Generalized Ehrenfeucht games

by

Martin Weese * (Berlin)

Abstract. Ehrenfeucht games give a necessary and sufficient condition for two structures to be elementary equivalent. Many generalizations of Ehrenfeucht games to nonelementary languages are known. In this article we show that for each language with Lindström quantifiers there exists a corresponding game. This game gives a sufficient condition for two structures to be equivalent with respect to the language with Lindström quantifiers. In case of monotone quantifiers we also get a necessary condition.

In 1954 Fraissé [6] developed an algebraic criterion for elementary equivalence of two structures. In 1961 Ehrenfeucht [4] formulated this criterion in game theoretic terms. Since that time this criterion received many generalizations to nonelementary languages, for instance to infinitary languages by Karp [11] and Benda [2], to languages with cardinality quantifiers by Lipner [18], Vinner [29], Brown [3] and Slomson [27], to languages with Malitz quantifier by Badger [1], to languages with Henkin quantifier by Krynicki [13], to monadic monotone quantifiers by Krawczyk and Krynicki [12] and Makowsky and Tulipani [23]; to higher order languages by Le Tourneau [15], Tenney [28] and Hauschild [9]; to topological languages by Ziegler [31], to positive and negative languages by Makowsky and Shelah [21], to stationary logic by Makowsky [20].

Hanf [8] gave a useful new formulation of the criterion. Lindström [17] showed, that the criterion is essential for characterization of elementary languages. The criterion was used to get decidability results, see for instance Läuchli and Leonard [14], Slomson [27], Tenney [28], Vinner [29], Weese [30].

In this article we show how the criterion can be generalized to languages with Lindström quantifiers.

§ 1. Notations. All languages we consider are languages with only relational and constant symbols as nonlogical symbols. L always denotes a first order language. Let $\mathscr{L}$ be any language. Then Form$(\mathscr{L})$ denotes the class of all formulas of $\mathscr{L}$ and Sent$(\mathscr{L})$ denotes the class of all sentences of $\mathscr{L}$. Mod$(\mathscr{L})$ denotes the class of all structures for $\mathscr{L}$. Suppose $\mathscr{L}$ has the relational symbols R_i $(i<\alpha)$ and constant symbols

* I like to thank Miss Ingrid Schiemann for finding out some mistakes in a first draft.

c_i $(i<\beta)$. Then $\tau = \langle\langle n_i\colon i<\alpha\rangle, \beta\rangle$ is the *type* of $\mathscr{L}$, if for each $i<\alpha$, n_i is the arity of R_i. The type of $\mathscr{L}$ is denoted by $\tau_{\mathscr{L}}$. If τ is any similarity type, then L_τ denotes the corresponding first order language L with $\tau_L = \tau$. The type $\tau = \langle\langle n_i\colon i<\alpha\rangle, \beta\rangle$ is *simple* if $\beta = 0$; then we also write $\langle n_i\colon i<\alpha\rangle$ instead of $\langle\langle n_i\colon i<\alpha\rangle, 0\rangle$.

A *sequence* $\bar a$ will be a function from an ordinal $\mathrm{lh}(\bar a)$, and its ith element will be $a_i = \bar a(i)$. We write $\bar a \in A$ if for each $i<\mathrm{lh}(\bar a)$, $\bar a(i) \in A$.

Let φ be any formula; then φ^0 denotes φ and φ^1 denotes $\neg\varphi$. When writing $\varphi(\bar x)$ we always mean that all free variables of $\varphi(\bar x)$ occur in $\bar x$.

Let $\mathfrak{A} \in \mathrm{Mod}(\mathscr{L})$, $\varphi(\bar x) \in \mathrm{Form}(\mathscr{L})$. Then $\mathrm{Rel}^{\mathfrak{A}}\varphi$ denotes the set

$$\{\bar a \in |\mathfrak{A}|\colon \mathfrak{A} \models \varphi(\bar a)\}\,.$$

$\bar x \neq \bar y$ stands for $\bigvee\limits_{i<\mathrm{lh}(\bar x)} x_i \neq y_i$.

§ 2. **Preliminaries.** In 1966 Lindström [16] introduced a very general concept of generalized quantifiers. Let L be any elementary language, $\tau = \langle n_0, \ldots, n_{m-1}\rangle$ a simple finite type. A *generalized* (*indetermined*) *quantifier* Q *of type* τ is a logical operation which binds some variables and makes a formula out of m other formulas. More precisely, if $\bar x_i$, $\bar w$ $(i<m, \mathrm{lh}(\bar x_i) = n_i)$ are sequences of distinct variables, $\varphi_i(\bar x_i, \bar w)$ $(i<m)$ are formulas, then

$$\chi \equiv Q\bar x_0 \ldots \bar x_{m-1}[\varphi_0, \ldots, \varphi_{m-1}]$$

is a formula. We also write $Q\bar x[\varphi_i\colon i<m]$ or $Q\bar x[\bar\varphi]$ for χ. The free variables of χ are those variables which are free in some φ_i $(i<m)$ and do not occur in $\bar x$; thus they are from $\bar w$. We set $\mathrm{lh}(Q) = \sum\limits_{i<m} n_i$ and $\mathrm{sc}\ Q = \mathrm{lh}(\bar x) = m$.

$L(Q)$ denotes the language that we get from L by adding the quantifier Q. Form $(L(Q))$ is the least class F such that

(i) all atomic formulas are in F;

(ii) if $\varphi, \psi \in F$, then $\varphi\wedge\psi$, $\neg\varphi \in F$;

(iii) if $\varphi \in F$, x is any variable, then $\exists x\varphi \in F$;

(iv) if $\bar x_i$ $(i<m)$, $\bar w$ are sequences of distinct variables, Q is of type $\langle \mathrm{lh}(\bar x_i)\colon i<m\rangle$, $\varphi_i(\bar x_i, \bar w) \in F$ for each $i<m$, then $Q\bar x[\bar\varphi] \in F$.

$\mathrm{Sent}(L(Q))$ is the set of all $\varphi \in \mathrm{Form}(L(Q))$ without free variables.

Now let $\boldsymbol{K}$ be a class of structures of type τ, closed under isomorphism, $\mathfrak{A} \in \mathrm{Mod}(L)$, $\bar a \in |\mathfrak{A}|$, $\mathrm{lh}(\bar a) = \mathrm{lh}(\bar w)$. Then we set

$$\mathfrak{A} \models_{\boldsymbol{K}} Q\bar x[\bar\varphi](\bar a) \quad\text{iff}\quad \langle|\mathfrak{A}|, R_0, \ldots, R_{m-1}\rangle \in \boldsymbol{K}\,,$$

where

$$R_i = \{\bar b\colon \mathfrak{A} \models_{\boldsymbol{K}} \varphi_i(\bar b, \bar a)\} \quad\text{for each } i<m\,.$$

Then $\boldsymbol{K}$ gives an interpretation for Q. A generalized quantifier Q with an interpretation $\boldsymbol{K}$ is denoted by $Q_{\boldsymbol{K}}$ and is called *determined.* If Q is a determined generalized quantifier, then the corresponding class of algebras giving the interpretation is denoted by $\boldsymbol{K}_Q$. We also write $\mathfrak{A} \models Q_{\boldsymbol{K}}\bar x[\bar\varphi]$ instead of $\mathfrak{A} \models_{\boldsymbol{K}} Q\bar x[\bar\varphi]$.

Let $\{Q_i\colon i\in I\}$ be a set of generalized indetermined quantifiers, $\mathscr{K} = \{\boldsymbol{K}_i\colon i\in I\}$ a family of interpretations. Then we define $L\{Q_i\colon i\in I\}$, $\mathrm{form}(L\{Q_i\colon i\in I\})$ and $\models_{\mathscr{K}}$ analogous to the above definitions. We make the following convention:

If $\{Q_i\colon i\in I\}$ is a set of generalized quantifiers, then for each $i\in I$, Q_i has the type $\langle n_{0,i}, \ldots, n_{m_i-1,i}\rangle$.

We give some examples of special generalized quantifiers:

1. Let $\tau = \langle 1\rangle$, $\boldsymbol{K}_1 = \{\langle A, P\rangle\colon P\subseteq A, P \neq \emptyset\}$. Then $Q_{\boldsymbol{K}_1}$ can be identified with $\exists$.

2. Let $\tau = \langle 1\rangle$, $\boldsymbol{K}_2^\alpha = \{\langle A, P\rangle\colon P\subseteq A, \mathrm{card}P \geqslant \aleph_\alpha\}$. Then $Q_{\boldsymbol{K}_2^\alpha}$ can be identified with Q_α.

3. Let $\tau = \langle 1, 1\rangle$, $\boldsymbol{K}_3 = \{\langle A, P_0, P_1\rangle\colon P_0\subseteq A,\ P_1\subseteq A,\ \mathrm{card}P_0 = \mathrm{card}P_1\}$. Then $Q_{\boldsymbol{K}_3}$ can be identified with the Härtig quantifier I (see [7]).

4. Let $\tau = \langle 1, 1\rangle$, $\boldsymbol{K}_4 = \{\langle A, P_0, P_1\rangle\colon P_0\subseteq A, P_1\subseteq A, \mathrm{card}P_0<\mathrm{card}P_1\}$. Then $Q_{\boldsymbol{K}_4}$ can be identified with the Rescher quantifier R (see [24]).

5. Let $\tau = \langle n\rangle$, $\boldsymbol{K}_5^{\alpha,n} = \{\langle A, P\rangle\colon P\subseteq A^n$, there is $S\subseteq A$ with $S^n\subseteq P$ and $\mathrm{card}\,S \geqslant \aleph_\alpha\}$. Then $Q_{\boldsymbol{K}_5^{\alpha,n}}$ can be identified with the Malitz quantifier Q^n_∞ (see [19]).

6. Let $\tau = \langle 2\rangle$, $\boldsymbol{K}_6 = \{\langle A, P\rangle\colon P$ is a well-ordering of $A\}$. $Q_{\boldsymbol{K}_6}$ is denoted by Q^{wo}.

7. Let $\tau = \langle 2\rangle$, $\boldsymbol{K}_7^\alpha = \{\langle A, P\rangle\colon P\subseteq A^2$, P is an equivalence with at least $\aleph_\alpha$ equivalence classes$\}$. The quantifier $Q_{\boldsymbol{K}_7^\alpha}$, denoted by Q^E_α, was first introduced by Feferman [5].

8. Let $\tau = \langle 2\rangle$, $\boldsymbol{K}_8^\alpha = \{\langle A, P\rangle\colon P\subseteq A^2$, P is a linear order of cofinality $\aleph_\alpha\}$. The quantifier $Q_{\boldsymbol{K}_8^\alpha}$, denoted by Q^c_α, was first introduced by Shelah [26].

9. Let $\tau = \langle m+n\rangle$, $\boldsymbol{K}_9^{m,n} = \{\langle A, P\rangle\colon P\subseteq A^{m+n}$, there is $Y\subseteq A^n$ such that $\mathrm{card}\,Y\leqslant\aleph_0$ and

$$\forall x_0, \ldots, x_{m-1}[\exists y_0 \ldots y_{n-1}P(\bar x, y) \rightarrow (\exists \bar y \in Y)P(\bar x, \bar y)]\,.$$

The quantifier $Q_{\boldsymbol{K}_9^{m,n}}$, denoted by $Q^{B_{n,m}}$, was introduced in [22].

10. Let $\tau = \langle 2\rangle$, $\boldsymbol{K}_{10} = \{\langle A, P\rangle\colon P\subseteq A^2$, P is a dense linear ordering with a countable dense subset$\}$. The quantifier $Q_{\boldsymbol{K}_{10}}$, denoted by Q^D, was introduced in [22].

11. Let $\tau = \langle 4\rangle$, $\boldsymbol{K}_{11} = \{\langle A, P\rangle\colon P\subseteq A^4$, there are $F\subseteq A^2$, $G\subseteq A^2$ such that F, G are functions from A in A and $F\times G\subseteq P\}$. $Q_{\boldsymbol{K}_{11}}$ can be identified with the Henkin quantifier Q_H (see [10]).

Let L be an elementary language, $\{Q_i\colon i\in I\}$ a set of generalized quantifiers. We define by induction a function $\mathrm{qr}\colon \mathrm{Form}(L\{Q_i\colon i\in I\}) \rightarrow \omega$ as follows:

$\mathrm{qr}\,\varphi = 0$, if φ is atomic;

$\mathrm{qr}\,\neg\varphi = \mathrm{qr}\,\varphi$;

$\mathrm{qr}\,\varphi\wedge\psi = \max\{\mathrm{qr}\,\varphi, \mathrm{qr}\,\psi\}$;

$\mathrm{qr}\,\exists x\varphi = \mathrm{qr}\,\varphi + 1$;

$\mathrm{qr}\,Q\bar x[\bar\varphi] = \max\{\mathrm{qr}\,\varphi_i\colon i<\mathrm{lh}(\bar\varphi)\}+1$.

We define an equivalence relation $\underset{e}{=}$ on $\mathrm{Form}(L\{Q_i\colon i\in I\})$ as follows:

$\varphi(\bar x)\underset{e}{=}\psi(\bar x)$ iff for any $\mathfrak{A}\in\mathrm{Mod}(L)$, any interpretation $\mathscr{K}$ of $\{Q_i\colon i\in I\}$ and any $\bar a\in|\mathfrak{A}|$ with $\mathrm{lh}(\bar a)=\mathrm{lh}(\bar x)$, $\mathfrak{A}\models_{\mathscr{K}}\varphi(\bar a)$ iff $\mathfrak{A}\models_{\mathscr{K}}\psi(\bar a)$.

Let L be a language with only a finite number of relational and constant symbols and let m and n be natural numbers. Then it is easy to see that there is a finite set $\Phi\subseteq\mathrm{Form}(L\{Q_i\colon i\in I\})$ such that for any $\varphi\in\mathrm{Form}(L\{Q_i\colon i\in I\})$ with $\mathrm{qr}\,\varphi<m$ and the free variables of φ are among $\{v_i\colon i<n\}$, there is $\psi\in\Phi$ with $\varphi\underset{e}{=}\psi$. Further on it is also easy to see that there is a recursive function $\varepsilon\colon {}^2\omega\to\omega$ such that for each m,n there is a set Φ with the above properties and card $\Phi\leqslant\varepsilon(m,n)$.

§ 3. **Main results.** Let L be any elementary language, $\mathfrak{A},\mathfrak{B}\in\mathrm{Mod}(L)$, $\{Q_i\colon i\in I\}$ a set of generalized quantifiers, $\mathscr{K}_1=\{K_i^1\colon i\in I\}$, $\mathscr{K}_2=\{K_i^2\colon i\in I\}$ interpretations of $\{Q_i\colon i\in I\}$, n a natural number. We define

(1) $\mathfrak{A}^{\mathscr{K}_1}\equiv_n{}^{\mathscr{K}_2}\mathfrak{B}$ if for any $\varphi\in\mathrm{Sent}(L\{Q_i\colon i\in I\})$ with $\mathrm{qr}\,\varphi\leqslant n$, $\mathfrak{A}\models_{\mathscr{K}_1}\varphi$ iff $\mathfrak{B}\models_{\mathscr{K}_2}\varphi$;

(2) $\mathfrak{A}^{\mathscr{K}_1}\sim_n{}^{\mathscr{K}_2}\mathfrak{B}$ if there is a sequence Γ with $\mathrm{lh}(\Gamma)=n+1$ such that for any $i\leqslant n$,

(i) Γ_i is a set of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ and $\varnothing\in\Gamma_0$;

(ii) if $i<n$, $\mathscr{C}\in\Gamma_i$, $j\in I$, $\mathfrak{R}_1\in K_j^1$, $|\mathfrak{R}_1|=|\mathfrak{A}|$, then there exists $\mathfrak{R}_2\in K_j^2$ with: $|\mathfrak{R}_2|=|\mathfrak{B}|$, for each $l<m_j$ ($=\mathrm{sc}\,Q_j$) and each $\bar d\in|\mathfrak{B}|$ with $\mathrm{lh}(\bar d)=n_{l,j}$ there exists $\bar c\in|\mathfrak{A}|$ with $\mathrm{lh}(\bar c)=\mathrm{lh}(\bar d)$ such that $\bar c\in R_{l,j}^{\mathfrak{R}_1}$ iff $\bar d\in R_{l,j}^{\mathfrak{R}_2}$ and $\mathscr{C}\cup\{(c_m,d_m)\colon m<n_{l,j}\}\in\Gamma_{i+1}$;

(iii) if $i<n$, $\mathscr{C}\in\Gamma_i$, $j\in I$, $\mathfrak{R}_2\in K_j^2$, $|\mathfrak{R}_2|=|\mathfrak{B}|$, then there exists $\mathfrak{R}_1\in K_j^1$ with: $|\mathfrak{R}_1|=|\mathfrak{A}|$, for each $l<m_j$ and each $\bar c\in|\mathfrak{A}|$ with $\mathrm{lh}(\bar c)=n_{k,j}$ there exists $\bar d\in|\mathfrak{B}|$ with $\mathrm{lh}(\bar d)=\mathrm{lh}(\bar c)$ such that $\bar c\in R_{l,j}^{\mathfrak{R}_1}$ iff $\bar d\in R_{l,j}^{\mathfrak{R}_2}$ and $\mathscr{C}\cup\{(c_m,d_m)\colon m<n_{l,j}\}\in\Gamma_{i+1}$;

(iv) if $i<n$, $\mathscr{C}\in\Gamma_i$, $c\in|\mathfrak{A}|$, then there exists $d\in|\mathfrak{B}|$ such that $\mathscr{C}\cup\{(c,d)\}\in\Gamma_{i+1}$;

(v) if $i<n$, $\mathscr{C}\in\Gamma_i$, $d\in|\mathfrak{B}|$, then there exists $c\in|\mathfrak{A}|$ such that $\mathscr{C}\cup\{(c,d)\}\in\Gamma_{i+1}$.

(3) $\mathfrak{A}^{\mathscr{K}_1}\underset{w}{\sim}_n{}^{\mathscr{K}_2}\mathfrak{B}$ if there is a sequence Γ with $\mathrm{lh}(\Gamma)=n+1$ such that for any $i<n$,

(i) Γ_i is a set of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ and $\varnothing\in\Gamma_0$;

(ii) if $i<n$, $\mathscr{C}\in\Gamma_i$, $j\in I$, $\mathfrak{R}_1\in K_j^1$, $|\mathfrak{R}_1|=|\mathfrak{A}|$ then there exists $\mathfrak{R}_2\in K_j^2$ with: $|\mathfrak{R}_2|=|\mathfrak{B}|$, for each $l<m_j$ and each $\bar d\in|\mathfrak{B}|$ with $\mathrm{lh}(\bar d)=n_{l,j}$ and $\bar d\in R_{l,j}^{\mathfrak{R}_2}$ there exists $\bar c\in|\mathfrak{A}|$ with $\mathrm{lh}(\bar c)=\mathrm{lh}(\bar d)$, $\bar c\in R_{l,j}^{\mathfrak{R}_1}$ and $\mathscr{C}\cup\{(c_m,d_m)\colon m<n_{l,j}\}\in\Gamma_{i+1}$;

(iii) if $i<n$, $\mathscr{C}\in\Gamma_i$, $j\in I$, $\mathfrak{R}_2\in K_j^2$, $|\mathfrak{R}_2|=|\mathfrak{B}|$, then there exists $\mathfrak{R}_1\in K_j^1$ with: $|\mathfrak{R}_1|=|\mathfrak{A}|$, for each $l<m_j$ and each $\bar c\in|\mathfrak{A}|$ with $\mathrm{lh}(\bar c)=n_{l,j}$ and $\bar c\in R_{l,j}^{\mathfrak{R}_1}$ there exists $\bar d\in|\mathfrak{B}|$ with $\mathrm{lh}(\bar d)=\mathrm{lh}(\bar c)$, $\bar d\in R_{l,j}^{\mathfrak{R}_2}$ and $\mathscr{C}\cup\{(c_m,d_m)\colon m<n_{l,j}\}\in\Gamma_{i+1}$;

(iv) if $i<n$, $\mathscr{C}\in\Gamma_i$, $c\in|\mathfrak{A}|$, then there exists $d\in|\mathfrak{B}|$ such that $\mathscr{C}\cup\{(c,d)\}\in\Gamma_{i+1}$;

(v) if $i<n$, $\mathscr{C}\in\Gamma_i$, $d\in|\mathfrak{B}|$, then there exists $c\in|\mathfrak{A}|$ such that $\mathscr{C}\cup\{(c,d)\}\in\Gamma_{i+1}$.

We write

$$\mathfrak{A}^{\mathscr{K}_1}\equiv{}^{\mathscr{K}_2}\mathfrak{B}\qquad(\mathfrak{A}^{\mathscr{K}_1}\sim{}^{\mathscr{K}_2}\mathfrak{B},\ \mathfrak{A}^{\mathscr{K}_1}\underset{w}{\sim}{}^{\mathscr{K}_2}\mathfrak{B}\text{ respectively})$$

if

$$\mathfrak{A}^{\mathscr{K}_1}\equiv_n{}^{\mathscr{K}_2}\mathfrak{B}\qquad(\mathfrak{A}^{\mathscr{K}_1}\sim_n{}^{\mathscr{K}_2}\mathfrak{B},\ \mathfrak{A}^{\mathscr{K}_1}\underset{w}{\sim}_n{}^{\mathscr{K}_2}\mathfrak{B}\text{ respectively})\quad\text{for any } n<\omega.$$

THEOREM 1. *Let L be any elementary language, $\mathfrak{A},\mathfrak{B}\in\mathrm{Mod}(L)$, $\{Q_i\colon i\in I\}$ a set of generalized quantifiers, $\mathscr{K}_1$, $\mathscr{K}_2$ interpretations of $\{Q_i\colon i\in I\}$. Then, for any n, if*

$$\mathfrak{A}^{\mathscr{K}_1}\sim_n{}^{\mathscr{K}_2}\mathfrak{B},\quad\textit{then}\quad\mathfrak{A}^{\mathscr{K}_1}\equiv_n{}^{\mathscr{K}_2}\mathfrak{B}.$$

Proof. We assume without loss of generality that $\exists\in\{Q_i\colon i\in I\}$ and is standard interpreted. Let Γ be as in (2). We show:

Claim. For each $k\leqslant n$, $\mathscr{C}=\{(a_l,b_l)\colon l<\mathrm{lh}(\mathscr{C})\}$ with $\mathscr{C}\in\Gamma_{n-k}$, we have

$$\langle\mathfrak{A},\bar a\rangle^{\mathscr{K}_1}\equiv_k{}^{\mathscr{K}_2}\langle\mathfrak{B},\bar b\rangle$$

with

$$\bar a=\langle a_i\colon i<\mathrm{lh}(\mathscr{C})\rangle,\qquad\bar b=\langle b_i\colon i<\mathrm{lh}(\mathscr{C})\rangle.$$

Let $k=0$. Then the claim follows immediately from the definition of Γ. Suppose the claim is proved for some $k<n$. We show that the claim is true for $k+1$ too. Suppose not. Then there are

$$\mathscr{C}=\{(a_l,b_l)\colon l<\mathrm{lh}(\mathscr{C})\}\in\Gamma_{n-k-1}\quad\text{and}\quad\chi(w_0,\ldots,w_{\mathrm{lh}(\mathscr{C})-1}\in\mathrm{From}(L\{Q_i\colon i\in I\})$$

with $\mathrm{qr}\,\chi=k+1$ such that $\mathfrak{A}\models_{\mathscr{K}_1}\chi(\bar a)$ and $\mathfrak{B}\models_{\mathscr{K}_2}\neg\chi(\bar b)$. Without loss of generality we assume that $\chi\equiv Q\bar x[\bar\varphi]$ for some $Q\in\{Q_i\colon i\in I\}$. Let $\mathfrak{R}_1=\langle|\mathfrak{A}|,\bar R^1\rangle$ with $\mathrm{lh}(\bar R^1)=\mathrm{sc}\,Q$ and $R_i^1=\{\bar c\colon\mathfrak{A}\models\varphi_i(\bar c,\bar a)\}$ for any $i<\mathrm{sc}\,Q$. Then $\mathfrak{R}_1\in K_a^1$. Let $\mathfrak{R}_2=\langle|\mathfrak{B}|,\bar R^2\rangle$ be as described in (2)(ii) and let $\mathfrak{R}_2^*=\langle|\mathfrak{B}|,\bar R^*\rangle$ with $R_i^*=\{\bar d\colon\mathfrak{B}\models\varphi_i(\bar d,\bar b)\}$ for any $i<\mathrm{sc}\,Q$. Then $\mathfrak{R}_2^*\notin K_Q^2$ and thus $\mathfrak{R}_2\not\cong\mathfrak{R}_2^*$. Thus there are $j<\mathrm{sc}\,Q$ and $\bar d\in|\mathfrak{B}|$ such that

(i) $\bar d\in R_j^2$ and $\bar d\notin R_j^*$ or

(ii) $\bar d\notin R_j^2$ and $\bar d\in R_j^*$.

In either of the two cases there cannot be a $\bar c\in|\mathfrak{A}|$ with $\bar c\in R_j^1$ iff $\bar d\in R_j^2$ and

$$\langle\mathfrak{A},\bar a^\frown\bar c\rangle^{\mathscr{K}_1}\equiv_k{}^{\mathscr{K}_2}\langle\mathfrak{B},\bar b^\frown\bar d\rangle.$$

Thus the theorem is proved.

Let Q be a determined quantifier. Q is *monotone* if K_Q satisfies the following condition:

if $\mathfrak{A}=\langle|\mathfrak{A}|,\bar R\rangle\in K_Q$, $\mathfrak{A}^*=\langle|\mathfrak{A}|,\bar R^*\rangle$ and $R_i\subseteq R_i^*$ for each $i<\mathrm{sc}\,Q$, then $\mathfrak{A}^*\in K_Q$.

When we have to do only with monotone quantifiers, then we can weaken the assumptions of Theorem 1:

THEOREM 2. *Let L be an elementary language, $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}(L)$, $\{Q_i\colon i \in I\}$ a set of generalized quantifiers, $\mathscr{K}_1$, $\mathscr{K}_2$ interpretations of $\{Q_i\colon i \in I\}$ such that each quantifier is monotone under its interpretation. Then for any n,*

$$\text{if} \quad \mathfrak{A}^{\mathscr{K}_1} \underset{w}{\sim}_n {}^{\mathscr{K}_2}\mathfrak{B}, \quad \text{then} \quad \mathfrak{A}^{\mathscr{K}_1} \equiv_n {}^{\mathscr{K}_2}\mathfrak{B}.$$

Proof. We assume without loss of generality that $\exists \in \{Q_i\colon i \in I\}$ and is standard interpreted. (Remember that $\exists$ is a monotone quantifier.)

Let Γ be as described in (3). We show:

Claim. For each $k \leqslant n$, $\mathscr{C} = \{(a_l, b_l)\colon l < \mathrm{lh}(\mathscr{C})\}$ with $\mathscr{C} \in \Gamma_{n-k}$, we have

$$\langle \mathfrak{A}, \bar{a}\rangle^{\mathscr{K}_1} \equiv_k {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}\rangle$$

with

$$\bar{a} = \langle a_i\colon i < \mathrm{lh}(\mathscr{C})\rangle, \qquad \bar{b} = \langle b_i\colon i < \mathrm{lh}(\mathscr{C})\rangle.$$

Let $k = 0$. Then the claim follows immediately from the definition of Γ. Suppose now that the claim is proved for some $k < n$. We show that the claim is true for $k+1$ too. Suppose not. Then there are

$$\mathscr{C} = \{(a_l, b_l)\colon l < \mathrm{lh}(\mathscr{C})\} \in \Gamma_{n-k-1} \quad \text{and} \quad \chi(w_0, \ldots, w_{\mathrm{lh}(\mathfrak{C})-1}) \in \mathrm{Form}(L\{Q_i\colon i \in I\})$$

with $\mathrm{qr}\,\chi = k+1$ such that

$$\mathfrak{A} \models_{\mathscr{K}_1} \chi(\bar{a}) \quad \text{and} \quad \mathfrak{B} \models_{\mathscr{K}_2} \neg\chi(\bar{b}).$$

Without loss of generality we assume that $\chi \equiv Q\bar{x}[\bar{\varphi}]$ for some $Q \in \{Q_i\colon i \in I\}$.

Let $\mathfrak{R}_1 = \langle |\mathfrak{A}|, \bar{R}^1\rangle$ with $R_i^1 = \{\bar{c}\colon \mathfrak{A} \models \varphi_i(\bar{c}, \bar{a})\}$ for each $i < \mathrm{sc}\,Q$. Then $\mathfrak{R}_1 \in \boldsymbol{K}_Q^1$. Let $\mathfrak{R}_2 = \langle |\mathfrak{B}|, \bar{R}^2\rangle$ be as described in (3)(ii) and let $\mathfrak{R}_2^* = \langle |\mathfrak{B}|, \bar{R}^*\rangle$ with $R_i^* = \{\bar{d}\colon \mathfrak{B} \models \varphi_i(\bar{d}, \bar{b})\}$ for each $i < \mathrm{sc}\,Q$. Then $\mathfrak{R}_2^* \notin \boldsymbol{K}_Q^2$. Now it follows from the monotony of Q that there are $j < \mathrm{sc}\,Q$ and $\bar{d} \in |\mathfrak{B}|$ such that $\bar{d} \in R_j^2$ and $\bar{d} \notin R_j^*$. Thus there cannot be $\bar{c} \in |\mathfrak{A}|$ with $\bar{c} \in R_j^1$ such that $\langle \mathfrak{A}, \bar{a}^\frown\bar{c}\rangle^{\mathscr{K}_1} \equiv_k {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}^\frown\bar{d}\rangle$ and the theorem is proved.

Now we are looking for conditions that are sufficient to prove the inverse of Theorem 1:

THEOREM 3. *Let L be an elementary language with only a finite number of relational and constant symbols, $\{Q_i\colon i \in I\}$ a finite set of quantifiers, $\mathscr{K}_1$, $\mathscr{K}_2$ interpretations of $\{Q_i\colon i \in I\}$ such that each quantifier is monotone. Then there is a recursive function $F\colon \omega \to \omega$ such that for each $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}(L)$ and each n,*

$$\text{if} \quad \mathfrak{A}^{\mathscr{K}_1} \equiv_{F(n)} {}^{\mathscr{K}_2}\mathfrak{B}, \quad \text{then} \quad \mathfrak{A}^{\mathscr{K}_1} \sim_n {}^{\mathscr{K}_2}\mathfrak{B}.$$

Proof. Let $l_0 = \max\{\mathrm{lh}\,Q_i\colon i \in I\}$ and let $\varepsilon\colon \omega^2 \to \omega$ be a recursive function with: for each m, n there is a set $\Phi \subseteq \mathrm{Form}(L\{Q_i\colon i \in I\})$ with $\mathrm{card}\,\Phi \leqslant \varepsilon(m, n)$ such that for each $\varphi \in \mathrm{Form}(L\{Q_i\colon i \in I\})$ with $\mathrm{qr}\,\varphi < m$ and the free variables among $\{v_0, \ldots, v_{n-1}\}$, there is $\psi \in \Phi$ with $\varphi \underset{e}{=} \psi$. To prove the theorem it is enough to show for any natural number l_1 the following:

if $\bar{a} \in |\mathfrak{A}|$, $\bar{b} \in |\mathfrak{B}|$, $\mathrm{lh}(\bar{a}) = \mathrm{lh}(\bar{b}) = l_1$,

$$k^* = k \cdot 2^{\varepsilon(k, l_0 + l_1)} \cdot 2^{l_0 \cdot l_0} + k + 1,$$

$$\langle \mathfrak{A}, \bar{a}\rangle^{\mathscr{K}_1} \equiv_{k^*} {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}\rangle,$$

then

(i) for each $Q \in \{Q_i\colon i \in I\}$, $\mathfrak{R}_1 \in \boldsymbol{K}_Q^1$ with $|\mathfrak{R}_1| = |\mathfrak{A}|$, there exists $\mathfrak{R}_2 \in \boldsymbol{K}_Q^2$ with $|\mathfrak{R}_2| = |\mathfrak{B}|$ such that for each $j_0 < \mathrm{sc}\,Q$ and each $\bar{d} \in |\mathfrak{B}|$ with $\mathrm{lh}(\bar{d}) = n_{j_0}$ there exists $\bar{c} \in |\mathfrak{A}|$ with $\mathrm{lh}(\bar{c}) = n_{j_0}$ such that $\bar{c} \in R_{j_0}^1$ iff $\bar{d} \in R_{j_0}^2$ and

$$\langle \mathfrak{A}, \bar{a}^\frown\bar{c}\rangle^{\mathscr{K}_1} \equiv_k {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}^\frown\bar{d}\rangle;$$

(ii) for each $Q \in \{Q_i\colon i \in I\}$, $\mathfrak{R}_2 \in \boldsymbol{K}_{Q^2}$ with $|\mathfrak{R}_2| = |\mathfrak{B}|$, there exists $\mathfrak{R}_1 \in \boldsymbol{K}_Q^1$ with $|\mathfrak{R}_1| = |\mathfrak{A}|$ such that for each $j_0 < \mathrm{sc}\,Q$ and each $\bar{c} \in |\mathfrak{A}|$ with $\mathrm{lh}(\bar{c}) = n_{j_0}$ there exists $\bar{d} \in |\mathfrak{B}|$ with $\mathrm{lh}(\bar{d}) = n_{j_0}$ such that $\bar{c} \in R_{j_0}^1$ iff $\bar{d} \in R_{j_0}^2$ and

$$\langle \mathfrak{A}, \bar{a}^\frown\bar{c}\rangle^{\mathscr{K}_1} \equiv_k {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}^\frown\bar{d}\rangle;$$

(iii) for each $c \in |\mathfrak{A}|$ there exists $d \in |\mathfrak{B}|$ such that

$$\langle \mathfrak{A}, \bar{a}^\frown c\rangle^{\mathscr{K}_1} \equiv_k {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}^\frown d\rangle;$$

(iv) for each $d \in |\mathfrak{B}|$ there exists $c \in |\mathfrak{A}|$ such that

$$\langle \mathfrak{A}, \bar{a}^\frown c\rangle^{\mathscr{K}_1} \equiv_k {}^{\mathscr{K}_2}\langle \mathfrak{B}, \bar{b}^\frown d\rangle.$$

We only show (i), the other cases are similar. For each l, let $G(l)$ be the set of all functions $g \in {}^{l \cdot l}2$ such that

(i) $g(i, j) = g(j, i)$ for all $i, j < l$;

(ii) $g(i, i) = 0$ for all $i < l$;

(iii) for all $i, j, k < 1$, if $g(i, j) = g(j, k) = 0$, then $g(i, k) = 0$.

Let $\mathrm{lh}(\bar{x}) = l$, $g \in G(l)$; we set

$$\ddot{A}q[g](\bar{x}) = \bigwedge_{i,j<l} (x_i = x_j)^{g(i,j)}.$$

Let $Q \in \{Q_i\colon i \in I\}$, $\mathfrak{R}_1 \in \boldsymbol{K}_Q^1$ with $|\mathfrak{R}_1| = |\mathfrak{A}|$. For each $i < m$ let $\Phi_i = \{\varphi_{i,j}\colon j < k_i\}$ be a set of formulas such that for each $\varphi \in \mathrm{Form}(L\{Q_i\colon i \in I\})$ with $\mathrm{qr}\,\varphi < k$, the free variables of φ are among $\{x_0, \ldots, x_{n_i-1}, w_0, \ldots, w_{l_i-1}\}$, there exists $\psi \in \Phi_i$ with $\varphi \underset{e}{=} \psi$. For each $f \in {}^{k_i}2$ let

$$\Phi[i, f](\bar{x}) = \bigwedge_{j<k_i} \varphi_{i,j}(\bar{x}, \bar{a})^{f(i)};$$

$$J_i = \{f \in {}^{k_i}2\colon R_i^1 \cap \mathrm{Rel}^{\mathfrak{A}}\Phi[i, f] \neq \emptyset\}$$

and

$$M_i(f) = \{g \in G(n_i)\colon \mathfrak{A} \models \exists\bar{x}(\ddot{A}q[g](\bar{x}) \wedge \neg R_i(\bar{x}) \wedge \Phi[i, f](\bar{x}))\}.$$

Let

$$\psi \equiv \bigwedge_{i<m} \bigwedge_{f \in J_i} \bigwedge_{g \in M_i(f)} \exists\bar{x}_{i,f,g}(\ddot{A}q[g](\bar{x}_{i,f,g})) \wedge \Phi[i, f](\bar{x}_{i,f,g} \wedge$$
$$\wedge\, Q\bar{y}[\bigvee_{f \in J_i} \Phi[i, f](\bar{x}_{i,f,g}) \vee \bigwedge_{f \in J_i} \bigwedge_{g \in M_i(f)} \bar{y}_i \neq \bar{x}_{i,f,g}\colon i < m]).$$

It follows from the monotony of Q that $\mathfrak{A} \models \psi$. An easy computation shows that $\mathrm{qr}\,\psi \leqslant k^*$. Thus $\mathfrak{B} \models \psi$. Now let $\bar{b}_{i,f,g}$ $(i<m,\ f \in J_i,\ g \in M_i(f))$ be such that

$$\mathfrak{B} \models \bigwedge_{i<m} \bigwedge_{f \in J_i} \bigwedge_{g \in M_i(f)} \ddot{A}q[g](\bar{b}_{i,f,g}) \wedge \Phi[i,f](\bar{b}_{i,f,g}) \wedge$$
$$\wedge Q\bar{y}[\bigvee_{f \in J_i} \Phi[i,f](\bar{y}_i) \wedge \bigwedge_{f \in J_i} \bigwedge_{g \in M_i(f)} \bar{y}_i \neq \bar{b}_{i,f,g}:\ i<m].$$

We set $\mathfrak{K}_2 = \langle |\mathfrak{B}|, \bar{R}^* \rangle$ with

$$R_i^* = \{\bar{d} \in |\mathfrak{B}|:\ \mathfrak{B} \models \bigvee_{f \in J_i} \Phi[i,f](\bar{d}) \wedge \bigwedge_{f \in J_i} \bigwedge_{g \in M_i(f)} \bar{d} \neq \bar{b}_{i,f,g}\}$$

for each $i<m$. Then it follows immediately from the definition of ψ, that for each $i<m$ and each $\bar{d} \in |\mathfrak{B}|$ with $\mathrm{lh}(\bar{d}) = n_i$, there is $\bar{c} \in |\mathfrak{A}|$ with $\mathrm{lh}(\bar{c}) = n_i$ such that $\bar{c} \in R_i^1$ iff $\bar{d} \in R_i^*$ and

$$\langle \mathfrak{A}, \bar{a}^\frown \bar{c} \rangle^{\mathcal{K}_1} \equiv_k {}^{\mathcal{K}_2}\langle \mathfrak{B}, \bar{b}^\frown \bar{d} \rangle .$$

Thus the theorem is proved.

§ 4. **Applications.** It is possible to formulate the results of § 3 in game theoretic terms. Let L be an elementary language, $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}(L)$, $\{Q_i:\ i \in I\}$ a set of generalized quantifiers, $\mathcal{K}_1$, $\mathcal{K}_2$ interpretations of $\{Q_i:\ i \in I\}$. Then for any natural number n, the game $G(n, \mathfrak{A}, \mathfrak{B}, \mathcal{K}_1, \mathcal{K}_2)$ is a game with two players, player I and player II and n moves. The ith move goes as follows: player I begins and at first decides whether it will be an $\exists$-move or a Q-move. In case of an $\exists$-move player I chooses one of the two structures, for instance $\mathfrak{A}$, and in this structure an element a_i. After that player II chooses the other structure, in our case $\mathfrak{B}$, and in this structure an element b_i. Now the ith move is finished; the set $\mathfrak{Z}_i = \{(a_i, b_i)\}$ is the result of the ith move. In case of a Q-move player I chooses some $j \in I$, one of the two structures, for instance $\mathfrak{B}$, and a structure $\mathfrak{K}_2 \in K_{Q_j}^2$ with $|\mathfrak{K}_2| = |\mathfrak{B}|$. Now player II chooses the other structure, in our case $\mathfrak{A}$, and a structure $\mathfrak{K}_1 \in K_{Q_j}^1$ with $|\mathfrak{K}_1| = |\mathfrak{A}|$. After that player I chooses some $k<m_j$ and $\bar{a} \in |\mathfrak{A}|$ with $\mathrm{lh}(\bar{a}) = n_{k,j}$. That means, player I is now in the structure which was chosen by player II. At least player II chooses $\bar{b} \in |\mathfrak{B}|$ with $\mathrm{lh}(\bar{b}) = n_{k,j}$ such that $\bar{a} \in R_{k,j}^1$ iff $\bar{b} \in R_{k,j}^2$. Now the ith move is finished; the set $\mathfrak{Z}_i = \{(a_l, b_l):\ l<n_{k,j}\}$ is the result of the ith move. After the $(n-1)$-th move the game is finished.

The game $H(n, \mathfrak{A}, \mathfrak{B}, \mathcal{K}_1, \mathcal{K}_2)$ has the same rules as the game

$$G(n, \mathfrak{A}, \mathfrak{B}, \mathcal{K}_1, \mathcal{K}_2)$$

with the only distinction that the sequences $\bar{a}$ and $\bar{b}$ have to be such that $\bar{a} \in R_{k,j}^1$ and $\bar{b} \in R_{k,j}^2$.

Player II wins the game if $\bigcup_{i<n} \mathfrak{Z}_i$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. If player II can always win the game $G(n, \mathfrak{A}, \mathfrak{B}, \mathcal{K}_1, \mathcal{K}_2)$ (the game $H(n, \mathfrak{A}, \mathfrak{B}, \mathcal{K}_1, \mathcal{K}_2)$), then we write

$$\mathfrak{A}^{\mathcal{K}_1} \underset{G}{\simeq}_n {}^{\mathcal{K}_2}\mathfrak{B} \quad (\mathfrak{A}^{\mathcal{K}_1} \underset{H}{\simeq}_n {}^{\mathcal{K}_2}\mathfrak{B}) .$$

Now we have immediately:

THEOREM 4. *Let L be an elementary language, $\mathfrak{A}, \mathfrak{B} \in \mathrm{Mod}(L)$, $\{Q_i:\ i \in I\}$ a set of generalized quantifiers, $\mathcal{K}_1$, $\mathcal{K}_2$ interpretations of $\{Q_i:\ i \in I\}$, n a natural number. Then*

(i) $\mathfrak{A}^{\mathcal{K}_1} \sim_n {}^{\mathcal{K}_2}\mathfrak{B}$ *iff* $\mathfrak{A}^{\mathcal{K}_1} \underset{G}{\simeq}_n {}^{\mathcal{K}_2}\mathfrak{B}$;

(ii) $\mathfrak{A}^{\mathcal{K}_1} \underset{w}{\simeq}_n {}^{\mathcal{K}_2}\mathfrak{B}$ *iff* $\mathfrak{A}^{\mathcal{K}_1} \underset{H}{\simeq}_n {}^{\mathcal{K}_2}\mathfrak{B}$.

The H-games can be applied to the quantifiers Q_α, Q_α^n and Q_H. In [18] Lipner describes the corresponding game for Q_α and in [1] Badger describes the corresponding game for Q_α^n. In [12] a game is described for a quantifier closely related to the Rescher quantifier R.

Let L be an elementary language, $\mathfrak{A} \in \mathrm{Mod}(L)$, $\{Q_i:\ i \in I\}$ a set of generalized quantifiers, $\mathcal{K}$ an interpretation of $\{Q_i:\ i \in I\}$. Let $\mathfrak{B} = \langle |\mathfrak{A}|, \bar{R} \rangle$ be a structure with finite $\mathrm{lh}(\bar{R})$. $\mathfrak{B}$ is *definable in* $\mathfrak{A}$ *with respect to* $\mathcal{K}$, if for each $j<\mathrm{lh}(\bar{R})$ there are $\bar{a} \in |\mathfrak{A}|$ and $\varphi(\bar{x}, \bar{w}) \in \mathrm{Form}(L\{Q_i:\ i \in I\})$ such that

$$R_i = \{\bar{b} \in |\mathfrak{A}|:\ \mathfrak{A} \models \varphi(\bar{b}, \bar{a})\} .$$

Let $\mathcal{D}_{\mathcal{K}}(\mathfrak{A})$ be the set of all structures $\mathfrak{B}$ which are definable in $\mathfrak{A}$ with respect to $\mathcal{K}$. It is easy to see that Theorem 4 remains valid if we replace in the games the classes $K_{Q_j}^1$, $K_{Q_j}^2$ $(j \in I)$ by any classes $K_{Q_j}^{*1}$, $K_{Q_j}^{*2}$ with

$$K_{Q_j}^1 \cap \mathcal{D}_{\mathcal{K}_1}(\mathfrak{A}) \subseteq K_{Q_j}^{*1} \subseteq K_{Q_j}^1 \quad \text{and} \quad K_{Q_j}^2 \cap \mathcal{D}_{\mathcal{K}_2}(\mathfrak{A}) \subseteq K_{Q_j}^{*2} \subseteq K_{Q_j}^2$$

(see also [25]). Sometimes these restrictions are useful for applications.

It is strightforeward to generalize the results of the article to languages $L_{\varkappa,\lambda}(\{Q_i:\ i \in I\})$ (where $\bigwedge\Phi$, $\bigvee\Phi$ are allowed for sets of formulas Φ with card $\Phi<\varkappa$ and $\exists\bar{x}\varphi$, $\forall\bar{x}\varphi$ are allowed for sequences of quantifiers with $\mathrm{lh}(\bar{x})<\lambda$). The corresponding results for languages $L_{\varkappa,\lambda}$ are contained in [2] and [11].

References

[1] L. Badger, *The Malitz quantifier meets its Ehrenfeucht game*, Ph. D. Thesis, Colorado, 1975.
[2] M. Benda, *Reduced products and non-standard logics*, J. Symb. Logic 34 (1969), pp. 424–436.
[3] M. Brown, *Infinitary languages, generalized quantifiers and generalized products*, Ph. D. Thesis, Dartmouth, 1971.
[4] A. Ehrenfeucht, *An application of games to the completeness problem for formalized theories*, Fund. Math. 49 (1961), pp. 129–141.
[5] S. Feferman, *Applications of many-sorted interpolation theorems*, in: Proceedings of the Tarski Symposium, Proc. Symp. Pure Math. 25, Providence 1974, pp. 205–223.
[6] R. Fraïssé, *Sur quelques classifications des systèmes de relations*, Publ. Sci. Univ. Alger Ser. A1 (1954), pp. 35–182.

[7] K. Härtig, *Über einen Quantifikator mit zwei Wirkungsbereichen*, Colloque sur les Fondements des Mathématiques et leurs Applications, Tihany 1962, Paris 1965, pp. 31–36.
[8] W. Hanf, *Model theoretic methods in the study of elementary logic*, in: The theory of Models, Amsterdam 1965, pp. 132–145.
[9] K. Hauschild, *Über zwei Spiele und ihre Anwendungen in der Modelltheorie*, Wiss. Z. Humboldt-Univ. Berlin, Math.-Nath. R. 24 (1975), pp. 783–788.
[10] L. Henkin, *Some remarks on infinitely long formulas*, in: Infinitistic Methods, London 1961, pp. 167–183.
[11] C. Karp, *Finite quantifier equivalence*, in: The Theory of Models, Amsterdam 1965, pp. 407–412.
[12] A. Krawczyk, and M. Krynicki, *Ehrenfeucht games and generalized quantifiers*, in: Set Theory and Hierarchy Theory, A Memorial Tribute to Andrzej Mostowski, Bierutowice 1975, Springer Lecture Notes 537 (1976), pp. 145–152.
[13] M. Krynicki, *Henkin quantifier and decidability*, to appear in: Proceedings of the Symposium in Helsinki 1975.
[14] H. Läuchli, and J. Leonard, *On the elementary theory of linear order*, Fund. Math. 49 (1966), pp. 109–116.
[15] J. Le Tourneau, *Decision problems related to the concept of operation*, Ph. D. Thesis, Berkeley 1968.
[16] P. Lindström, *First order logic and generalized quantifiers*, Theoria 32 (1966), pp. 186–195.
[17] — *On extensions of elementary logic*, Theoria 35 (1969), pp. 1–11.
[18] L. Lipner, *Some aspects of generalized quantifiers*, Ph. D. Thesis, Berkeley, 1970.
[19] M. Magidor, and I. Malitz, *Compact extensions of $L(Q)$*, to appear in: Ann. Math. Logic.
[20] J. Makowsky, preprint, 1977.
[21] — and S. Shelah, *The theorems of Beth and Craig in abstract model theory*, preprint 1976.
[22] — — and J. Stavi, *Δ-logics and generalized quantifiers*, to appear in Ann. Math. Logic.
[23] — and S. Tulipani, *Some model theory for monotone quantifiers*, preprint 1976.
[24] N. Rescher, *Plurality quantification*, J. Symb. Logic 27 (1962), pp. 373–374.
[25] I. Schiemann, *Untersuchungen an Logiken mit Lindström-Quantoren*, Dissertation, Berlin 1977.
[26] S. Shelah, *Generalized quantifiers and compact logic*, Trans. Amer. Math. Soc. 204 (1975), pp. 342–364.
[27] A. Slomson, *Generalized quantifiers and well orderings*, Arch. Math. Logic 15 (1972), pp. 57–73.
[28] R. Tenney, *Second-order Ehrenfeucht games and the decidability of the second-order theory of an equivalence relation*, J. Austral. Math. Soc. 20 (1975), pp. 323–331.
[29] S. Vinner, *A generalization of Ehrenfeucht's game and some applications*, Israel J. Math. 12 (1972), pp. 279–298.
[30] M. Weese, *Entscheidbarkeit in speziellen uniformen Structuren bezüglich Sprachen mit Mächtigkeitsquantoren*, Z. Math. Logik Grundl. Math. 22 (1976), pp. 215–230.
[31] M. Ziegler, *A language for topological structures which satisfies a Lindström theorem*, Bull. Amer. Math. Soc. 82 (1976), pp. 568–570.

Accepté par la Rédaction le 13. 3. 1978

Dimension of non-normal spaces

by

Keiô Nagami (Matsuyama)

Abstract. Let X be a general topological space and $\dim X$ the covering dimension of X due to Katetov defined by means of finite cozero covers. If V is a cozero set of X, then $\dim V \leqslant \dim X$. If $\{V_i\}$ is a countable cozero cover of X, then $\dim X = \sup \dim V_i$. Several applications of the subset and sum theorems thus stated are also given.

0. Introduction. Let X be a topological space. Then $\dim X \leqslant n$ if each finite cozero cover of X is refined by a finite cozero cover of order $\leqslant n+1$. This definition of covering dimension for general topological spaces stems from Katětov [4] and coincides with the usual definition of covering dimension for normal spaces. There has been a great amount of studies for the dimension of normal spaces in many aspects. On the contrary we have only a few for non-normal case. Especially, concerning subset and sum theorems we have had nothing with the exception of those due to Katětov [4]. Sections 1 and 2 below constitute the body of the paper where subset and sum theorems for non-normal spaces will respectively be given. In Sections 3 and 4 we give product and inverse limiting theorems for non-normal or normal spaces which will refine known results. In this paper all spaces are non-empty topological spaces and maps are continuous.

1. Subset theorem.

1.1. THEOREM. *Let V be a cozero set of a space X. Then* $\dim V \leqslant \dim X$.

Proof. When $\dim X$ is infinite the inequality is clear. Consider the case when $\dim X = n$. Let $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ be an arbitrary finite cozero cover of V. It is to be noted that each U_α is cozero in X since V is cozero in X. Let f be an element of $C(X, I)$ with $V = \{x \in X: f(x) > 0\}$. Set

$$V_i = \{x \in X: f(x) > 1/i\}, \quad F_i = \{x \in X: f(x) \geqslant 1/i\}.$$

Then $V = \bigcup_{i=1}^{\infty} V_i$ and $V_i \subset F_i \subset V_{i+1}$ for each i. Set

$$\mathcal{W}_1 = \{W_{1\alpha} = U_\alpha \cup (X - F_2): \alpha \in A\}.$$

Then $\mathcal{W}_1$ is a finite cozero cover of X. Let $\mathcal{U}_1 = \{U_{1\alpha}: \alpha \in A\}$ be a cozero cover of X such that $U_{1\alpha} \subset W_{1\alpha}$ for each $\alpha \in A$ and order $\mathcal{U}_1 \leqslant n+1$. Set

$$\mathcal{W}_2 = \{W_{2\alpha} = (U_{1\alpha} \cap V_2) \cup (U_\alpha - F_1) \cup (X - F_3): \alpha \in A\}.$$