Slippery Cantor sets in E^n

by

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Abstract. We prove the existence of wild Cantor sets in E^n ($n \geqslant 3$) which can be pushed off an arbitrary 1-dimensional compactum in E^n by an arbitrarily small homeomorphism of E^n. This answers in the negative a conjecture of R. J. Daverman.

1. Introduction. Wild Cantor sets in E^n behave geometrically in E^n much like polyhedra of dimension $n-2$ [12]. Thus two wild Cantor sets in "general position" in E^n might be expected to intersect like two polyhedra of dimension $n-2$. Such an intersection of polyhedra would, in general, be nonempty precisely in the range $n \geqslant 4$. These considerations motivate the following conjecture of R. J. Daverman [3].

CONJECTURE 1.1. A Cantor set C in E^n, $n \geqslant 4$, is tame if and only if, for each Cantor set D in E^n and each $\varepsilon > 0$, there is an ε-homeomorphism $h\colon E^n \to E^n$ such that $(hC) \cap D = \emptyset$. (The ε-homeomorphism h is to be thought of as an adjustment putting hC and D in "general position".)

Our goal is to supply counterexamples to Conjecture 1.1. We say that a set X in E^n can be slipped off the set Y in E^n if, for each $\varepsilon > 0$, there is an ε-homeomorphism $h\colon E^n \to E^n$ such that $(hX) \cap Y = \emptyset$; otherwise we say that X *cannot be slipped off* Y. No wild Cantor set can be slipped off every two dimensional compactum; nevertheless we exhibit wild Cantor sets in E^n, $n \geqslant 3$, which can be slipped off every 1-dimensional compactum in E^n, thus certainly off every Cantor set. This contradicts Conjecture 1.1. L. O. Cannon has suggested the term *slippery* for the sets that can be slipped off every 1-dimensional compactum.

Antoine's Necklace, the standard wild Cantor set in E^3 [1], is not slippery (Section 3); for it cannot be slipped off the tangled one-dimensional continuum described by McMillan and Row [8]. Antoine's Necklace seems to be too rigid to be slippery. We suspect that many wild Cantor sets in E^n, $n > 3$, are also too rigid to be slippery, but we have been unable to confirm our suspicion. Thus, the following alternative to Conjecture 1.1 also suggested by Daverman [3], while unlikely, is still a possibility.

CONJECTURE 1.2. A Cantor set in E^n can be slipped off every Cantor set in E^n.

Conjectures 1.1 and 1.2 arose in Daverman's early attempts to improve the results of his paper [3]. Conjecture 1.2 is known to be true for $n \leqslant 3$ [7], and it is equivalent in all dimensions with the following conjecture [13].

CONJECTURE 1.3. For two Cantor sets C and D in E^n with $C \cap D$ a single point, C can be slipped off D.

The key to our construction of wild slippery sets is to make them so knotted that they fit through any conceivable hole. A tame arc in E^3 is not slippery [8] (can the slippery lawyer go straight?); we construct a slippery arc α in E^3 by putting every possible knot in every subarc of α. All of our slippery constructions simply amplify this idea of multiple knotting.

2. Definitions and notation. We use S^n, B^n and E^n to denote the n-sphere, the n-ball, and Euclidean n-space, respectively. We let I^n, J^n, and Σ^n denote the piecewiselinear manifolds $[-1, 1]^n$, $[-2, 2]^n$, and the boundary of I^{n+1}, respectively. Suppose Y is a space with metric ϱ. A subset of Y is said to be ε-*small* if its diameter is less than or equal to ε. If f, g are maps of a space X into Y, we say f is ε-*close* to g if $\varrho(f(x), g(x)) \leqslant \varepsilon$ for each x in X. A map of Y into itself is called an ε-*map* if it is ε-close to the identity map. We let id_X: $X \to X$ be the identity map and omit the X when it causes no confusion. We use $\mathrm{Int}\, M$ and $\mathrm{Bd}\, M$ to denote the interior and boundary, respectively, of a manifold M.

We let W^n denote the quotient space $(B^2 \times S^{n-2})/(B^2 \times \{x\})$ where $x \in S^{n-2}$. A space W homeomorphic with W^n is called a P-*manifold* (pinched-manifold) of dimension n, and if $n = 3$ W is called a *pinched solid torus*. If π: $B^2 \times S^{n-2} \to W^n$ is the natural projection and h: $W^n \to W$ is a homeomorphism, then $h \circ \pi(B^2 \times \{x\}) = p$ is called the *pinchpoint* of W. Notice that $W - \{p\}$ is a manifold. We also call $h \circ \pi(\{0\} \times S^{n-2})$ a *core* of W. More generally, if U is an open subset of the set X which, in turn, is a closed subset of the compact metric space Y, and Y is homeomorphic with the quotient space of $B^2 \times X$ with $B^2 \times \{x\}$ identified to a point for each $x \in X - U$, the homeomorphism sending x to $(0, x)$, then X is a *core* for Y. If X is a compact subset of E^n and U is an open subset of X which is also a flat $(n-2)$-manifold in E^n, then by a *thickening of X relative to $X - U$* we mean a set Y in E^n with core X so that $Y - (X - U)$ is a B^2-product neighborhood of U; furthermore $\mathrm{Bd}(Y - (X - U))$ is required to be flat in E^n.

3. Antoine's necklace. We review the construction of Antoine's necklace [1] and prove a lemma. We then show that Antoine's necklace is a wild Cantor set in E^3 which cannot be slipped off every 1-dimensional continuum.

A *solid torus* is a topological space homeomorphic to $B^2 \times S^1$. The inverse of $\{0\} \times S^1$ under some homeomorphism is a core of the solid torus. Consider the embedding of $2k$ $(k > 2)$ solid tori $A_1, \ldots, A_{2k}$ in a solid torus A with core J. The A_i are small regular neighborhoods of the boundaries of the disks D_i shown in Figure 1. We call this embedding an Antoine embedding and for future reference pay particular attention to the manner in which J lies in the union of the D_i. The boundaries of the D_i serve as cores for the A_i.

We construct a Cantor set $X = \bigcap M_i$ in E^3 where for each positive integer i, M_i is a collection of disjoint solid tori. We let M_1 be any solid torus in E^3. The collection M_{i+1} is obtained by taking an Antoine embedding of solid tori in each component of M_i. By choosing k large enough we can suppose that the diameter of each component of M_{i+1} is as small as we like. If the diameters of the components of M_i approach zero as i gets large, then X will be a Cantor set.

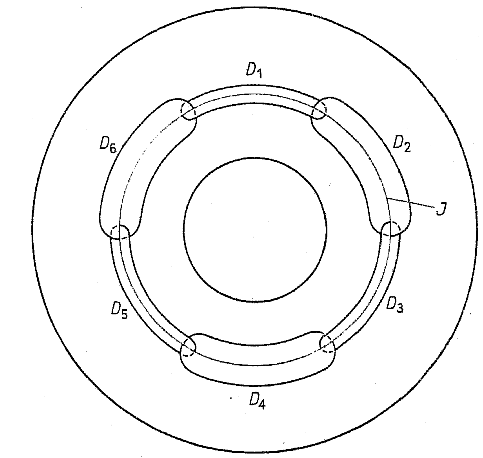

Fig. 1

LEMMA 3.1. *Suppose $Y \subset E^3$ is compact, and suppose that any unknotted simple closed curve in $E^3 - Y$ is null-homotopic in $E^3 - Y$. Let $X = \bigcap M_i$ be the Cantor set described above. If the cores of the components of M_{i+1} miss Y, there is a homeomorphism of E^3 fixed outside M_i which takes Y off the cores of the components of M_i.*

Proof. Suppose the solid torus of Figure 1 is a component of M_i. By assumption $\mathrm{Bd}\, D_i \cap Y = \emptyset$. For $i = 1, 3$ $\mathrm{Bd}\, D_i$ is unknotted and hence there are singular disks D_i' in $E^3 - Y$ with $\mathrm{Bd}\, D_i' = \mathrm{Bd}\, D_i$ and in general position with respect to D_2. Hence, we can find paths $\gamma_i \subset D_i' \cap D_2$ from $\mathrm{Bd}\, D_i \cap D_2$ to $\mathrm{Bd}\, D_2$. Connecting the paths γ_i by a path in $\mathrm{Bd}\, D_2$, we conclude the existence of an arc in $D_2 - Y$ which runs between $\mathrm{Bd}\, D_1 \cap D_2$ and $\mathrm{Bd}\, D_3 \cap D_2$. In the disks with even subscripts we find arcs like the one just constructed. In the disks with odd subscripts we choose arcs in the boundaries which connect the arcs of the first collection. The union of all the arcs is a new core for A which misses Y. We complete the proof by repeating the above process in each component of M_i and taking a homeomorphism which takes the new cores to the old cores.

McMillan and Row [8] have shown the existence of a 1-dimensional continuum Y in E^3 which fails to have the arc pushing property; furthermore, Y cannot be homotopically linked by an unknotted simple closed curve. The failure of Y to have the arc pushing property implies the existence of a polygonal simple closed curve L which cannot be slipped off Y.

THEOREM 3.1. *There is a Cantor set X in E^3 which cannot be slipped off Y.*

Proof. Let $\varepsilon>0$ be such that L cannot be moved off Y by a 4ε-homeomorphism of E^3. Let M_1 be a solid torus in E^3 with core L. We use M_1 to construct an Antoine's necklace $X = \bigcap M_i$ and require, in addition, that the components of M_2 and their associated disks are ε-small. If X can be slipped off Y, there is an ε-homeomorphism h of E^3 onto E^3 which takes Y off X. Hence, $X \cap h(Y) = \emptyset$ and $M_i \cap h(Y) = \emptyset$ for some i. Repeated applications of Lemma 3.1 give rise to an ε-homeomorphism f which is fixed outside M_2 and moves $h(Y)$ off the centerlines of the components of M_2. Now using the proof of the lemma we find a 2ε-homeomorphism g which takes $f\circ h(Y)$ off L. This can be accomplished because the disks associated with M_2 are ε-small. The 4ε-homeomorphism $(g\circ f\circ h)^{-1}$ takes L off Y which is a contradiction.

4. A slippery arc in E^3. The McMillan–Row example shows that a tame arc in E^3 cannot be slipped off an arbitrary 1-dimensional compactum. We now construct a wild arc in E^3 which can be slipped off every 1-dimensional compactum. The basic idea for the construction of slippery Cantor sets is contained in the construction of the arc.

Consider the 3-ball B in E^3 which is the suspension of $I^2\times\{0\}$ with suspension points $a = (0, 0, 1)$ and $b = (0, 0, -1)$. Let A be the straight line interval from a to b and σ_i, $i = 1, 2, 3, \ldots$, be the 1-simplexes of a triangulation of $\operatorname{Int} A$. We denote the endpoint of σ_i which is closer to a by a_i and the other endpoint by b_i. The arc σ_i is properly embedded in the 3-ball $B_i = (E^2\times\sigma_i)\cap B$. In each B_i we find a properly embedded polygonal arc A_i with the same endpoints as σ_i. The arcs A_i are chosen so that all possible knots are represented; i.e., if A' is a properly embedded polygonal arc in B from a to b, then there is an orientation preserving homeomorphism $h\colon B_i\to B$ for some i sending A_i, a_i, b_i to A', a, b, respectively. The orientations on B_i and B are induced by some fixed orientation of E^3.

We set τ_j, $j = 1, 2, 3, \ldots$, to be the 1-simplexes of some triangulation of $\bigcup A_i$. For each τ_j we find a 3-ball W_j by choosing a small polygonal disk which contains the midpoint of τ_j and is perpendicular to τ_j and suspending the disk by the endpoint of τ_j. This is done so that $W_j\cap W_k = \tau_j\cap\tau_k$ for $j\neq k$. We let $W = \bigcup W_j\cup\{a, b\}$. The pair (B, W) has the following important property.

LEMMA 4.1. *Suppose R is a 1-dimensional compactum in B which misses $\{a, b\}$. Then there is a homeomorphism of B onto itself, fixing the boundary, which takes W off R.*

Proof. Since R is 1-dimensional, there is a properly embedded polygonal arc A' in B from a to b which misses R. Because $\bigcup A_i$ contains all possible knots, there is a homeomorphism of B onto itself, fixed on the boundary, which takes $\bigcup A_i$ into a small neighborhood of A'. We may also assume that W is taken into the neighborhood and that the neighborhood misses R.

We now construct the slippery arc α as the intersection of nested compact sets X_i. Each X_i is the disjoint union of a countable number of 3-balls and points. We set $X_1 = B$, $X_2 = W$, and inductively define X_{n+1} as the union of all previous suspension points and the result of repeating the previous construction in each 3-ball of X_n; i.e., if $\tilde{B}$ is a 3-ball of X_n, we construct a set inside $\tilde{B}$ similar to the manner that W was constructed in B. If the diameters of the 3-balls of X_i approach zero as i gets large, then $\bigcap X_i = \alpha$ is an arc.

THEOREM 4.1. *The arc α can be slipped off every 1-dimensional compactum in E^3.*

Proof. Let $R\subset E^3$ be a 1-dimensional compactum and $\varepsilon>0$ be given. We choose i large enough so that the 3-balls of X_i are all $\frac{1}{2}\varepsilon$-small. Since R is nowhere dense in E^3, there is a $\frac{1}{2}\varepsilon$-homeomorphism h of E^3 onto itself which moves R off the suspension points of X_i. By repeated applications of Lemma 4.1 there is a homeomorphism f of E^3 onto itself, fixing points outside that 3-balls of X_i, which takes X_{i+1} off $h(R)$. The ε-homeomorphism $h^{-1}\circ f$ is the desired homeomorphism.

5. A slippery wild Cantor set in E^3. The Cantor set of Section 3 could not be slipped off every 1-dimensional compactum in E^3. We now modify the construction of Section 4 to obtain a wild Cantor set in E^3 which can be slipped off every 1-dimensional compactum. The Cantor set is wild because of its similarity with Antoine's necklace. The Cantor set can be slipped off 1-dimensional compacta because of its similarity with the arc of Section 4. We give only the construction but no proofs. However, we will give proofs later when we show how to generalize this construction in E^n $(n\geqslant 3)$.

If T is a pinched solid torus with pinchpoint p, we let g be a map of the 3-ball B of Section 4 onto T so that $g|B-\{a, b\}$ is a homeomorphism onto $T-\{p\}$ and $g(a) = g(b) = p$. The simple closed curve $\{p\}\cup g(\bigcup A_i)$ is the core for a pinched solid torus T' in T with pinchpoint p. We say that T' is placed in T by the *slippery construction*.

Once again, if T is a pinched solid torus with pinchpoint p, we construct a countably infinite collection of disjoint pinched solid tori in E^3 which follow the core of T and converge down to p. The cores of these pinched solid tori are indicated in Figure 2. We say the collection of pinched solid tori has been placed in T using the *Antoine construction*.

We now define the slippery wild Cantor set $Q\subset E^3$ as the intersection of nested compact sets T_i. Each T_i is the countable union of disjoint pinched solid tori and points. We let T_1 be any pinched solid torus in E^3. We define T_{i+1} inductively. If i is odd, we let T_{i+1} be the union of all previous pinchpoints plus the result of placing pinched solid tori inside the pinched solid tori of T_i by the slippery construction. If i is even, T_{i+1} is the union of all previous pinchpoints plus the result of placing pinched solid tori in the pinched solid tori of T_i using the Antoine construction.

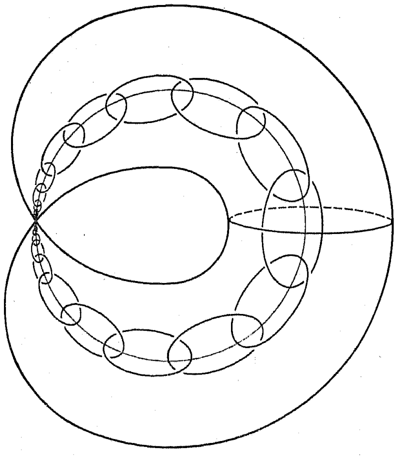

Fig. 2

If care is taken to insure that the diameters of the pinched solid tori go to zero as i gets large, then Q is a Cantor set.

6. Some generalizations. In this section we make the appropriate generalizations of our previous results to aid in the construction of slippery wild Cantor sets in E^n ($n \geqslant 3$). We used earlier, without proof, the fact that a 1-dimensional compactum R in B^3 cannot separate two points in the boundary. In fact, if $a, b \in (\mathrm{Bd}\, B^3) - R$ there is a polygonal properly embedded arc in B^3 from a to b which misses R. We now note the generalization of this result.

LEMMA 6.1. *Suppose $I^n = I^2 \times I^{n-2}$ and X is a 1-dimensional compactum in $I^n - (I^2 \times \Sigma^{n-3})$. Then there is a compact orientable piecewise-linear manifold M in $I^n - X$ bounded by $\{0\} \times \Sigma^{n-3}$ which has an $\mathrm{Int}(I^2)$-product neighborhood $(\mathrm{Int}\, I^2) \times M$, the product structures on $\{0\} \times \Sigma^{n-3}$ in $(\mathrm{Int}\, I^2) \times M$ and I^n agreeing.*

Proof. Consider the projection map $\pi\colon I^n \to I^2$. We first show the existence of a piecewise-linear map $p\colon I^n \to I^2$ such that $p(x) = \pi(x)$ if $x \in \mathrm{Bd}\, I^n$, $p(x) \in I^2 - \{0\}$ if $x \in X$, p is simplicial with respect to triangulations K and T of I^n and I^2 respectively, and 0 is the barycenter of a 2 simplex $\sigma \in T$. This is because $\pi | X$ can be approximated by two maps, the first taking $X - \mathrm{Bd}\, I^n$ into the (1-dimensional) nerve of a covering of $X - \mathrm{Bd}\, I^n$, the second taking the nerve into $I^2 - \{0\}$. This procedure defines $p|X$ and $p|\mathrm{Bd}\, I^n$. The Tietze extension theorem allows for the extension of p to all of I^n. The p we have constructed is now modified by a simplicial approximation theorem to obtain the remaining properties.

We let v_1, v_2, v_3 be the vertices of the 2-simplex σ. We now show that $M = p^{-1}(0)$ has a $\mathring{\sigma}$-product neighborhood $M \times \mathring{\sigma}$ in I^n. We define the homeomorphism $h\colon p^{-1}(\mathring{\sigma}) \to M \times \mathring{\sigma}$. Suppose $\tau \in K$ is a simplex and $p(\mathring{\tau}) = \mathring{\sigma}$. Then τ is the join of three faces τ_1, τ_2, τ_3, $f(\tau_i) = v_i$, $i = 1, 2, 3$. A point ω in $\mathring{\tau}$ can be written $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $x_i \in \tau_i$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. We define

$$h(\omega) = (\tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{3}x_3,\ \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3)\,.$$

The polyhedron M is a cell complex [10]. The cells of M are given by $\xi \cap M$ where ξ ranges over the simplexes of K. Suppose v is a vertex of M, then v is the barycenter of a 2-simplex $\gamma \in K$. Let Σ be the link of γ in K and $f\colon \gamma * \Sigma \to v * \Sigma$ be a simplicial map given by $f(\gamma) = v$ and $f|\Sigma =$ identity. Then f restricted to $M \cap (\gamma * \Sigma)$ is one-to-one and goes onto a neighborhood of v in the $(n-2)$-ball $v * \Sigma$. Hence, M is a manifold at its vertices. By uniqueness of links, M is an $(n-2)$-manifold. It is now easy to modify the product neighborhood so that the product structures on $\{0\} \times \Sigma^{n-3}$ in both I^n and the neighborhood of M agree.

The above lemma is essentially a classical result but was made known to us by R. D. Edwards who used a smooth map p and the transversality theorem to do the second part of the theorem.

We now generalize the slippery construction of Section 5. We let

$$I^n = I^2 \times I^{n-2} \subset J^2 \times J^{n-2} = J^n\,.$$

In I^n we choose a countable dense set D and let $\{M_i\}$ be the collection of all properly embedded, compact, piecewise-linear $(n-2)$-manifolds with $\mathrm{Bd}\, M_i = \{0\} \times \Sigma^{n-3}$ and the vertices of M_i contained in D. Furthermore, the M_i must have $\mathrm{Int}\, I^2$ product neighborhoods $(\mathrm{Int}\, I^2) \times M_i$, the product structures on $\{0\} \times \Sigma^{n-3}$ in both I^n and $(\mathrm{Int}\, I^2) \times M_i$ agreeing. It is clear that $\{M_i\}$ is countable and that if M satisfies all of the conditions imposed on the M_i with the exception of the condition on the vertices, then a homeomorphism of I^n, fixing the boundary, will take M onto some M_i.

Let σ_i be the $(n-2)$-simplexes of some triangulation of $\mathrm{Int}\, J^{n-2}$. We let h_i be piecewise-linear homeomorphisms of J^{n-2} onto itself, fixing the boundary, with $h_i(\sigma_i) = I^{n-2}$. We use h_i to construct a homeomorphism g_i of J^n onto itself, fixing the boundary, satysfying $g_i | I^2 \times J^{n-2} = \mathrm{id} \times h_i$. We let $\mathscr{L} = \bigcup g_i^{-1}(M_i) \cup$ $\cup (I^2 \times \mathrm{Bd}\, J^{n-2})$. The set $\mathscr{L} \cap \mathrm{Int}\, J^n$ is a piecewise-linear $(n-2)$-manifold which is flat in $\mathrm{Int}\, J^n$. The pair $(J^n, \mathscr{L})$ has the following important property.

LEMMA 6.2. *Suppose R is a 1-dimensional compactum which misses $I^2 \times \mathrm{Bd}\, J^{n-2}$. Then there is a homeomorphism of J^n onto itself, fixing the boundary, which takes $\mathscr{L}$ off R.*

Proof. We may assume that $R \cap I^2 \times (J^{n-2} - \mathrm{Int}\, I^{n-2}) = \emptyset$. Let M be an $(n-2)$-manifold in I^n which misses $R \cap I^n$ as promised by Lemma 6.1. We may assume that $M = M_i$ for some i. Hence $g_i(\mathscr{L}) \subset M_i \cup I^2 \times (J^{n-2} - \mathrm{Int}\, I^2)$ and $g_i(\mathscr{L}) \cap R = \emptyset$.

Let W be a P-manifold of dimension n with pinchpoint p. We let $f\colon J^n \to W$ be a map such that $f(J^2 \times \mathrm{Bd}\, J^{n-2}) = p$ and f restricted to the complement of $J^2 \times \mathrm{Bd}\, J^{n-2}$ is a homeomorphism onto $W - \{p\}$. We call $f(\mathscr{L})$ a *knotted strand* in W. The following lemma is an easy consequence of Lemma 6.2.

LEMMA 6.3. *Suppose K is a knotted strand in a P-manifold W and $R \subset W$ is a 1-dimensional compactum which misses the pinchpoint. Then there is a homeomorphism of W, fixing* $\mathrm{Bd}(W-\{p\})$ *which takes K off R.*

7. The Daverman-Edwards construction. Daverman and Edwards [5] have shown that a (topologically) flat embedding of a compact piecewise-linear $(n-2)$-dimensional manifold in E^n can be "approximated" by a Cantor set. The key to their argument is the following theorem which we state without proof.

THEOREM 7.1. *Suppose U is a neighborhood of a compact piecewise-linear $(n-2)$-dimensional manifold N which is flatly embedded in E^n $(n \geqslant 3)$. For $\varepsilon>0$ there exists a flatly embedded compact piecewise-linear manifold N' whose components are ε-small and such that a loop γ in E^n-U is null homotopic in E^n-N if and only if γ is null homotopic in E^n-N'.*

THEOREM 7.2 (Generalized Daverman–Edwards construction). *Suppose $X \subset E^n$ is compact, $U \subset E^n$ is open $(n \geqslant 3)$, $X-U$ is countable, and $U \cap X$ is a flatly embedded $(n-2)$-dimensional piecewise-linear manifold. Then for each $\varepsilon>0$ there is a compact set X' which satisfies*:

(1) $X'-U = X-U$,

(2) *for each loop γ in the complement of $X \cup U$, γ is null homotopic in E^n-X if and only if γ is null homotopic in E^n-X',*

(3) *$X' \cap U$ is a flatly embedded $(n-2)$-dimensional piecewise-linear manifold with compact ε-small components, and*

(4) *X' is the disjoint union of A and B. The set A is compact and can be pushed off the 2-skeleton of any triangulation of E^n by an ε-homeomorphism. The set B is a countable union of $(n-2)$-sphere components of $X' \cap U$.*

Proof. We assume familiarity with the Daverman–Edwards construction. Let K be a triangulation of $X \cap U$ with a fine mesh. The set X' is the union of $X-U$ and the result of applying the Daverman–Edwards construction locally to the triangulation K. The properties (1), (2), and (3) follow from the construction of Daverman and Edwards. Now, in that construction, small $(n-2)$-spheres of X' "approximate" the 0-skeleton of K, one for each vertex of K, and the remaining manifold components of X' "approximate" the dual $(n-3)$-skeleton of K. The set B is the union of those $(n-2)$-spheres approximating the 0-skeleton. The set A is the remainder of X'. Let L be the $(n-3)$-dimensional polyhedron which is the dual of the 0-skeleton of K. We set $\bar{L} = L \cup (X-U)$. If the mesh of K has been chosen fine enough, we can find $\frac{1}{2}\varepsilon$-homeomorphisms of E^n, fixed outside U, which move A as close as we like to $\bar{L}$. Hence, it will be sufficient to show that $\bar{L}$ can be slipped off the 2-skeleton of an arbitrary triangulation of E^n. But $\bar{L}-L$ can always be slipped off the 2-skeleton of an arbitrary triangulation. Furthermore, by [6] for $n = 4$ and [2] for $n \geqslant 5$ we may assume that L is a subpolyhedron of any triangulation of E^n and hence L can be slipped off the 2-skeleton by this assumption and general position. These two facts together imply that $\bar{L}$ can be slipped off the 2-skeleton of an arbitrary triangulation of E^n. We note that the argument given in [2] requires Miller's theorem [9] which was not known at the time. For our purposes we could have given an alternative proof for $n \geqslant 5$ which avoids the use of [2] by pushing $\bar{L}$ off the 0, 1, and 2-skeleta, respectively, using the piecewise-linear unknotting of ball pairs [14].

We now give a construction which we will alternate with the generalized Daverman–Edwards construction.

Suppose $X \subset E^n$ $(n \geqslant 3)$ is compact, $U \subset E^n$ is open, $U \cap X$ is a flatly embedded $(n-2)$-manifold, and each component of $X \cap U$ is an $(n-2)$-sphere. Let P be a set which contains exactly one point in each of the $(n-2)$-spheres and V be a thickening of $X \cap U$ relative to P which is contained in U. Let X' be the union of $X-U$ and the result of placing a knotted strand in each P-manifold component of V.

LEMMA 7.1. *If γ is a loop in E^n-U, then γ is null homotopic in E^n-X if and only if γ is null homotopic in E^n-X'.*

Proof. Let γ be a loop in E^n-U. If γ is null homotopic in E^n-X', there is a map $\hat{\gamma}$: $B^2 \to E^n$ such that $\hat{\gamma}|\mathrm{Bd}\,B^2 = \gamma$, $\hat{\gamma}(B^2) \cap X' = \emptyset$, and $\hat{\gamma}^{-1}(\mathrm{Bd}(V-P))$ is a finite collection of disjoint simple closed curves. Let D be a disk in B^2 bounded by a simple closed curve of this collection. Then $\hat{\gamma}(\mathrm{Bd}\,D)$ is contained in $\mathrm{Bd}\,W$ for some P-manifold component of V and $\hat{\gamma}(D)$ misses the knotted strand in W. Hence, by linking arguments [11, pp. 257–268] $\hat{\gamma}(\mathrm{Bd}\,D)$ is trivial in $\mathrm{Bd}\,W_i$. The map $\hat{\gamma}$ can now be modified on a small neighborhood of D by mapping D into $\mathrm{Bd}\,W$ and then using $\mathrm{Bd}\,W$ to push the image of the neighborhood of D to a side of $\mathrm{Bd}\,W$. Repeating the above process, we may assume $\hat{\gamma}(D) \cap \mathrm{Bd}(V-P) = \emptyset$ and, hence, $\hat{\gamma}(D) \cap X = \emptyset$. The proof of the other direction is trivial.

8. Slippery wild Cantor sets. We now construct slippery wild Cantor sets in E^n $(n \geqslant 3)$ as the intersection of nested compact sets M_i with cores C_i. Let $C_1 = S^{n-2} \subset E^n$, M_1 be a thickening of C_1, and γ be a loop in E^n-M_1 which links S^{n-2}. We define C_{n+1} and M_{n+1} inductively. If n is odd, C_{n+1} is the result of applying the generalized Daverman–Edwards construction to the compact set C_n with respect to the open set U which is the interior of M_n. Then M_{n+1} is a thickening of C_{n+1} relative to the set consisting of all points of $C_{n+1}-U$ plus exactly one point from each $(n-2)$-sphere promised by condition 4 of the generalized Daverman–Edwards construction. Notice, the set M_{n+1} has some P-manifolds as components. If n is even C_{n+1} is just C_n with the exception of the cores of the P-manifolds which are replaced by knotted strands, and M_{n+1} is just M_n with the exception of the P-manifolds which are replaced by thickening the knotted strands relative the pinchpoints. Using the generalized Daverman–Edwards construction to be sure that the components of M_i get small, $\bigcap M_i = X$ is a Cantor set. By Theorem 7.2 and Lemma 7.1, γ is not null homotopic in E^n-X. Therefore, X is wild.

THEOREM 8.1. *The Cantor set X is a slippery wild Cantor set.*

Proof. Let R be a 1-dimensional compactum in E^n and $\varepsilon>0$ be given. We pick an integer k so that the components of M_k are all $\frac{1}{2}\varepsilon$-small.

Let V_k be the union of the P-manifold components of M_k. Inductively we define V_{n+1} $(n \geqslant k)$ by taking all the P-manifold components of M_{n+1} which are not contained in the union of the previous V_j. The Cantor set $X' = \bigcap_{n=k}^{\infty}(M_n - \bigcup_{i=k}^{n} V_i)$ can be slipped off the $(n-2)$-skeleton of any triangulation of E^n by property 4 of the generalized Daverman–Edwards construction. Hence, X' is tame [13] and there is a $\frac{1}{2}\varepsilon$-homeomorphism h of E^n onto itself which takes X' plus the union of all the pinchpoints of the V_i off R. Therefore, for some n, $h(R) \cap (M_n - \bigcup_{i=k}^{n} V_i) = \emptyset$. Let g be a homeomorphism which is fixed outside $\bigcup_{i=k}^{n} V_i$ and moves $M_i \cap V_i$ off $h(R)$. This is accomplished by moving the knotted strand inside each component of V_i off $h(R)$, fixing the boundary, and then pulling the intersection of M_i with that component close to the image of the knotted strand. We now have $g(X) \cap h(R) = \emptyset$, and $h^{-1} \circ g$ is the desired homeomorphism.

References

[1] L. Antoine, *Sur l'homeomorphie de deux figures et de leurs voisanages*, J. Math. Pures Appl. 86 (1921), pp. 221–325.
[2] J. L. Bryant and C. L. Seebeck III, *Locally nice embeddings in codimension three*, Quart. J. of Math. (Oxford) (2) 19 (1968), pp. 257–274.
[3] R. J. Daverman, *On the absence of tame disks in certain wild cells*, Geometric Topology, Proceedings of the Geometric Topology Conference held at Park City, Utah, 1974 (edited by L. C. Glaser and T. B. Rushing), Springer-Verlag (New York), pp. 142–155.
[4] — *On the scarcity of tame disks in certain wild cells*, Fund. Math. 79 (1973), pp. 63–77.
[5] — and R. D. Edwards, in preparation.
[6] H. Gluck, *Unknotting S^1 in S^4*, Bull. Amer. Math. Soc. 69 (1963), pp. 91–94.
[7] D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, II, Trans. Amer. Math. Soc. 126 (1967), pp. 217–224.
[8] — and W. H. Row, *Tangled embeddings of one-dimensional continua*, Proc. Amer. Math. Soc. 22 (1969), pp.378–385.
[9] R. T. Miller, *Approximating codimension three embeddings*, Ann. of Math. 95 (2) (1972), pp. 406–416.
[10] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise Linear Topology*, Springer-Verlag, 1972.
[11] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Chelsea, New York 1947.
[12] M. A. Štan'ko, *The embedding of compacta in Euclidean space*, Mat. Sbornik 83 (125) (1970), pp. 234–255 (Math. USSR Sbornik 12 (1970), pp. 234–254).
[13] D. G. Wright, *Pushing a Cantor set off itself*, Houston J. Math 2 (1976), pp. 439–447.
[14] E. C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. 78 (1963), pp. 501–526.

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A fixed point principle for locally expansive multifunctions

by

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Abstract. Let (X, d) be a well-chained metric space and F a uniformly open multifunction in $X\times X$ with complete graph so that there exist $\alpha>0$ and an isotone φ: $[0, \alpha)\to[0, \infty)$ such that $\varphi(t)>t$ for $0<t<\alpha$ and $d(u, v)\geqslant\varphi[d(x, y)]$ whenever $d(x, y)<\alpha$, $u\in F(x)$, $v\in F(y)$. Then $p\in F(p)$ for some p. In particular, every locally expansive, open multifunction with closed graph on a compact, connected metric space has a fixed point.

1. Introduction. Let (X, d) be a metric space and f: $X\to X$. F is *expansive* on a set B if

(1) $$d(fx, fy)>d(x, y) \quad \text{for all } x, y \text{ in } B \text{ with } x\neq y.$$

f is *contractive* on B if (1) holds with the inequality reversed. f is a *local expansion* (*local contraction*) if every point in X has a neighborhood B on which f is expansive (resp., contractive).

We seek here a fixed point principle that will provide a common base for the following pair of dual theorems: Let (X, d) be a compact, connected metric space. (i) Every continuous, open, local expansion f on X has a fixed point. (ii) Every local contraction g on X has a fixed point. Theorem (i) generalizes a theorem of Rosenholtz [3] who proved (i) for local expansions with the condition $d(fx, fy)\geqslant\lambda d(x, y)$ for some $\lambda>1$ replacing the less stringent inequality in (1). Theorem (ii) is a variant of a theorem of Edelstein [1].

We can unite (i) and (ii) in a single theorem if we formulate it in terms of multifunctions (i.e. binary relations).

Let F be a subset of $X\times X$. Let $F(x)$ be the set of all y such that $(y, x)\in F$. Let $F(B)$ be the set of all y such that $(y, x)\in F$ for some x in B. F is *expansive* on B if $d(u, v)>d(x, y)$ whenever $x, y\in B$, $x\neq y$, $u\in F(x)$, and $v\in F(y)$.

The definition of local expansion is then the same as for single-valued mappings.

With $F=f$ in (i) and $F=g^{-1}$ in (ii), both (i) and (ii) are subsumed by