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On the remainder term of the prime number formula I. On a problem of Littlewood

by

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1. In the present paper we shall deal with the prime number formula and other well known sums depending on the prime numbers. Let us define

$$(1.1)\qquad \Delta_1(x) \stackrel{\text{def}}{=} \pi(x)-\operatorname{li}x \stackrel{\text{def}}{=} \sum_{p\leqslant x} 1-\int_0^x \frac{dr}{\log r},$$

$$(1.2)\qquad \Delta_2(x) \stackrel{\text{def}}{=} \Pi(x)-\operatorname{li}x \stackrel{\text{def}}{=} \sum_{\nu\geqslant 1}\frac{1}{\nu}\,\pi(x^{1/\nu})-\operatorname{li}x,$$

$$(1.3)\qquad \Delta_3(x) \stackrel{\text{def}}{=} \theta(x)-x \stackrel{\text{def}}{=} \sum_{p\leqslant x}\log p - x,$$

$$(1.4)\qquad \Delta_4(x) \stackrel{\text{def}}{=} \psi(x)-x \stackrel{\text{def}}{=} \sum_{n\leqslant x}\Lambda(n) - x.$$

All these sums depend on the nontrivial zeros $\varrho = \beta+i\gamma$ $(0<\beta<1)$ of $\zeta(s)$. The corresponding formula has the simplest character in the case of $\Delta_4(x)$ where the formula of Riemann–von Mangoldt states: if

$$(1.5)\qquad \tilde{\Delta}_4(x) = \psi_0(x)-x = \frac{\psi(x+0)+\psi(x-0)}{2}-x \qquad (x>1)$$

then we have

$$(1.6)\qquad \tilde{\Delta}_4(x) = -\sum_{\varrho}\frac{x^{\varrho}}{\varrho}-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)-\frac{\zeta'}{\zeta}(0)$$

where $\varrho=\beta+i\gamma$ stand for the zeros with $0<\beta<1$. This shows that

$$(1.7)\qquad \Delta_4(x) = -\sum_{\varrho}\frac{x^{\varrho}}{\varrho}+O(\log x).$$

The corresponding formula for $\Delta_1(x)$ or $\Delta_2(x)$ is more complicated, namely, with the modification

$$\tilde{\Delta}_2(x) = \frac{\Delta_2(x+0)+\Delta_2(x-0)}{2}, \tag{1.8}$$

we have the explicit formula (see e. g. Landau [10], p. 36)

$$\tilde{\Delta}_2(x) = -\sum_{\gamma>0}\big(\mathrm{li}(x^{\varrho})+\mathrm{li}(x^{1-\varrho})\big)+\int_x^{\infty}\frac{dt}{(t^2-1)t\log t}-\log 2 \tag{1.9}$$

and an analogous, but much more sophisticated formula can be derived from this for $\tilde{\Delta}_1(x)$, owing to the relation

$$\pi(x) = \sum_{r=1}^{\infty}\mu(r)\Pi(x^{1/r}). \tag{1.10}$$

Thus, from (1.9), we have

$$\Delta_2(x) = -\sum_{\gamma>0}\big(\mathrm{li}(x^{\varrho})+\mathrm{li}(x^{1-\varrho})\big)+O(1). \tag{1.11}$$

Further (1.11) and the elementary estimation

$$\Delta_2(x)-\Delta_1(x) = \Pi(x)-\pi(x) = O\left(\frac{\sqrt{x}}{\log x}\right) \tag{1.12}$$

imply the relation

$$\Delta_1(x) = -\sum_{\gamma>0}\big(\mathrm{li}(x^{\varrho})+\mathrm{li}(x^{1-\varrho})\big)+O\left(\frac{\sqrt{x}}{\log x}\right). \tag{1.13}$$

2. In this paper and in the next one we shall investigate the possibility to give lower bounds for the functions $\Delta_i(x)$ in terms of the zeros of $\zeta(s)$, and to clear somewhat the connection between the order of magnitude of $\Delta_i(x)$ and the configuration of the ζ-roots.

Let us denote by θ the least upper bound of the real parts of the ζ-zeros. Phragmèn proved the relation

$$\Delta_4(x) = \Omega_{\pm}(x^{\theta-\varepsilon}) \tag{2.1}$$

and Erhard Schmidt [14] in 1903 proved

$$\Delta_4(x) = \Omega_{\pm}(\sqrt{x}) \tag{2.2}$$

which is better than (2.1) in case $\theta = 1/2$.

3. However, these results have a curious ineffective character. They do not give any explicit X values for which with at least one $x \leqslant X$ value the inequality

$$\Delta_4(x) > cx^{\theta-\varepsilon} \tag{3.1}$$

or

$$\Delta_4(x) < -cx^{\theta-\varepsilon} \tag{3.2}$$

would hold. The reason for it is a basic one. Namely, all these proofs use an ineffective theorem concerning Dirichlet's integrals which was proved in a final form by Landau.

4. These facts induced Littlewood [11] to write in 1937: "... Those familiar with the theory of the Riemann zeta-function in connection with the distribution of primes may remember that the interference difficulty arises with the function

$$f(x) = \sum_{\varrho}\frac{x^{\varrho}}{\varrho} = \sum\frac{x^{\beta+i\gamma}}{\beta+i\gamma}$$

(where the ϱ's are the complex zeros of $\zeta(s)$). There exist proofs that if θ is the upper bound of the β's (so that $\theta = \frac{1}{2}$ if Riemann hypothesis is true) then $f(x)$ is of order at least $x^{\theta-\varepsilon}$ in x. But these proofs are curiously indirect: if ($\theta > \frac{1}{2}$ and) we are given a particular $\varrho = \varrho_0$ for which $\beta = \beta_0 > \frac{1}{2}$, they provide no explicit X depending only upon β_0, γ_0 and ε such that $|f(x)| > X^{\beta_0-\varepsilon}$ for some x in $(0, X)$. There are no known ways of showing (for any explicit X) that the single term $x^{\beta_0+i\gamma_0}/(\beta_0+i\gamma_0)$ of f is not interfered with by other terms of the series over the range $(0, X)$".

5. Such a theorem was proved by Turán [17] in 1950 using the powersum method.

THEOREM (Turán). *If $\varrho_0 = \beta_0+i\gamma_0$, $\beta_0 \geqslant \frac{1}{2}$, is an arbitrary non-trivial zero of $\zeta(s)$, then for*

$$T > \max\big(c_0, c_1(\varrho)\big) \tag{5.1}$$

one has

$$\max_{1\leqslant x\leqslant T}|\Delta_4(x)| > \frac{T^{\beta_0}}{|\varrho_0|^{\frac{10\log T}{\log\log T}}}\exp\left(-c_1\frac{\log T\log_3 T}{\log\log T}\right), \tag{5.2}$$

where the c_i's as in the following always are explicitly calculable positive constants.

This theorem says at the first sight nothing on the value of x, for which the inequality (5.2) (i.e. the corresponding inequality without max) is valid. But the trivial remark

$$\Delta_4(x) = O(x) \tag{5.3}$$

tells us that the inequality is trivially true for the tightened interval $[T^{\beta_0-\varepsilon}, T]$ for $T > c_1'(\varrho, \varepsilon)$.

So the question arises whether one can get analogous statements with better localized x values. This is really possible, due to the following theorem proved by W. Staś [16] in 1959 (applying Turán's method).

THEOREM (Staś). *If $\varrho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geqslant \frac{1}{2}$, is an arbitrary non-trivial zero of $\zeta(s)$, then for*

(5.4) $$T > \max(c_2, \exp\exp(2|\varrho_0|))$$

one has

(5.5) $$\max_{x \in I} |\Delta_4(x)| > T^{\beta_0} \exp\left(-8 \frac{\log T}{\log\log T}\right)$$

where

(5.6) $$I = \left[T \exp\left(-\frac{\log T \log_3 T}{(\log\log T)^2}\right) - 1,\ T\right].$$

We note that the corresponding inequality follows for $\Delta_2(x)$ immediately by partial summation. On the other hand for the more interesting case $\beta_0 > \frac{1}{2}$ (because if the Riemann hypothesis is supposed to be true, other methods furnished already better results, as e.g. the theorem of E. Schmidt in (2.2)) the same statements for $\Delta_1(x)$ and $\Delta_2(x)$ and further for $\Delta_3(x)$ and $\Delta_4(x)$ are equivalent because the order of their difference is $O\left(\frac{\sqrt{x}}{\log x}\right)$ and $O(\sqrt{x})$ respectively.

6. However, two interesting problems remained also open after the quoted theorem of Staś.

Firstly we note that the absolute value of the "true error caused by a zero $\varrho_0 = \beta_0 + i\gamma_0$ $(\beta_0 \geqslant \frac{1}{2})$" is

(6.1) $$\frac{x^{\beta_0}}{|\varrho_0|}$$

in case of $\Delta_3(x)$ and $\Delta_4(x)$, and it is

(6.2) $$|\mathrm{li}(x^{\varrho_0})| \sim \frac{x^{\beta_0}}{|\varrho_0| \log x} \quad (\text{for } x \to \infty)$$

in case of $\Delta_1(x)$ and $\Delta_2(x)$. At the same time the estimates occurring in (5,2) and (5.5) are weaker.

Thus it would be important to show that the error term of the prime number formula can have the same order of magnitude — even in an explicitly calculable interval $(0, X(\varrho_0))$ for a suitable x — as the error (6.1) and (6.2) respectively "caused" by the single zero ϱ_0.

The theorems of Phragmèn (2.1) and E. Schmidt (2.2) and other similar results of Littlewood [12] and Ingham [4] are "one-sided" theorems,

i.e. these theorems assure that the remainder term can have infinitely many times a large absolute value with prescribed sign. On the other hand our quoted theorems are only "two-sided". The problem to find one-sided theorems instead of two-sided ones was already stated as one of the main problems in Turán's book [18].

These difficulties were essentially overwhelmed by the one-sided powersum theorems developed by Turán, Knapowski, Dancs. These results had a large scale of application already in the investigation of the distribution of primes in arithmetic progressions (see Knapowski–Turán [7], [8]). However, it turned out (Knapowski–Turán [9]) that even two-sided powersum theorems can lead to one-sided theorems.

The first difficulty is caused by the fact that by the use of powersum theorems we usually cannot estimate the powersum purely by the largest term, but also a factor depending on the number of terms occurs, which is usually relatively small owing to the large number of terms. This problem can be solved either by a powersum theorem of Atkinson [1] which, however, has the constant factor $1/6$, or by a slightly modified form of a powersum theorem due to Montgomery [13] (for the theorem and its proof see the Appendix, Theorem 1) which in this formulation has only the factor $1-\varepsilon$. (We note that actually also Dirichlet's classical theorem could be used, but we should not gain anything by it and the localization of the corresponding x value would be extremely bad in dependence of ϱ_0.)

Thus as an answer to Littlewood's problem we can prove the following theorem, which is completely satisfactory concerning the lower estimate.

THEOREM 1. *Let $0 < \varepsilon \leqslant 1/50$ and let us assume the existence of a $\varrho_0 = \beta_0 + i\gamma_0$ zero of $\zeta(s)$ with*

$$\beta_0 = \tfrac{1}{2} + \delta_0 > \tfrac{1}{2} \quad \text{and} \quad \gamma_0 > 400/\varepsilon^2.$$

Then for $1 \leqslant i \leqslant 4$ and for every H satisfying

(6.3) $$\min(H^{\delta_0/2} H^{\varepsilon^2/4}) > \max(\gamma_0, c_3)$$

(where c_3 is an absolute constant explicitly calculable, positive) we have in the interval

(6.4) $$I = [H, H^{\frac{4\cdot 10^4}{\varepsilon^2}\log\gamma_0}]$$

an $x_i' \in I$ and an $x_i'' \in I$ for which

(6.5) $$\begin{aligned} \Delta_i(x_i') &> (1-\varepsilon)\frac{(x_i')^{\beta_0}}{|\varrho_0| \log x_i'} \\ \Delta_i(x_i'') &< -(1-\varepsilon)\frac{(x_i'')^{\beta_0}}{|\varrho_0| \log x_i''} \end{aligned} \qquad (i = 1, 2)$$

and

$$\text{(6.6)}\qquad \begin{aligned} \Delta_i(x_i') &> (1-\varepsilon)\frac{(x_i')^{\beta_0}}{|\varrho_0|} \\ \Delta_i(x_i'') &< -(1-\varepsilon)\frac{(x_i'')^{\beta_0}}{|\varrho_0|} \end{aligned} \qquad (i=3,4)$$

respectively hold.

7. First we note that it is enough to prove the inequalities (6.5) and (6.6) for $i=2$ and $i=4$ with the factor $1-9\varepsilon/10$ instead of $1-\varepsilon$. Namely, owing to the elementarily provable inequalities

$$\text{(7.1)}\qquad 0 < \Delta_2(x)-\Delta_1(x) < \frac{100\sqrt{x}}{\log x} \qquad (x \geqslant 2)$$

and

$$\text{(7.2)}\qquad 0 < \Delta_4(x)-\Delta_3(x) < 100\sqrt{x} \qquad (x \geqslant 2)$$

and considering (6.3), we have

$$\text{(7.3)}\qquad x_i \geqslant H > (\gamma_0^2)^{1/\delta_0} > \left(\frac{1000}{\varepsilon}|\varrho_0|\right)^{1/\delta_0},$$

i.e.

$$\text{(7.4)}\qquad \frac{\varepsilon}{10}\cdot\frac{x_i^{\beta_0}}{|\varrho_0|} > 100\sqrt{x},$$

which assures (6.5) and (6.6) for the cases $i=1$ and $i=3$ too.

The second natural question would be whether it is enough to prove the theorem (perhaps with a slight modification) either only for $i=2$ or only for $i=4$. It would be advantageous, if it were sufficient to prove this for the easier treatable case $i=4$. This would be really the case if we considered the two-sided analogon of Theorem 1, owing to the relations

$$\text{(7.5)}\qquad \Delta_2(x) = \frac{\Delta_4(x)}{\log x} + \int_2^x \frac{\Delta_4(u)}{u\log^2 u}\,du + O(1)$$

and

$$\text{(7.6)}\qquad \Delta_4(x) = \Delta_2(x)\log x - \int_2^x \frac{\Delta_2(u)}{u}\,du + O(1)$$

which can be proved easily by partial summation (for (7.5) see e.g. Ingham [3], p. 64).

However, dealing with one-sided theorems the situation already changes in an unpleasant way. It turns out that if we prove for the case $i=2$ the inequalities (6.5) with the factor $1-\dfrac{8\varepsilon}{10}$ instead of $1-\varepsilon$ and with the stronger localization

$$\text{(7.7)}\qquad I' = [H^2, H^{\frac{4\cdot 10^4}{\varepsilon^2}\log\gamma_0}]$$

(which in this case is no problem) then the inequality (6.6) is satisfied with $1-9\varepsilon/10$ instead of $1-\varepsilon$ and with the original interval I in (6.4).

Namely, if we suppose to the contrary to the above that for every $u \in I$ one has

$$\text{(7.8)}\qquad \Delta_4(u) < \left(1-\frac{9\varepsilon}{10}\right)\frac{u^{\beta_0}}{|\varrho_0|},$$

then by (7.5) we get for every $x \in I'$

$$\text{(7.9)}\qquad \Delta_2(x)$$

$$< \left(1-\frac{9\varepsilon}{10}\right)\frac{x^{\beta_0}}{|\varrho_0|\log x} + \int_H^x \frac{\left(1-\frac{9\varepsilon}{10}\right)\frac{u^{\beta_0}}{|\varrho_0|}}{u\log^2 u}\,du + O\left(\int_2^H \frac{u}{u\log^2 u}\,du\right)$$

$$< \left(1-\frac{9\varepsilon}{10}\right)\frac{x^{\beta_0}}{|\varrho_0|\log x}\left(1+\frac{2}{\beta_0\log^2 x}\right) + O\left(\frac{\sqrt{x}}{\log^2 x}\right) < \left(1-\frac{8\varepsilon}{10}\right)\frac{x^{\beta_0}}{|\varrho_0|\log x}$$

in contradiction to the modified form of (6.5). (Proof is naturally the same for the other part of (6.6).)

But this way of argument breaks down if we want to prove that the inequality (6.6) for $i=4$ implies (6.5) for $i=2$. Namely, a short reflection on the formula (7.6) tells us that even large negative values of $\Delta_2(u)$ near to x can "effect" large positive values for $\Delta_4(x)$.

Thus the curious situation occurs that we must prove Theorem 1 for the more difficult case $i=2$ and this implies the same result for $i=4$ but not conversely.

8. Though Theorem 1 is very satisfactory regarding the lower estimate, the localization of x in (6.4) is rather bad. So it would be important to prove a theorem which combines relatively good localization with an only somewhat weaker lower estimate than that given in Theorem 1.

Now, using the so-called second main theorem of the powersum theory due to T. Sós and P. Turán [15] (see Appendix, Theorem 2) instead of Montgomery's powersum theorem (Appendix, Theorem 1), it is really possible to prove such a result.

THEOREM 2. *Let $0 < \varepsilon \leqslant 1/50$, and let us assume the existence of a $\varrho_0 = \beta_0 + i\gamma_0$ zero of $\zeta(s)$ with $\beta_0 = \frac{1}{2}+\delta_0 > \frac{1}{2}+\varepsilon$ and*

$$\text{(8.1)}\qquad \gamma_0 > \exp\exp\left(\frac{10^{12}}{\varepsilon^3}\right).$$

Then for $1 \leqslant i \leqslant 4$ and for every H satisfying

$$H^{\varepsilon^4/10^7} > \max(\gamma_0, c_4) \tag{8.2}$$

we have in the interval

$$I = [H, H^{1+\varepsilon}] \tag{8.3}$$

an $x_i' \in I$ and $x_i'' \in I$ for which

$$\begin{aligned} \Delta_i(x_i') &> \frac{(x_i')^{\beta_0}}{\gamma_0^{1+\varepsilon}\log x_i'} \\ \Delta_i(x_i'') &< -\frac{(x_i'')^{\beta_0}}{\gamma_0^{1+\varepsilon}\log x_i''} \end{aligned} \qquad (i = 1, 2) \tag{8.4}$$

and

$$\begin{aligned} \Delta_i(x_i') &> \frac{(x_i')^{\beta_0}}{\gamma_0^{1+\varepsilon}} \\ \Delta_i(x_i'') &< -\frac{(x_i'')^{\beta_0}}{\gamma_0^{1+\varepsilon}} \end{aligned} \qquad (i = 3, 4) \tag{8.5}$$

hold respectively.

This theorem is necessary if we want to investigate the dependence of the remainder terms $\Delta_i(x)$ on infinitely many ζ-zeros and to prove connections between the remainder terms and zerofree regions of $\zeta(s)$. This application of Theorem 2 we shall discuss in the next paper.

Let us note that here even the case $i = 2$ does not imply the case $i = 4$. So it would be necessary to prove it for both cases $i = 2, 4$ and from this one can infer the corresponding result for $i = 1, 3$ as it was shown in Section 7. We shall not give a full proof even for $i = 2$; we only point out the differences with respect to Theorem 1. Further we do not discuss the case $i = 4$; we only give the corresponding starting formulae, because the whole proof runs on the same lines as for $i = 2$ (it is, as can be expected, even simpler than for $i = 2$).

We want to make it perfectly clear that in Theorems 1 and 2 we suppose the existence of a zero with a real part $\beta_0 > \frac{1}{2}$, i.e. we suppose the Riemann hypothesis is not true. This is due to the already mentioned fact that if the Riemann hypothesis is supposed to be true, then by the famous results of Littlewood [12] the case is settled even with lower estimates for every $\Delta_i(x)$. However, Littlewood's proof in its original form did not furnish any localization, but later this problem too was solved by Ingham [4].

It is also worth noting that as all the zeros in the critical strip lie on the line $\sigma = \frac{1}{2}$ up to the height 10^6 if we choose $\varepsilon = 1/50$ any possible zero $\varrho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > \frac{1}{2}$ certainly satisfies the condition $\gamma_0 \geqslant 400/\varepsilon^2$ and so the inequalities (6.5) and (6.6) hold with the constant 0.98 instead of $1-\varepsilon$ for suitable x_i' and x_i'' in I (given by (6.4)).

Finally we mention the interesting corollary of Theorem 2, namely the

COROLLARY 1. *To every $\varepsilon > 0$ there exists an ineffective constant $Y_0(\varepsilon)$, such that for $Y > Y_0(\varepsilon)$ every $\Delta_i(x)$ changes its sign in the interval*

$$I = [Y, Y^{1+\varepsilon}]. \tag{8.6}$$

The result trivially follows from Theorem 2 if the Riemann hypothesis is not true. If it is true, then the stronger theorem of Ingham [4] gives the result even for any interval of the form

$$I = [Y, CY] \tag{8.7}$$

with a constant C.

From Corollary 1 follows directly that denoting the number of sign changes of $\Delta_i(x)$ in $[2, Y]$ by $V_i(Y)$ we have the

COROLLARY 2.

$$\lim_{Y\to\infty} \frac{V_i(Y)}{\log\log Y} = \infty \qquad (1 \leqslant i \leqslant 4).$$

This result is better than the first result of Knapowski [5], [6] stating

$$\lim_{Y\to\infty} \frac{V_1(Y)}{\log\log Y} > 0. \tag{8.8}$$

But recently the latter was improved by Knapowski and Turán [9] to

$$\lim_{Y\to\infty} \frac{V_1(Y)}{\log^{1/4} Y (\log\log Y)^{-4}} > 0. \tag{8.9}$$

Further the author obtained the following improvement of (8.9) (see part IV of this series):

$$\lim_{Y\to\infty} \frac{V_i(Y)}{(\log Y)(\log\log Y)^{-3}} > 0 \qquad (1 \leqslant i \leqslant 4). \tag{8.10}$$

We note that it is possible to prove analogous results to that contained in this series for algebraic number fields and Dedekind zeta-functions too. To these problems we shall return later.

9. Now we shall prove Theorem 1 for the case $i = 2$ with the already mentioned slight modifications, that we shall construct x_2' and x_2'' in I' given by (7.7) for which the inequalities (6.5) hold with $1 - 8\varepsilon/10$ instead of $1 - \varepsilon$. Our proof shall follow the line of Knapowski–Turán [9].

We shall deal with the function

$$\lg x = \begin{cases} 0 & \text{for} \quad 0 < x < 2, \\ \sum_{2\leqslant n\leqslant x} \frac{1}{\log n} & \text{for} \quad x \geqslant 2 \end{cases} \tag{9.1}$$

due to Gauss.

Obviously

$$\lg x = \operatorname{li} x + O(1).$$

Let $\varrho_1 = \beta_1 + i\gamma_1$ be a ζ-zero with the maximal real part β_1 among those satisfying the inequality

$$0 < \gamma_1 \leqslant \gamma_0. \tag{9.2}$$

(Naturally $\varrho_1 = \varrho_0$ is possible.)

Let further

$$\varepsilon' \stackrel{\text{def}}{=} \frac{\varepsilon}{20} \;\left(\leqslant \frac{1}{1000}\right), \quad \lambda \stackrel{\text{def}}{=} \frac{1}{\varepsilon'}. \tag{9.3}$$

Let k be a real number to be determined later for which

$$3(\varepsilon')^2 \log H \leqslant k \leqslant 90 \log H \log \gamma_0. \tag{9.4}$$

Let further

$$\mu \stackrel{\text{def}}{=} \frac{k}{(\varepsilon')^2} = k\lambda^2, \tag{9.5}$$

$$A \stackrel{\text{def}}{=} \exp(\mu - 3k\lambda) = \exp\big(\mu(1-3\varepsilon')\big), \tag{9.6}$$

$$B \stackrel{\text{def}}{=} \exp(\mu + 3k\lambda) = \exp\big(\mu(1+3\varepsilon')\big), \tag{9.7}$$

$$f(x) \stackrel{\text{def}}{=} \Pi(x) - \lg x \pm \frac{x^{\beta_1}(1-10\varepsilon')}{\gamma_0 \mu}, \tag{9.8}$$

$$H(s) \stackrel{\text{def}}{=} \frac{\zeta'}{\zeta}(s) + \zeta(s) - 1 \mp \frac{\beta_1(1-10\varepsilon')}{(s-\beta_1)^2 \gamma_0 \mu}, \tag{9.9}$$

where both in (9.8) and (9.9) the upper signs or the lower ones are meant.

Let us assume that we have for all $x \in I'$

$$\Delta_2(x) < \left(1 - \frac{8\varepsilon}{10}\right) \frac{x^{\beta_0}}{|\varrho_0| \log x} \tag{9.10}$$

or

$$\Delta_2(x) > -\left(1 - \frac{8\varepsilon}{10}\right) \frac{x^{\beta_0}}{|\varrho_0| \log x}. \tag{9.11}$$

Then these inequalities also hold in

$$I'' \stackrel{\text{def}}{=} [A, B] \subset I'. \tag{9.12}$$

But if e.g. (9.11) were satisfied, then we should have for all $x \in I''$

$$\begin{aligned} \Pi(x) - \lg x &= \Delta_2(x) + O(1) > -\left(1 - \frac{3\varepsilon}{4}\right) \frac{x^{\beta_0}}{|\varrho_0| \log x} \\ &> -(1-15\varepsilon') \frac{x^{\beta_0}}{\gamma_0(1-3\varepsilon')\mu} > -(1-10\varepsilon') \frac{x^{\beta_1}}{\gamma_0 \mu} \end{aligned} \tag{9.13}$$

from which for $x \in I''$

$$f(x) = \Pi(x) - \lg x + (1-10\varepsilon') \frac{x^{\beta_1}}{\gamma_0 \mu} > 0 \tag{9.14}$$

would follow, i.e. $f(x)$ would not change its sign in the interval I''. From this we get a contradiction, which proves Theorem 1.

10. By partial integration one can easily prove the formula

$$\int_1^\infty f(x) \frac{d}{dx}(x^{-s} \log x)\, dx = H(s) \tag{10.1}$$

valid for $\sigma > 1$.

Further we shall use the formula ($A > 0$, B arbitrary complex)

$$\begin{aligned} \frac{1}{2\pi i} \int_{(2)} e^{As^2 + Bs} ds &= \exp\left(-\frac{B^2}{4A}\right) \frac{1}{2\pi i} \int_{(2)} e^{\left(\sqrt{A}s + \frac{B}{2\sqrt{A}}\right)^2} ds \\ &= \exp\left(-\frac{B^2}{4A}\right) \cdot \frac{1}{\sqrt{A}} \cdot \frac{1}{2\pi i} \int_{\left(2\sqrt{A} + \frac{B}{2\sqrt{A}}\right)} e^{r^2} dr = \exp\left(-\frac{B^2}{4A}\right) \cdot \frac{1}{2\pi\sqrt{A}} \int_{-\infty}^{+\infty} e^{-t^2} dt \\ &= \exp\left(-\frac{B^2}{4A}\right) \cdot \frac{1}{2\sqrt{\pi A}}. \end{aligned} \tag{10.2}$$

Replacing s by $s + i\gamma_1$ in (10.1), multiplying both sides by $e^{ks^2 + \mu s}$ and integrating with respect to s along the line $\sigma = 2$ and changing the order of integrations we get

$$\begin{aligned} U &= \frac{1}{2\pi i} \int_{(2)} H(s + i\gamma_1) e^{ks^2 + \mu s} ds \\ &= \frac{1}{2\pi i} \int_{(2)} \int_1^\infty f(x) \frac{d}{dx} (x^{-s-i\gamma_1} \log x\, e^{ks^2 + \mu s})\, dx\, ds \end{aligned} \tag{10.3}$$

$$= \int_1^\infty f(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\log x\cdot\frac{1}{2\pi i}\int_{(2)} e^{ks^2+(\mu-\log x)s}\,ds\right\}dx$$

$$= \frac{1}{2\sqrt{\pi k}}\int_1^\infty f(x)\frac{d}{dx}\left\{x^{-i\gamma_1}\log x\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\right\}dx$$

$$= \frac{1}{2\sqrt{\pi k}}\int_1^\infty \frac{f(x)}{x}\left\{\left(-i\gamma_1\log x+1+\right.\right.$$

$$\left.\left.+\log x\frac{(\mu-\log x)}{2k}\right)x^{-i\gamma_1}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)\right\}dx.$$

11. The basic idea of the proof is that supposing that $f(x)$ does not change its sign in I'' one can deduce an upper bound for the absolute value of the right side of (10.3); on the other hand one can give a lower estimate for the absolute value of the left side of (10.3) by suitable choice of k satisfying (9.4), and these two estimations will contradict each other.

12. First we split the integral U on the right side of (10.3) into 3 parts,

(12.1) $$U = U_1+U_2+U_3$$

where

(12.2) $$U_1 = \int_1^A, \quad U_2 = \int_A^B, \quad U_3 = \int_B^\infty.$$

Considering

$$\gamma_0 > \frac{400}{\varepsilon^2} = \left(\frac{1}{\varepsilon'}\right)^2 = \lambda\cdot\frac{1}{\varepsilon'}$$

and our notations in (9.3), (9.5)–(9.7) we have

(12.3)

$$|U_2| \leqslant \frac{1}{2\sqrt{\pi k}}\int_A^B \frac{|f(x)|\log x}{x}\left(\gamma_1+\frac{1}{\log x}+\frac{|\mu-\log x|}{2k}\right)\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx$$

$$\leqslant \frac{1}{2\sqrt{\pi k}}\int_A^B \frac{|f(x)|\mu(1+3\varepsilon')}{x}\left(\gamma_0+\frac{\varepsilon'}{100}+\frac{3k\lambda}{2k}\right)\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx$$

$$\leqslant \frac{(1+5\varepsilon')\mu\gamma_0}{2\sqrt{\pi k}}\int_A^B \frac{|f(x)|}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx$$

$$= \frac{(1+5\varepsilon')\mu\gamma_0}{2\sqrt{\pi k}}\left|\int_A^B \frac{f(x)}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\right|.$$

On the other hand we can trivially estimate

(12.4) $$|U_4| \overset{\text{def}}{=} \left|\int_B^\infty \frac{f(x)}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\right|$$

$$\leqslant \int_B^\infty \exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx = \int_{3k\lambda}^\infty \exp\left(\mu+y-\frac{y^2}{4k}\right)dy$$

$$\leqslant \int_{3k\lambda}^\infty \exp(\mu+y-2y-2k\lambda^2)\,dy < \int_0^\infty e^{-\mu-y}\,dy = e^{-\mu} = o(1)$$

and analogously

(12.5) $$|U_5| = \left|\int_1^A \frac{f(x)}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx\right|$$

$$\leqslant \int_0^{e^\mu} \exp\left(-\frac{9k^2\lambda^2}{4k}\right) = e^{\mu-\frac{9}{4}\mu} = o(1).$$

Naturally, *mutatis mutandis*, we have

(12.6) $$U_1 = o(1) \quad\text{and}\quad U_3 = o(1).$$

Thus, using (12.3)–(12.6), we can change the intervals in the left and right side of (12.3) from $[A, B]$ to $[1, \infty]$ and so with the notation

(12.7) $$K \overset{\text{def}}{=} \frac{1}{2\sqrt{\pi k}}\int_1^\infty \frac{f(x)}{x}\exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx$$

we get

(12.8) $$|U| = |U_2|+o(1) \leqslant (1+5\varepsilon')\mu\gamma_0|K|+o(1).$$

13. Now we shall estimate $|K|$ from above. For $\sigma>1$ one gets easily the formula

(13.1) $$\int_1^\infty \frac{f(x)}{x^{s+1}}\,dx = \frac{1}{s}\left\{\int_2^s\left(\frac{\zeta'}{\zeta}(z)+\zeta(z)\right)dz+h\right\} \pm \frac{(1-10\varepsilon')}{(s-\beta_1)\mu\gamma_0}$$

$$\overset{\text{def}}{=} \varphi(s) \pm \frac{1-10\varepsilon'}{(s-\beta_1)\mu\gamma_0}$$

where h is a constant.

Multiplying both sides by $\frac{1}{2\pi i}e^{ks^2+\mu s}$ and integrating along $\sigma = 2$ we have

$$K = \pm\frac{1-10\varepsilon'}{\mu\gamma_0}\cdot\frac{1}{2\pi i}\int_{(2)}\frac{\exp(ks^2+\mu s)}{s-\beta_1}\,ds + \frac{1}{2\pi i}\int_{(2)}\varphi(s)\exp(ks^2+\mu s)\,ds \stackrel{\text{def}}{=} \pm\frac{1-10\varepsilon'}{\mu\gamma_0}K_1+K_2. \tag{13.2}$$

Shifting the line of integration to $\sigma = 0$ we get

$$K_1 = \exp(k\beta_1^2+\mu\beta_1)+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-t^2k+\mu it}}{it-\beta_1}\,dt = \exp(k\beta_1^2+\mu\beta_1)+O(1). \tag{13.3}$$

We define the broken line l for $t \geqslant 0$ by

$$\begin{aligned} &I_1:\ \sigma = 5/4 &&\text{for } t > \lambda^2,\\ &I_2:\ \beta_1+1/\lambda^2 \leqslant \sigma \leqslant 5/4 &&\text{for } t = \lambda^2,\\ &I_3:\ \sigma = \beta_1+1/\lambda^2 &&\text{for } 14 < t < \lambda^2,\\ &I_4:\ 1/4 \leqslant \sigma \leqslant \beta_1+1/\lambda^2 &&\text{for } t = 14,\\ &I_5:\ \sigma = 1/4 &&\text{for } 0 \leqslant t < 14 \end{aligned} \tag{13.4}$$

and for $t \leqslant 0$ by reflexion on the real axis. Since by the choice of ϱ_1 in (9.2) $\varphi(s)$ is regular on the right of l, we have

$$K_2 = \frac{1}{2\pi i}\int_{(l)}\varphi(s)\exp(ks^2+\mu s)\,ds. \tag{13.5}$$

We shall use the fact, that if $\zeta(s)$ has no zero in the domain

$$\sigma \geqslant \beta, \quad |t| \leqslant T+1 \tag{13.6}$$

then for

$$\sigma \geqslant \beta+\alpha, \quad |t| \leqslant T \tag{13.7}$$

one has

$$\left|\frac{\zeta'}{\zeta}(z)\right| = O\left(\frac{\log T}{\alpha^2}\right). \tag{13.8}$$

(This is a somewhat modified form of Theorem 20 in Ingham [3].)

Further we use the classical estimate

$$|\zeta(z)| = O(\sqrt{t}) \quad \text{for} \quad \sigma \geqslant \tfrac{1}{2},\ |t| \geqslant 10. \tag{13.9}$$

From (13.8) and (13.9) we get by easy computations for the integrals $\mathscr{F}_i$ on the interval I_i $(i = 1, 2, \ldots, 5)$ the estimates (considering $\mu = k\lambda^2 = k(\varepsilon')^{-2}$ and $k, \mu \to \infty$)

(13.10)

$$\begin{aligned} |\mathscr{F}_1| &= O\big(\exp(k\cdot\tfrac{25}{16}-k\lambda^2+\tfrac{5}{4}\mu)\big) \leqslant \tfrac{1}{10}\exp(\tfrac{1}{3}\mu),\\ |\mathscr{F}_2| &= O\big(\lambda^4\log\lambda\exp(k\cdot\tfrac{25}{16}-k\lambda^2+\tfrac{5}{4}\mu)\big) \leqslant \tfrac{1}{10}\exp(\tfrac{1}{3}\mu),\\ |\mathscr{F}_3| &= O\Big(\lambda^4\log\lambda\exp\big(-194k+\mu(\beta_1+1/\lambda^2)\big)\Big) \leqslant \tfrac{1}{10}\varepsilon'^{-5}\exp(\mu\beta_1-192k),\\ |\mathscr{F}_4| &= O\Big(\exp\big(-194k+\mu(\beta_1+1/\lambda^2)\big)\Big) \leqslant \tfrac{1}{10}\exp(\mu\beta_1-192k),\\ |\mathscr{F}_5| &= O\big(\exp(\tfrac{1}{16}k+\tfrac{1}{4}\mu)\big) \leqslant \tfrac{1}{10}\exp(\tfrac{1}{3}\mu) \end{aligned}$$

and analogously for the integrals in the domain $t \leqslant 0$.

Then, taking into account (6.3) and (9.4), we have:

$$\begin{aligned} \min(e^{\mu(\beta_1-1/3)}, e^{192k}) &= e^{192k} > 2k\cdot e^{190k} \geqslant 2kH^{(190\cdot 3\varepsilon^2)/400}\\ &> 2k(H^{\varepsilon^2/4})^{5.5} > 2k\gamma_0^{3/2}\cdot\gamma_0^4 > 2k\gamma_0^{3/2}\cdot\frac{1}{\varepsilon'^8}\\ &= \frac{1}{\varepsilon'^5}\cdot\frac{1}{\frac{1}{2}\varepsilon'}\cdot\mu\gamma_0^{3/2}. \end{aligned} \tag{13.11}$$

From (13.10) and (13.11) we get:

$$|K_2| \leqslant \frac{1}{\varepsilon'^5}\max(e^{\mu\beta_1-192k}, e^{\mu/3}) \leqslant \frac{\varepsilon'}{2}\,\frac{\exp(k\beta_1^2+\mu\beta_1)}{\mu\gamma_0}. \tag{13.12}$$

This together with the estimate of K_1 in (13.3) gives for K in (13.2) the upper bound

$$|K| < \frac{1-9\varepsilon'}{\mu\gamma_0}\exp(k\beta_1^2+\mu\beta_1). \tag{13.13}$$

Now, replacing in (12.8) K by this upper estimate, we get for U the inequality

$$|U| < (1-4\varepsilon')\exp(k\beta_1^2+\mu\beta_1) \tag{13.14}$$

and so we have an upper estimate for U.

14. In order to get a lower bound for U we shall estimate the absolute value of the integral U on the left side of (10.3) by suitable choice of k, furnished by a powersum theorem. Shifting the line of integration to $\sigma = -\frac{1}{2}$ we get

$$(14.1)\quad U = \sum_{\varrho} \exp\{k[(\varrho - i\gamma_1)^2 + \lambda^2(\varrho - i\gamma_1)]\} \mp \mp \frac{(1-10\varepsilon')\beta_1}{\mu\gamma_0}\frac{d}{ds}\left(\exp(ks^2+\mu s)\right)_{s=\beta_1 - i\gamma_1} + + \frac{1}{2\pi i}\int_{(-\frac{1}{2})} H(s+i\gamma_1)\exp(ks^2+\mu s)\,ds.$$

Easy computation shows that the integral is $o(1)$ for $\mu \to \infty$. Further the second residuum is in absolute value less than

$$(14.2)\quad \frac{\beta_1(2k|\beta_1 - i\gamma_1| + \mu)}{\mu\gamma_0}\exp\left(k(\beta_1^2 - \gamma_1^2) + \mu\beta_1\right) < \frac{\varepsilon'}{2}\exp(k\beta_1^2 + \mu\beta_1).$$

Now, using that for $T > T_0$ the number of zeros in $T \leqslant t \leqslant T+1$ is (with the usual notation)

$$(14.3)\quad N(T+1) - N(T) \leqslant 15\log T$$

(see e.g. W.J. Ellison–M.Mendès France [2], p. 165), we get for the contribution of $\varrho = \beta + i\gamma$'s with $\gamma - \gamma_1 \geqslant 2\lambda$ the upper bound

$$(14.4)\quad \int_{2\lambda}^{\infty} 15\log(\gamma_1 + t)\exp\left(k\left(1-(t-1)^2\right) + \lambda^2 k\right) = o(1)$$

and analogously for the zeros with $\gamma - \gamma_1 \leqslant -2\lambda$.

15. So the number of remaining zeros with $|\gamma - \gamma_1| < 2\lambda$ is owing to (14.3)

$$(15.1)\quad N(\gamma_1 + 2\lambda) - N(\gamma_1 - 2\lambda) \leqslant 60\lambda\log(\gamma_1 + 2\lambda) < 90\lambda\log\gamma_0 = \frac{90}{\varepsilon'}\log\gamma_0.$$

Thus we can apply the powersum theorem of Montgomery (see the Appendix, Theorem 1) for the numbers

$$(15.2)\quad z_j = \exp\{3(\varepsilon')^2\log H[(\varrho_j - i\gamma_1)^2 + \lambda^2(\varrho_j - i\gamma_1)]\}$$

with the property

$$(15.3)\quad |\gamma_j - \gamma_1| < 2\lambda$$

where the number of terms is by (15.1)

$$(15.4)\quad 1 \leqslant n < \frac{90}{\varepsilon'}\log\gamma_0.$$

Thus we get the existence of a ν_0 for which

$$(15.5)\quad 1 \leqslant \nu_0 \leqslant \frac{1}{3\varepsilon'}n < \frac{30}{(\varepsilon')^2}\log\gamma_0$$

and

$$(15.6)\quad \left|\sum_{j=1}^{n} z_j^{\nu_0}\right| > (1-3\varepsilon')\max_{1\leqslant j\leqslant n}|z_j^{\nu_0}| \geqslant (1-3\varepsilon')|z_1^{\nu_0}|$$

hold, i.e. choosing

$$(15.7)\quad k \overset{\text{def}}{=} 3\varepsilon'^2\log H\cdot\nu_0$$

we get

$$(15.8)\quad |W| \overset{\text{def}}{=} \sum_{|\gamma-\gamma_1|<2\lambda} \exp\{k[(\varrho - i\gamma_1)^2 + \lambda^2(\varrho - i\gamma_1)]\} \geqslant (1-3\varepsilon')\exp(k\beta_1^2 + \mu\beta_1)$$

where k satisfies the inequality

$$(15.9)\quad 3(\varepsilon')^2\log H \leqslant k \leqslant 90\log H\log\gamma_0$$

required in (9.4).

And thus from (14.1), (14.2), (14.4) and (15.8) we have the inequality

$$(15.10)\quad |U| > |W| - \varepsilon'\exp(k\beta_1^2 + \mu\beta_1) > (1-4\varepsilon')\exp(k\beta_1^2 + \mu\beta_1)$$

which contradicts (13.14) and thus proves the theorem.

16. Now we shall sketch what changes are necessary in the proof of Theorem 1 to get a proof for Theorem 2 (in the case $i = 2$).

First let $\varrho_1' = \beta_1' + i\gamma_1'$ be a ζ-zero with the maximal real part β_1' among those satisfying

$$(16.1)\quad 0 < \gamma_1' \leqslant \gamma_0.$$

Let ϱ_2' be the zero with the maximal real part β_2' satisfying

$$(16.2)\quad \gamma_1' \leqslant \gamma_2 \leqslant \gamma_1' + \log\gamma_0, \qquad \beta_2' \geqslant \beta_1' + \frac{1}{\log H}$$

if such a zero exists, and let ϱ_{n+1}' be the zero with the maximal real part satisfying

$$(16.3)\quad \gamma_n' < \gamma_{n+1}' \leqslant \gamma_n' + \max(\log\gamma_n', \log\gamma_0), \qquad \beta_{n+1}' \geqslant \beta_n' + \frac{1}{\log H}$$

if such a zero exists.

Thus we get after no more than $\left[\frac{\log H}{2}\right] + 1$ steps a zero $\varrho_N' = \beta_N' + i\gamma_N' = \varrho_1 = \beta_1 + i\gamma_1$ for which $\gamma_1 < \gamma_1' + \log^2 H < \gamma_0 + \log^2 H$. Further the regions

(16.4) $$0 < |t| \leqslant \gamma_1, \quad \sigma > \beta_1$$

and

(16.5) $$\gamma_1 < |t| \leqslant \gamma_1 + \max(\log\gamma_1, \log\gamma_0), \quad \sigma \geqslant \beta_1 + \frac{1}{\log H}$$

are zerofree.

Now we have to distinguish two cases.

Case A:

$$\gamma_0^{\varepsilon^{4/3}/500} < \gamma_1 < \gamma_0 + \log^2 H$$

and

Case B:

$$\gamma_1 \leqslant \gamma_0^{\varepsilon^{4/3}/500}.$$

17. In Case A for $\gamma'_{n+1} \geqslant \gamma_0$ and for any x in I given by (8.3) we have the inequality

(17.1) $$\frac{x^{\beta'_{n+1}}}{\gamma'^{1+\varepsilon}_{n+1}} > \frac{x^{\beta'_n} e^{\frac{\log x}{\log H}}}{(2\gamma'_n)^{1+\varepsilon}} > \frac{5}{4} \cdot \frac{x^{\beta'_n}}{(\gamma'_n)^{1+\varepsilon}}$$

and for $\gamma'_{n+1} < \gamma_0$ the trivial inequality

(17.2) $$\frac{x^{\beta'_{n+1}}}{(\gamma'_{n+1})^{1+\varepsilon}} > \frac{x^{\beta_0}}{\gamma_0^{1+\varepsilon}}.$$

So if we work with $\varrho_1 = \beta_1 + i\gamma_1$ instead of $\varrho_0 = \beta_0 + i\gamma_0$, then as one can easily see (8.2) (if we replace 10^7 by $5\cdot 10^6$, say, in the exponent) remains true; thus we shall prove the estimate (8.4) in the form

(17.3) $$\Delta_2(x'_2) > \frac{(x'_2)^{\beta_1}}{\gamma_1^{1+\varepsilon}\log x'_2} \qquad \left(\geqslant \frac{(x'_2)^{\beta_0}}{\gamma_0^{1+\varepsilon}\log x_2}\right).$$

The necessary changes in the choice of our parameters (see (9.3)–(9.9)) and notations are the following

(17.4) $$\frac{1}{(\varepsilon')^2}\log H(1+4\varepsilon') \leqslant k \leqslant \log H(1+6\varepsilon')\cdot\frac{1}{(\varepsilon')^2},$$

(17.5) $$f(x) \overset{\text{def}}{=} \Pi(x) - \operatorname{lg} x \pm \frac{2x^{\beta_1}}{\gamma_1^{1+\varepsilon}\mu},$$

(17.6) $$H(s) \overset{\text{def}}{=} \frac{\zeta'}{\zeta}(s) + \zeta(s) - 1 \mp \frac{2}{(s-\beta_1)^2\gamma_1^{1+\varepsilon}\mu}.$$

Further instead of γ_0 in the course of the proof of Theorem 1 we always write γ_1.

The first part of the proof runs completely analogously to that of Theorem 1 and thus we get:

(17.7) $$|K| < \frac{3e^{k\beta_1^2+\mu\beta_1}}{\gamma_1^{1+\varepsilon}\mu}$$

and from this as in (13.14)

(17.8) $$|U| < \frac{4e^{k\beta_1^2+\mu\beta_1}}{\gamma_1^{\varepsilon}}.$$

18. With the upper estimation of U the integral in (14.1) is also $o(1)$; the residuum is now

(18.1) $$< \frac{\varepsilon'}{2}\,\frac{e^{k\beta_1^2+\mu\beta_1}}{\gamma_1^{\varepsilon}}$$

and the contribution of zeros with $|\gamma-\gamma_1| \geqslant 2\lambda$ to the infinite powersum is again $o(1)$ as in (14.4).

However, the remaining finitely many zeros we divide into two classes, defined as follows:

(18.2) $$\begin{aligned} C_1 &\overset{\text{def}}{=} \{\varrho;\ |\gamma-\gamma_1| \leqslant (\varepsilon')^{4/3},\ |\beta-\beta_1| \leqslant (\varepsilon')^{4/3}\},\\ C_2 &\overset{\text{def}}{=} \{\varrho;\ |\gamma-\gamma_1| < 2\lambda,\ \varrho \notin C_1\}. \end{aligned}$$

For the zeros of C_2

(18.3) $$|\gamma-\gamma_1| < 2\lambda = \frac{2}{\varepsilon'} = \frac{40}{\varepsilon} < \frac{1}{\varepsilon^2} \leqslant \log\gamma_0$$

and so by (16.4) and (16.5)

(18.4) $$\beta < \beta_1 + \frac{1}{\log H} < \beta_1 + \frac{2}{\mu}.$$

Thus we get for a $\varrho \in C_2$

(18.5) $$\exp\{k(\beta^2 - (\gamma-\gamma_1)^2) + \mu\beta\} < \exp\left\{k\left(\beta_1^2 + \frac{5}{\mu} - (\varepsilon')^{8/3}\right) + \mu\beta_1 + 2\right\}$$

and the number of zeros $\varrho \in C_2$ is, as in (15.4),

$$< \frac{90}{\varepsilon'}\log\gamma_1.$$

Hence the contribution of the zeros $\varrho \in C_2$ is

$$(18.6)\qquad < \frac{90}{\varepsilon'} \log\gamma_1 \exp\left(k\beta_1^2 + \mu\beta_1 - k(\varepsilon')^{8/3} + 7\right) < \gamma_1^{3/2} \exp\left(k\beta_1^2 + \mu\beta_1 - k(\varepsilon')^3\right)$$

$$= \frac{\gamma_1^{\varepsilon/2} \exp(k\beta_1^2 + \mu\beta_1)}{\exp\left(\mu(\varepsilon')^5\right)} < \frac{\gamma_1^{\varepsilon/2} \exp(k\beta_1^2 + \mu\beta_1)}{H^{\frac{\varepsilon^6}{32\cdot 10^5}}} < \frac{\exp(k\beta_1^2 + \mu\beta_1)}{\gamma_1^{\varepsilon}}.$$

On the other hand, we can apply Theorem 3 of the Appendix with the choice

$$(18.7)\qquad a = \beta_1 + \frac{1}{\log H}, \quad \tau = \gamma_1, \quad \eta = 2(\varepsilon')^{4/3}$$

and thus we get for the number n of zeros $\varrho \in C_1$ the estimate

$$(18.8)\qquad 1 \leqslant n < 2(\varepsilon')^{4/3} \log\gamma_1.$$

Now we can apply the continuous form of the second main theorem of the powersum theory (see the Appendix, Theorem 2):

$$(18.9)\qquad \max_{a \leqslant t \leqslant a+d} \frac{\left|\sum_{j=1}^{n} e^{\alpha_j t}\right|}{\max\limits_{1 \leqslant j \leqslant n} |e^{\alpha_j t}|} \geqslant \left(\frac{1}{8e\left(1 + \frac{a}{d}\right)}\right)^n$$

for the numbers

$$(18.10)\qquad \alpha_j = (\varrho_j - i\gamma_1)^2 + \lambda^2(\varrho_j - i\gamma_1) \qquad (\varrho_j \in C_1)$$

and choosing

$$(18.11)\qquad a = \frac{1}{(\varepsilon')^2} \log H(1 + 4\varepsilon'), \qquad d = \frac{2}{\varepsilon'} \log H$$

we get the existence of a k satisfying (17.4) for which

$$(18.12)\qquad |W| = \sum_{\varrho_j \in C_1} \exp\{k[(\varrho_j - i\gamma_1)^2 + \lambda^2(\varrho_j - i\gamma_1)]\}$$

$$\geqslant \frac{e^{k\beta_1^2 + k\lambda^2\beta_1}}{\left(\frac{20}{\varepsilon'}\right)^{2(\varepsilon')^{4/3}\log\gamma_1}} \geqslant \frac{6e^{k\beta_1^2 + \mu\beta_1}}{\exp\left(3\log\frac{1}{\varepsilon'} \cdot 2(\varepsilon')^{4/3}\log\gamma_1\right)} \geqslant \frac{6e^{k\beta_1^2 + \mu\beta_1}}{\gamma_1^{\varepsilon}}.$$

Thus we get from (18.1), (18.6) and (18.12) for U the lower bound

$$(18.13)\qquad |U| > |W| - \frac{2e^{k\beta_1^2 + \mu\beta_1}}{\gamma_1^{\varepsilon}} \geqslant \frac{4e^{k\beta_1^2 + \mu\beta_1}}{\gamma_1^{\varepsilon}}$$

which contradicts to (17.8).

19. In Case B in the definition of (17.5) and (17.6) in the last term instead of $\gamma_1^{1+\varepsilon}$ we write $\gamma_0^{1+\varepsilon}$, and in contrast to Case A the proof we leave γ_0 where it was in the proof of Theorem 1.

Thus we get the formulae (17.7), (17.8) and (18.1), further (18.6) with the only change that γ_1 is replaced by γ_0.

But perhaps γ_1 is not large enough (compared to ε') and thus Theorem 3 of the Appendix is not applicable and that is why the distinction of Case A and B was needful. However it does not make any difficulty, as here, using (14.3) for the number n of zeros $\varrho \in C_1'$ we have the estimate:

$$(19.1)\qquad n \leqslant 15\log\gamma_1 < \frac{15\varepsilon^{4/3}\log\gamma_0}{500} < 2(\varepsilon')^{4/3}\log\gamma_0$$

as in (18.8) and so we get in Case B also a contradiction in the same way as in Case A.

20. Now, as already mentioned, it would be necessary to prove the case $i = 4$ too, as we cannot infer this from the case $i = 2$.

The proof runs naturally along the same lines, it is even simpler. One must, namely, deal instead of $f(x)$ and $H(s)$ in (17.5), (17.6) with the functions

$$(20.1)\qquad \tilde{f}(x) = \psi(x) - x \pm \frac{2x^{\beta_1}}{\gamma_1^{1+\varepsilon}}$$

and

$$(20.2)\qquad \tilde{H}(s) = \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} \mp \frac{2s}{(s-\beta_1)\gamma_1^{1+\varepsilon}}$$

which are connected by the relation

$$(20.3)\qquad \int_1^{\infty} \tilde{f}(x) \frac{d}{dx}(x^{-s})\, dx = \tilde{H}(s)$$

which replaces both (10.1) and (13.1).

The cases $i = 1$ and $i = 3$ can be inferred from the above immediately owing to (8.2) as it was done in case of Theorem 1 (see (7.1)–(7.4)).

Appendix

The following theorem is the slightly modified form of a special case of a powersum theorem due to H. L. Montgomery [13].

THEOREM 1 (Montgomery). *For arbitrary complex numbers* z_j $(j = 1, 2, \ldots, n)$ *and for any* ε, $0 < \varepsilon < 1$, *one has*

$$\max_{1 \leqslant \nu \leqslant n/\varepsilon} \frac{\left|\sum_{j=1}^{n} z_j^{\nu}\right|}{\max\limits_{1 \leqslant j \leqslant n} |z_j|^{\nu}} \geqslant 1 - \varepsilon.$$

We give here a very short proof. The following lemma embodies the basic idea in Montgomery's original theorem.

LEMMA 1 (Montgomery). *Let*

$$P(r,\theta)=\sum_{\nu=1}^{N}\left(1-\frac{\nu}{N+1}\right)r^{\nu}\cos\nu\theta.$$

Then $P(r,\theta)\geqslant -1/2$ *for all* θ *and* $0\leqslant r\leqslant 1$, *further* $P(1,0)=N/2$.

Proof. Let $z=re^{i\theta}$. Then $P(r,\theta)$ is a harmonic function of z and

$$\tfrac{1}{2}+P(1,\theta)\geqslant 0$$

because of the properties of the Fejér kernel, which proves the first assertion. The second assertion is trivial.

For the proof of the theorem we may suppose $\max\limits_{1\leqslant j\leqslant n}|z_j|=|z_1|=z_1=1$. Let $z_j=r_je^{\theta_j}$ where now $0\leqslant r_j\leqslant 1$. Thus using Lemma 1 we have

$$\sum_{\nu=1}^{N}\left(1-\frac{\nu}{N+1}\right)\operatorname{Re}\left(\sum_{j=1}^{n}z_j^{\nu}\right)=\sum_{j=1}^{m}\sum_{\nu=1}^{n}\left(1-\frac{\nu}{N+1}\right)r_j^{\nu}\cos\nu\theta_j$$
$$=\sum_{j=1}^{n}P(r_j,\theta_j)\geqslant\frac{N}{2}-\frac{n-1}{2}$$

and thus

$$\frac{\sum\limits_{\nu=1}^{N}\left(1-\frac{\nu}{N+1}\right)\operatorname{Re}\left(\sum\limits_{j=1}^{N}z_j^{\nu}\right)}{\sum\limits_{\nu=1}^{N}\left(1-\frac{\nu}{N+1}\right)}\geqslant 1-\frac{n-1}{N}.$$

Now choosing $N=\left[\frac{n}{\varepsilon}\right]>\frac{n}{\varepsilon}-1>\frac{n-1}{\varepsilon}$ we get from this

$$\max_{1\leqslant\nu\leqslant N}\operatorname{Re}\left(\sum_{j=1}^{n}z_j^{\nu}\right)\geqslant 1-\frac{n-1}{N}>1-\varepsilon$$

which proves the theorem.

The following theorem is a special case of the so-called second main theorem of the powersum theory developed by Turán.

THEOREM 2 (T. Sós–Turán). *For arbitrary complex numbers* z_j

$$\max_{m\leqslant\nu\leqslant m+n}\frac{\left|\sum\limits_{j=1}^{n}z_j^{\nu}\right|}{\max\limits_{1\leqslant j\leqslant n}|z_j|^{\nu}}\geqslant\left(\frac{1}{8e\left(\frac{m}{n}+1\right)}\right)^{n}.$$

For the proof see T. Sós–Turán [15].

Now, choosing $m=a\dfrac{n}{d}$, $z_j=e^{a_j\frac{a}{m}}=e^{a_j\frac{d}{n}}$, we get from this

$$\max_{\frac{n}{d}a\leqslant\nu\leqslant(a+d)\frac{n}{d}}\frac{\left|\sum\limits_{j=1}^{n}e^{a_j\frac{d}{n}\nu}\right|}{\max\limits_{1\leqslant j\leqslant n}|z_j|^{\nu}}\geqslant\left(\frac{1}{8e\left(\frac{a}{d}+1\right)}\right)^{n}$$

and so we get the continuous form of it as

$$\max_{a\leqslant t\leqslant a+d}\frac{\left|\sum\limits_{j=1}^{n}e^{a_jt}\right|}{\max\limits_{1\leqslant j\leqslant n}|e^{a_jt}|}\geqslant\left(\frac{1}{8e\left(\frac{a}{d}+1\right)}\right)^{n}$$

and this naturally remains true if we replace $\max\limits_{1\leqslant j\leqslant n}|e^{a_jt}|$ by $|e^{a_jt}|$ with arbitrary fixed j.

Further we shall use the fact that in small squares "affixed from the left to big zero-free parallelograms" the number of zeros is not too large. Namely Turán [19] proved the following

THEOREM 3 (Turán). *Let*

$$\tfrac{1}{2}<a<1,\qquad 0<\eta<\tfrac{1}{20}\left(a-\tfrac{1}{2}\right)$$

and suppose $\zeta(s)$ *does not vanish in the parallelogram*

$$a\leqslant\sigma\leqslant 1,\qquad |t-\tau|\leqslant\log\tau.$$

Denoting by $M(\tau,a,\eta)$ *the number of zeros in the parallelogram*

$$a-\eta\leqslant\sigma\leqslant a,\qquad |t-\tau|\leqslant\eta/2$$

we have for

$$\eta\geqslant 500\,\frac{\log\log\log\tau}{\log\log\tau}$$

the estimation

$$M(\tau,a,\eta)<\eta\log\tau.$$

For the proof see the Appendix of Turán [19].

Note (added 18 November 1977). I am indebted to Professor Szalay for calling my attention to the powersum theorem of J. W. S. Cassels (*On the sums of powers of complex numbers*, Acta Math. Acad. Sci. Hungar. 7 (3–4) (1956), pp. 283–290) according to which for arbitrary complex numbers $z_1,\ldots,z_n$ the inequality

$$\max_{1\leqslant\nu\leqslant 2n-1}\frac{\left|\sum\limits_{i=1}^{n}z_i^{\nu}\right|}{\max\limits_{1\leqslant i\leqslant n}|z_i|^{\nu}}\geqslant 1$$

holds.

Using this theorem instead of the Theorem 1 of the Appendix we get Theorem 1 with the tightened interval

$$I_1 = [H, H^{\frac{10^4}{\varepsilon}\log\gamma_0}]$$

instead of I in (6.4). With some other simple modifications of the proof of Theorem 1 it is possible even to prove it with the interval

$$I_2 = [H, H^{100\log\gamma_0}].$$

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