0| - |\dot{f}_0| \delta_p)^{-1}, \\ |n_p \dot{f}_0| &\leq \varepsilon_p \delta_p |\dot{f}_0| (|f_0| - |\dot{f}_0| \delta_p)^{-1}. \end{aligned}$$

Hence, for p sufficiently large,

$$\begin{aligned} & \left| \int_0^{\delta_p} (n_p - n_0) (u_p(s) - u_0(s)) ds \right| \\ & \geq \left| \int_{A_p} n_p f_0 \right| - \left| \int_{A_p} n_p \dot{f}_0 s + (n_p - n_0) o(s) ds \right| \\ & \geq \frac{1}{2} a \varepsilon_p \delta_p^2 |f_0| (|f_0| - |\dot{f}_0| \delta_p)^{-1} - 2 a \varepsilon_p \delta_p^2 (|n_p - n_0| + |\dot{f}_0| \delta_p (|f_0| - |\dot{f}_0| \delta_p)^{-1}) \end{aligned}$$

and so, for p sufficiently large we get

$$(4) \quad \left| \int_0^{\delta_p} (n_p - n_0) (u_p(s) - u_0(s)) ds \right| \geq \frac{1}{3} a \varepsilon_p \delta_p^2.$$

Also for p sufficiently large we get

$$(5) \quad |n_p - n_0|^2 \leq 3 \varepsilon_p^2 \delta_p^2 |f_0|^{-2}.$$

It is clear that (4) and (5) imply (2).

Now consider the case $a = 0$.

From our construction we get $u_p(s) = f(s)$ for $s \in [0, \delta_p]$ and $u_0(s) = f(s)\chi_{A_0}(s)$ for $s \in [0, d_1]$. Write $A'_p = [0, \delta_p]$, then

$$\begin{aligned} & \int_0^{\delta_p} (n_p - n_0) (u_p(s) - u_0(s)) ds \\ & = \int_{A'_p} n_p f_0 + n_p \dot{f}_0 s + (n_p - n_0) o(s) ds + \int_{A_p} (n_0 - n_p) f(s) ds. \end{aligned}$$

Let us remark that (3) are also valid in this case, therefore (as in the case $a > 0$) we get (by an analogous calculation) for p sufficiently large

$$(6) \quad \left| \int_{A'_p} n_p f_0 + n_p \dot{f}_0 s + (n_p - n_0) o(s) ds \right| \geq \frac{1}{3} \varepsilon_p \delta_p^2.$$

From (5) (which is also valid) we get for k sufficiently large

$$(7) \quad \left| \int_{A_p} (n_0 - n_p) f(s) ds \right| \leq 2 \varepsilon_p \delta_p |f_0|^{\frac{1}{2}} m(A_p)$$

and hence using (6), (7) and " $a = 0$ " we get for p sufficiently large

$$(8) \quad \left| \int_0^{\delta_p} (n_p - n_0) (u_p(s) - u_0(s)) ds \right| \geq \frac{1}{4} \varepsilon_p \delta_p^2.$$

It is clear that (5) and (8) imply (2), and so the proof of (2) is complete.

It results from (2) that

$$|n_p - n_0|^{-1} |x_p(\delta_p) - x_0(\delta_p)| \rightarrow \infty \quad \text{as } p \rightarrow \infty.$$

Using this and [5] we get

$$|n_p - n_0|^{-1} \left| v(E(L, (0, 0), \delta_p), n_p) - v(E(L, (0, 0), \delta_p), n_0) \right| \rightarrow \infty \quad \text{as } p \rightarrow \infty$$

which implies (1), and so the proof of the lemma is complete.

Now we are going to finish the proof of the theorem in Case 2. In this case there exists $b \in]0, d'[\cap D_{i,j'}$. Define

$$K(t) = H(b+t) - h_{i'}(b+t) \cdot f(t) = h_{j'}(b+t) - h_{i'}(b+t)$$

and $L(t) = [0, f(t)]$ for $t \in [0, d' - b]$.

Obviously, it is sufficient to prove our assertion for the orientor field K . Let $w_0 \in S_{k-1} \cap P(b)$ be such that for $w \in S_{k-1} \cap P(b)$ in some neighbourhood of w_0 we have

$$(9) \quad \{u \in K(0): wu = \max\{wK(0)\}\} \subset L(0).$$

(We denote $\{yX\} = \{yx: x \in X\}$, where $y \in \mathcal{E}^k$ and $X \subset \mathcal{E}^k$.)

Note that $n(b)\dot{f}(0) \neq 0$ and choose $l > 0$ such that

$$n_0 = l[w_0 - (w_0\dot{f}(0))(n(b)\dot{f}(0))^{-1}n(b)] \in S_{k-1}.$$

From (9) and by the regularity assumptions it follows that there exist $d_1 \in]0, d' - b[$ and $\delta > 0$ such that

$$\{u \in K(t): un' = \max\{n'K(t)\}\} \subset L(t)$$

for $t \in [0, d_1]$ and $|n' - n_0| < \delta$, $n' \in S_{k-1}$. Hence

$$(10) \quad \max\{n'K(t)\} = \max\{n'L(t)\}.$$

Now, using the Lemma, we get that $E(L, (0, 0), d_1)$ is not uniformly convex. But from (10) and [5] it results that $E(K, (0, 0), d_1)$ is not uniformly convex, and so the proof of case 2, and hence of the Theorem 3 is complete.

We have (contrary to the case $k = 2$) the following:

COROLLARY OF LEMMA. Assume that $k \geq 3$.

Let A be a matrix function from $\mathcal{E}$ into M_k , whose elements are locally integrable, and let f and g be absolutely continuous functions from $\mathcal{E}$ into $\mathcal{E}^k$. Define

$$F(t, x) = A(t)x + [f(t), g(t)]$$

for $(t, x) \in \mathcal{E} \times \mathcal{E}^k$. Let (t_0, x_0) be a given point in $\mathcal{E} \times \mathcal{E}^k$. Then the sets $E(F, (t_0, x_0), t)$ are not uniformly convex for $t > t_0$.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES, ODDZIAŁ KRAKÓW
AND CENTRO DE INVESTIGACIÓN DEL I.P.N., MÉXICO

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