[10] J. Dieudonné, *Une generalization des espaces compacts*, J. Math. Pures et Appl. 23 (1944), pp. 65–76.

[11] R. D. Edwards, to appear.

[12] J. Eells and N. H. Kuiper, *Homotopy negligible subsets of infinite-dimensional manifolds*, Compositio Math. 21 (1969), pp. 155–164.

[13] N. Kroonenberg, *Characterization of finite-dimensional Z-sets*, Proc. Amer. Math. Soc. 43 (1974), pp. 421–427.

[14] C. Lacher, *Cell-like mappings I*, Pacific J. Math. 30 (1969), pp. 717–731.

[15] A. Marin and Y. M. Visetti, *A general proof of Bing's shrinkability criterion*, Proc. Amer. Math. Soc. 53 (1975), pp. 501–507.

[16] J. Mogilski, Ph. D. dissertation, Inst. of Math., Polish Acad. of Sci., 1978 (to appear).

[17] H. Toruńczyk, *An introduction to infinite-dimensional topology*, to appear.

[18] — *Concerning locally homotopy negligible sets and characterization of l_2-manifolds*, Fund. Math. 101 (1978), pp. 93–110.

[19] J. E. West, *Mapping Hilbert cube manifolds to* ANR'*s: a solution of a conjecture of Borsuk*, Ann. of Math. 106 (1977), pp. 1–18.

[20] — *Infinite products which are Hibert cubes*, Trans. Amer. Math. Soc. 150 (1970), pp. 1–25.

[21] M. Wojdysławski, *Retractes absolus et hyperspaces des continus*, Fund. Math. 32 (1939), pp. 184–192.

[22] R. Y. T. Wong, *A wild Cantor set in the Hilbert cube*, Pacific J. Math. 24 (1968), pp. 189–193.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Accepté par la Rédaction le 21. 4. 1977

Provability in arithmetic and a schema of Grzegorczyk

by

George Boolos (Cambridge, Mass.)

Abstract. S4Grz is the system that results when the schema

$$(((B \Rightarrow DA) \Rightarrow DA) \,\&\, ((-B \Rightarrow DA) \Rightarrow DA)) \Rightarrow DA$$

is added to the modal (propositional) logic S4. Let $\varnothing$ map sentence letters of modal logic to sentences in the language of P(eano) A(rithmetic). Define ${}^{\varnothing}A$, A a sentence of modal logic, by: ${}^{\varnothing}p = \varnothing(p)$; ${}^{\varnothing}(-A) = -({}^{\varnothing}A)$; ${}^{\varnothing}(A \,\&\, B) = ({}^{\varnothing}A \,\&\, {}^{\varnothing}B)$; ${}^{\varnothing}(DA) = (\mathrm{Bew}(\ulcorner {}^{\varnothing}A \urcorner) \,\&\, {}^{\varnothing}A)$, where $\mathrm{Bew}(x)$ is the standard provability predicate for PA and $\ulcorner S \urcorner$ is the numeral for the Gödel number of the sentence S.

THEOREM. *For all sentences A of modal logic, $\vdash_{\mathrm{S4Grz}} A$* iff *for all $\varnothing$*, $\vdash_{\mathrm{PA}} {}^{\varnothing}A$.

(This result was independently obtained by R. Goldblatt.)

We shall describe a connection between provability in PA (= Peano Arithmetic, classical first-order formal arithmetic with induction) and a system of (propositional) modal logic considered (1) by Grzegorczyk. We are interested in "readings" of the box (D) of modal logic that concern provability in PA. Accordingly, we let $\varnothing$ be a variable ranging over functions from the sentence letters of modal logic to sentences of PA and define the *provability translation* $A^{\varnothing}$ (under $\varnothing$) of a sentence A of modal logic as follows: if A is the sentence letter p, then $A^{\varnothing} = \varnothing(p)$; if $A = -B$, then $A^{\varnothing} = -(B^{\varnothing})$; if $A = (B \,\&\, C)$, then $A^{\varnothing} = (B^{\varnothing} \,\&\, C^{\varnothing})$ (and similarly for the other non-modal connectives); and if $A = DB$, then $A^{\varnothing} = \mathrm{Bew}(\ulcorner B^{\varnothing} \urcorner)$, where $\mathrm{Bew}(x)$ is the standard provability predicate for PA, and $\ulcorner S \urcorner$ is the numeral for the Gödel number of the sentence S of PA.

It is a well-known consequence of Gödel's incompleteness theorems and their proofs that not every provability translation of every theorem of the modal system S4 is a theorem of PA. For example, if $\varnothing(p)$ is the undecidable sentence S constructed by Gödel, then since $\vdash_{\mathrm{PA}}(S \leftrightarrow -\mathrm{Bew}(\ulcorner S \urcorner))$, $(Dp \to p)^{\varnothing}, = (\mathrm{Bew}(\ulcorner S \urcorner) \to S)$, is not a theorem of PA; and if $\varnothing(p) = 0 = 1$, then if $\vdash_{\mathrm{PA}}(Dp \to p)^{\varnothing}$ then $\vdash_{\mathrm{PA}}(\mathrm{Bew}(\ulcorner 0 = 1 \urcorner) \to 0 = 1)$, $\vdash_{\mathrm{PA}} -\mathrm{Bew}(\ulcorner 0 = 1 \urcorner)$, and (by the second incompleteness theorem) PA is inconsistent.

(1) In [2], p. 230.

Solovay has recently shown [2] that the sentences of modal logic all of whose provability translations are theorems of PA are precisely the theorems of a system of modal logic he calls G. The axioms of G are all tautologies and all instances of the schemata $D(A \to B) \to (DA \to DB)$, $DA \to DDA$, and $D(DA \to A) \to DA$; the rules of inference of G are modus ponens and necessitation. Of course, not $\vdash_G Dp \to p$.

An interesting question thus arises: is there a "natural" way to read the box of modal logic that has to do with provability in PA and on which all theorems of the familiar system S4 represent principles that are universally provable in PA?

Yes: if we define the *provability-truth translation* ${}^{\varnothing}A$ (of A under $\varnothing$) by setting ${}^{\varnothing}A = \varnothing(p)$ if A is the sentence letter p; ${}^{\varnothing}A = -({}^{\varnothing}B)$ if $A = -B$; ${}^{\varnothing}A = ({}^{\varnothing}B \,\&\, {}^{\varnothing}C)$ if $A = (B \,\&\, C)$; and ${}^{\varnothing}A = (\mathrm{Bew}(\ulcorner {}^{\varnothing}B \urcorner) \,\&\, {}^{\varnothing}B)$ if $A = DB$, then it is easy to see that for all sentences A of modal logic, if $\vdash_{S4} A$, then $\vdash_{PA} {}^{\varnothing}A$ for all $\varnothing$. But is the converse the case? And if not, what is the extension L of S4 such that $\vdash_L A$ iff $\vdash_{PA} {}^{\varnothing}A$ for all $\varnothing$? That is, what are the sentences of propositional modal logic all of whose provability-truth translations are theorems of PA?

Grzegorczyk studied [3] the modal system that results when the schema (Grz)

$$\big(((B \Rightarrow DA) \Rightarrow DA) \,\&\, ((-B \Rightarrow DA) \Rightarrow DA)\big) \Rightarrow DA$$

is added to S4. ($(C \Rightarrow D)$ may be regarded as abbreviating $D(C \to D)$.) We shall call this system S4Grz. Grzegorczyk showed that although S4Grz is stronger than S4 (but incomparable with S5), only the intuitionistic tautologies are theorems, when the intuitionistic connectives are interpreted in the usual way.

The aim of this note is to show that for all sentences A of modal logic, $\vdash_{S4Grz} A$ if and only if $\vdash_{PA} {}^{\varnothing}A$ for all $\varnothing$. Thus the sentences of modal logic all of whose provability-truth translations are theorems of PA are precisely the theorems of S4Grz.

The system, which we shall call S4Sob, obtained by adding to S4 the schema

(Sob) $$((A \Rightarrow DA) \Rightarrow A) \to A,$$

was studied [4] earlier by Sobociński. (Sobociński called the system S4Sob K1.1; Segerberg calls it S4Grz in his *Essay in Classical Modal Logic* [5]. Segerberg states, but does not show, that S4Grz and S4Sob are deductively equivalent. It is of interest to prove their equivalence, and to do so it clearly suffices to show that all instances of (Sob) are theorems of S4Grz and that all instances of (Grz) are theorems of S4Sob.

LEMMA 1. $\vdash_{S4Grz}((A \Rightarrow DA) \Rightarrow A) \to A$.

Proof. From $\vdash_{S4}(-A \Rightarrow A) \Rightarrow DA$ and $\vdash_{S4} DA \Rightarrow A$, we have

(1) $$\vdash_{S4}(-A \Rightarrow DA) \Rightarrow DA.$$

But then, since

(2) $$\vdash_{S4Grz}\big(((A \Rightarrow DA) \Rightarrow DA) \,\&\, ((-A \Rightarrow DA) \Rightarrow DA)\big) \Rightarrow DA,$$

(1) and (2) imply that

(3) $$\vdash_{S4Grz}((A \Rightarrow DA) \Rightarrow DA) \Rightarrow DA.$$

Now

(4) $$\vdash_{S4}(A \Rightarrow DA) \Rightarrow D(A \Rightarrow DA)$$

and

(5) $$\vdash_{S4}((A \Rightarrow DA) \Rightarrow A) \Rightarrow (D(A \Rightarrow DA) \Rightarrow DA).$$

From (4) and (5) we have

(6) $$\vdash_{S4}((A \Rightarrow DA) \Rightarrow A) \Rightarrow ((A \Rightarrow DA) \Rightarrow DA).$$

By (3) and (6), weakening $\Rightarrow$ to $\to$, we have

(7) $$\vdash_{S4Grz}((A \Rightarrow DA) \Rightarrow A) \to DA.$$

And since

(8) $$\vdash_{S4} DA \to A,$$

we have

(9) $$\vdash_{S4Grz}((A \Rightarrow DA) \Rightarrow A) \to A. \ \blacksquare$$

A *Kripke model* $\mathcal{M}$ is a triple $\langle W, R, P\rangle$ such that $W \neq \varnothing$, R is a binary relation on W, and P is a function that assigns a truth-value to each pair consisting of a member w (a "world") of W and a sentence letter. Truth at a world in a Kripke model is defined as usual, and validity in a model is, as always, truth at all worlds in the model. $\mathcal{M} \models_w A$ means A is true at w in $\mathcal{M}$.

A *partial ordering of* W is a transitive, antisymmetric relation on W that is reflexive on W.

LEMMA 2. *Let W be finite, R a partial ordering of W, and $\mathcal{M} = \langle W, R, P\rangle$. Then for all w in W and all modal sentences A, B.*

$$\mathcal{M} \models_w \big(((B \Rightarrow DA) \Rightarrow DA) \,\&\, ((-B \Rightarrow DA) \Rightarrow DA)\big) \Rightarrow DA.$$

Proof. Suppose that for some w in W, $\mathcal{M}$ not $\models_w$(Grz), i.e.,

$$\mathcal{M} \text{ not} \models_w \big(((B \Rightarrow DA) \Rightarrow DA) \,\&\, ((-B \Rightarrow DA) \Rightarrow DA)\big) \Rightarrow DA.$$

Then for some x such that wRx,

$$\mathcal{M} \models_x (B \Rightarrow DA) \Rightarrow DA, \quad \mathcal{M} \models_x (-B \Rightarrow DA) \Rightarrow DA \quad \text{and} \quad \mathcal{M} \text{ not} \models_x DA.$$

Since R is reflexive,

$$\mathcal{M} \models_x (B \Rightarrow DA) \to DA, \quad \mathcal{M} \models_x (-B \Rightarrow DA) \to DA, \quad \mathcal{M} \text{ not} \models_x B \Rightarrow DA,$$
$$\text{and} \quad \mathcal{M} \text{ not} \models_x -B \Rightarrow DA,$$

[2] Theorem 4.6 of [5]; cf. [1] also.
[3] Op. cit.
[4] In [4], p. 314, Sobociński formulates (Sob) with $\Rightarrow$ in place of $\to$.
[5] Cf. [3], pp. 168–169.

and so for some y' such that xRy' and some y'' such that xRy'',

$$\mathcal{M}\text{ not}\models_{y'}B\rightarrow DA,\quad \mathcal{M}\text{ not}\models_{y''}-B\rightarrow DA,\quad \mathcal{M}\models_{y'}B,\quad \mathcal{M}\text{ not}\models_{y'}DA,\quad \mathcal{M}\models_{y''}-B\quad\text{and}\quad \mathcal{M}\text{ not}\models_{y''}DA\,.$$

Thus $y' \neq y''$. Since R is transitive,

$$\mathcal{M}\models_{y'}\big(((B\Rightarrow DA)\Rightarrow DA)\,\&\,((-B\Rightarrow DA)\Rightarrow DA)\big)$$

and

$$\mathcal{M}\models_{y''}\big(((B\Rightarrow DA)\Rightarrow DA)\,\&\,((-B\Rightarrow DA)\Rightarrow DA)\big),$$

and thus $\mathcal{M}$ not $\models_{y'}$(Grz) and $\mathcal{M}$ not $\models_{y''}$(Grz). By the transitivity of R, wRy' and wRy''. But not both $w = y'$ and $w = y''$. Thus for some y, wRy, $w \neq y$, and $\mathcal{M}$ not $\models_y$(Grz).

Thus if $\mathcal{M}$ not $\models_w$(Grz) for some w in W, there exists a sequence $\{w_i\}_{i\in\omega}$ of elements of W such that w_iRw_{i+1} and $w_i \neq w_{i+1}$, and thus, since R is transitive and W is finite, for some i and some $j>i+1$, $w_iRw_{i+1}Rw_j$ and $w_i = w_j$, whence by the antisymmetry of R, $w_i = w_{i+1}$, contradiction. ∎

According to Theorem 3.2 of Chapter II of Segerberg's *Essay*, $\vdash_{S4Sob}A$ if and only if for all W, R, P, w with W finite and R a partial ordering of W, $\langle W, R, P\rangle \models_w A$. It thus follows from Lemma 2 that all instances of (Grz) are theorems of S4Sob. By Lemma 1, all instances of (Sob) are theorems of S4Grz. Thus for all modal sentences A, $\vdash_{S4Grz}A$ iff $\vdash_{S4Grz}A$, iff A is valid in all Kripke models $\langle W, R, P\rangle$, with W finite and R a partial ordering of W.

We define a mapping of modal sentences A to modal sentences ${}^hA$ as follows: if A is the sentence letter p, then ${}^hA = p$; if $A = -B$, then ${}^hA = -({}^hB)$; if $A = (B\,\&\,C)$, then ${}^hA = ({}^hB\,\&\,{}^hC)$; and if $A = DB$, then ${}^hA = (D^hB\,\&\,{}^hB)$.

A *strict partial ordering of* W is an irreflexive, transitive relation on W. The correspondence I defined by:

$$I(R) = R - \{\langle w, w\rangle \mid \langle w, w\rangle \in R\},$$

maps the set of partial orderings of W one-one onto the set of strict partial orderings of W.

LEMMA 3. *Let R be a partial ordering of W. Then $\langle W, R, P\rangle \models_w A$ iff*

$$\langle W, I(R), P\rangle \models_w {}^hA\,.$$

Proof. Induction on the complexity of A. The only slightly non-trivial case is that in which $A = DB$. So suppose $A = DB$. Then $\langle W, R, P\rangle \models_w A$ iff for all x such that wRx, $\langle W, R, P\rangle \models_x B$, iff for all x such that $wI(R)x$ or $w = x$, $\langle W, R, P\rangle \models_x B$, iff (by the induction hypothesis) for all x such that $wI(R)x$ or $w = x$, $\langle W, I(R), P\rangle \models_x {}^hB$, iff $\langle W, I(R), P\rangle \models_w D^hB$ and $\langle W, I(R), P\rangle \models_w {}^hB$, iff $\langle W, I(R), P\rangle \models_w (D^hB\,\&\,{}^hB)$, iff $\langle W, I(R), P\rangle \models_w {}^hA$. ∎

According to Theorem 2.2 of Chapter II of Segerberg's *Essay*, $\vdash_G A$ if and only if for all W, S, P, w with W finite and S a strict partial ordering of W, $\langle W, S, P\rangle \models_w A$. (Segerberg refers to the system G as K4W.) By Lemma 3 and the fact that if S is a strict partial ordering of W, then there is a partial ordering R of W such that $I(R) = S$, the completeness theorems for S4Grz and G quoted from Segerberg immediately imply

LEMMA 4. *For all modal sentences A, $\vdash_{S4Grz}A$ iff $\vdash_G {}^hA$.*

LEMMA 5. *For all modal sentences A, ${}^{\varnothing}A = ({}^hA)^{\varnothing}$.*

Proof. Again, all cases except for $A = DB$ are trivial. And if $A = DB$, then

$$\begin{aligned}{}^{\varnothing}A &= (\mathrm{Bew}(\ulcorner {}^{\varnothing}B\urcorner)\,\&\,{}^{\varnothing}B) \underset{\text{i.h.}}{=} (\mathrm{Bew}(\ulcorner({}^hB)^{\varnothing}\urcorner)\,\&\,({}^hB)^{\varnothing})\\ &= ((D^hB)^{\varnothing}\,\&\,({}^hB)^{\varnothing}) = (D^hB\,\&\,{}^hB)^{\varnothing} = ({}^hA)^{\varnothing}.\ \blacksquare\end{aligned}$$

Solovay's theorem on provability translations (6) is that $\vdash_G B$ iff $\vdash_{PA} B^{\varnothing}$ for all $\varnothing$. By Lemmas 4 and 5, then, $\vdash_{S4Grz}A$ iff $\vdash_G {}^hA$, iff $\vdash_{PA}({}^hA)^{\varnothing}$ for all $\varnothing$, iff $\vdash_{PA}{}^{\varnothing}A$ for all $\varnothing$. We have thus established.

THEOREM. *For all modal sentences A, $\vdash_{S4Grz}A$ iff $\vdash_{PA}{}^{\varnothing}A$ for all $\varnothing$.*

References

[1] G. Boolos, *The Unprovability of Consistency*, Cambridge 1979.

[2] A. Grzegorczyk, *Some relational systems and the associated topological spaces*, Fund. Math. 60 (1967), pp. 223–231.

[3] K. Segerberg, *An Essay in Classical Modal Logic*, Uppsala 1971.

[4] B. Sobociński, *Family ϰ of the non-Lewis modal systems*, Notre Dame J. Formal Logic 5 (1964), pp. 313–318.

[5] R. Solovay, *Provability Interpretations of Modal Logic*, Israel J. Math. 25 (1976), pp. 287–304.

DEPARTMENT OF LINGUISTICS AND PHILOSOPHY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, Massachusetts

Accepté par la Rédaction le 2. 5. 1977

(6) [5], op. cit.