On homotopy types of 2-dimensional polyhedra

by

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Abstract. To every polyhedron of dimension $\leqslant 2$ a polyhedron with the same homotopy type and with an especially regular structure is assigned. The topological type of this last polyhedron is determined by a numerical scheme $\mathfrak{S}$ and it is shown that for every given scheme $\mathfrak{S}$ there exist polyhedra of dimension $\leqslant 2$ with $\mathfrak{S}$ as their scheme.

1. Introduction. The problem to classify all polyhedra of dimension $\leqslant 2$ from the point of view of their homotopy properties has been studied by many authors (see [2] and [4] and the there given bibliography). In those studies an important role of the fundamental group for this problem has been exhibited. In our approach we limit ourselves to very elementary geometric consideration with the aim to find for a given polyhedron of dimension $\leqslant 2$ a polyhedron of the same homotopy type (i.e. of the same shape), but with an especially regular structure. Thus our aim is closely related to the problem to find a reasonable representative of a given shape (compare [1], p. 357). We give also a numerical scheme which allows us to determine the topological type of each polyhedron which has a such regular structure.

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2. Notations. By *polyhedra* we understand here always polyhedra of dimension $\leqslant 2$, i.e. compacta for which there exists a finite *triangulation* T consisting of (curvilinear) simplexes of dimensions $\leqslant 2$. Thus T consists of *vertices*, *edges* (i.e. 1-dimensional simplexes) and *triangles* (i.e. 2-dimensional simplexes). An edge L is said to be an *n-edge* if there exist in T exactly n triangles adjacent to L. In particular, 0-edges will be said *free*, 1-edges — *sharp*, 2-edges — *smooth* and n-edges with $n>2$ will be said *edges of ramification*.

By a *subpolyhedron* P' of a polyhedron P we understand any set which is the union of some simplexes of any triangulation of P. In the sequel we need the following well known facts:

(2.1) *If P' is a subpolyhedron of a polyhedron P, then the quotient-space P/P' is a polyhedron.*

(2.2) *If P' is a subpolyhedron of a polyhedron P, then the closure of each component of $P\setminus P'$ is a polyhedron.*

By a *simple graph* of a triangulation T, we understand either a set consisting of only one vertex of T, or a set with trivial shape which is the union of some edges of T. Let us observe that

(2.3) *If T is a triangulation of a connected polyhedron, then there exists a simple graph of T containing all vertices of T.*

By a *bouquet* we understand here always a finite bouquet, i.e. a pointed continuum (X, c) which is the union of a finite number of subcontinua $X_1, \ldots, X_k$ such that $X_i \cap X_j = (c)$ for $i \neq j$. The continua $X_1, \ldots, X_k$ are said to be the *leaves* and the point c — the *center* of this bouquet. By a *degenerate bouquet*, we understand the singleton (c).

3. Surfaces. By surfaces we understand continua $S \neq \emptyset$ such that for every point $x \in S$ there exists a neighborhood of x in S homeomorphic to the plane E^2. It is well known that every surface is a polyhedron nad that two surfaces S and S' are homeomorphic if and only if

$$p_1(S) = p_1(S') \quad \text{and} \quad p_2(S) = p_2(S'),$$

where $p_1(X)$ and $p_2(X)$ denote the first and the second Betti numbers of a space X. Any non-negative integer can be the value of $p_1(S)$, but the value of $p_2(S)$ is either 1 (if S is orientable), or 0 (if S is non-orientable).

Let us recall the following well-known facts:

(3.1) *If D is a disk lying on a surface S, then there is a disk $D' \subset S$ containing D in its interior.*

(3.2) *If D and D' are two disks lying on two homeomorphic surfaces S and S' respectively, then for every homeomorphism $h\colon D \to D'$ there is a homeomorphism M $\hat{h}\colon S \to S'$ such that $\hat{h}(x) = h(x)$ for every point $x \in D$.*

(3.3) *Let $\vec{D}_1, \ldots, \vec{D}_k$ be oriented and disjoint one to another disks lying on a surface S. If S is non-orientable, assume that the orientations of $\vec{D}_i$ are arbitrary, but if S is orientable, assume that the orientations of $\vec{D}_i$ are all induced by a given orientation of S. In both cases (of non-orientable or orientable surface) there exists in S an oriented disk $\vec{D}$ containing all $\vec{D}_i$ in its interior and oriented in the same sense as all $\vec{D}_i$.*

Consider now a surface S, a point $c \in S$ and two oriented disks $\vec{D}_1$, $\vec{D}_2$ lying on S and containing c in their interiors. The orientation of $\vec{D}_1$ is said to be *the same as the orientation of* $\vec{D}_2$ if there exists an oriented disk D lying in the interiors of D_1 and of D_2 and which is oriented in the same sense as $\vec{D}_1$ and $\vec{D}_2$. It is clear that the collection of all oriented disks on S containing c in their interiors decomposes into two classes, where two such oriented disks belong to the same class if and only if they have the same orientation. If we select one of those classes, then we give to S a *local orientation at the point* c.

4. Rosettes on surfaces. By a *rosette* Z lying on a surface S we understand a bouquet with center $c \in S$ and with k leaves $D_1, \ldots, D_k$ which are disks lying on S. The singleton (c) will be said to be a rosette with center $c \in S$ and with 0 leaves. If $\dot{D}_i$ denotes the boundary of D_i, then the curves $\dot{D}_1, \ldots, \dot{D}_k$ constitute a bouquet $\dot{Z}$ with center c, called the *boundary of the rosette* Z. The set $\mathring{Z} = Z \setminus \dot{Z}$ is said to be the *interior* of Z.

Using proposition (3.1), one shows easily the following proposition:

(4.1) *Let Z be a rosette lying on a surface S. Then for every neighborhood U of Z (in S) there is a disk $D \subset U$ containing Z in its interior.*

If Z is a rosette with center c and leaves $D_1, \ldots, D_k$ lying on a surface S and if $D \subset S$ is a disk containing Z in its interior, then a local orientation of S at c induces an orientation $\vec{D}$ of D, and consequently also an orientation $\vec{D}_i$ of each disk D_i. By an *oriented rosette* $\vec{Z}$ (on S) we understand a rosette Z in which the leaves are given in a fixed order and their orientations are induced by a fixed local orientation of S at c.

Consider now a pair $(C, (a_1, \ldots, a_k))$ consisting of a simple closed curve C and of an ordered system of k different points lying on C. Let $(C', (a'_1, \ldots, a'_k))$ be another pair consisting of another simple closed curve C' and of another ordered system of k different points lying on C'. Let us say that $(a_1, \ldots, a_k)$ has on C the same *position* as $(a'_1, \ldots, a'_k)$ on C' if there exists a homeomorphism $h\colon C \to C'$ such that

$$h(a_i) = a'_i \quad \text{for} \quad i = 1, \ldots, k.$$

Let us say that two pairs $(C, (a_1, \ldots, a_k))$ and $(C', (a'_1, \ldots, a'_k))$ belong to the same *class* if and only if the position of $(a_1, \ldots, a_k)$ on C is the same as the position of $(a'_1, \ldots, a'_k)$ on C'. One shows easily that the number $\sigma(k)$ of all such classes is given by the formulas:

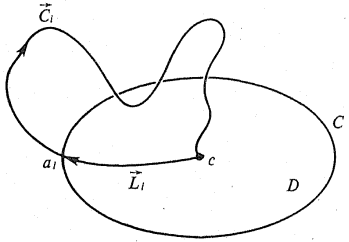
$$\sigma(0) = \sigma(1) = \sigma(2) = 1$$

and

$$\sigma(k) = \tfrac{1}{2}(k-1)! \quad \text{if} \quad k > 2.$$

Thus to every class of pairs $(C, (a_1, \ldots, a_k))$ one can assign in one-to-one manner an index ω equal to one of the numbers $1, \ldots, \sigma(k)$.

If $\vec{Z}$ is an oriented rosette with center c lying on a surface S, then $\vec{Z}$ is given by the ordered system of oriented disks $\vec{D}_1, \ldots, \vec{D}_k$, for which the orientations are induced by a given local orientation of S at c. Consider a disk $D \subset S$ containing the point c in its interior and such that no disk $\vec{D}_i$ $(i = 1, \ldots, k)$ is contained in D. Let C denote the boundary of D and $\vec{C}_i$ denote the oriented boundary of $\vec{D}_i$. Then for every $i = 1, \ldots, k$ there is a point $a_i \in C$ which belongs to $\vec{C}_i$ and is the endpoint of an arc $\vec{L}_i$ starting from c, lying on the boundary $\vec{C}_i$ of D_i, oriented by the orientation of $\vec{C}_i$ and such that the interior of $\vec{L}_i$ lies in the interior of D. In this way we assing to $\vec{Z}$ and ordered system $(a_1, \ldots, a_k)$ of different points lying on the boundary C of the disk D.



Let $\vec{Z}'$ be another oriented rosette with center c' and with the ordered system of oriented leaves $\vec{D}'_1, \ldots, \vec{D}'_k$ lying on another surface S' locally oriented at c'. As previously, we select a disk $D' \subset S'$ containing c' in its interior and such that no disk $\vec{D}'_i$ is contained in D', and we assign to $\vec{Z}'$ and ordered system of points $(a'_1, \ldots, a'_k)$ lying on the boundary C' of the disk D'. We say that the *position* of the oriented rosette $\vec{Z}'$ on S' is the *same* as the position of the oriented rosette Z on S if the position of $(a'_1, \ldots, a'_k)$ on C' is the same as the position of $(a_1, \ldots, a_k)$ on C. One sees easily that for this relation the choice of the local orientations of S (at c) and of S' (at c'), and also the choice of the disks D and D' are immaterial.

Since the position of $(a_1, \ldots, a_n)$ on C can be determined by a coefficient ω (with value equal to one of the numbers $1, \ldots, \sigma(k)$), we infer that also the position of an oriented rosette $\vec{Z}$ lying on a surface S is determined by the coefficient ω.

Consider a disk $\check{D} \subset S$ containing the oriented rosette $\vec{Z}$ in its interior and a disk $\check{D}' \subset S'$ containing $\vec{Z}'$ in its interior. One sees easily that

(4.2) *If the position of $\vec{Z}$ on S is the same as the position of $\vec{Z}'$ on S', then for every homeomorphism $h\colon Z \to Z'$ which maps $\vec{D}_i$ onto D'_i for $i = 1, \ldots, k$, there exists a homeomorphism $\check{h}\colon \check{D} \to \check{D}'$ such that $\check{h}(x) = h(x)$ for every point x of $\vec{Z}$.*

5. Pinched surfaces. By a *pinched surface* we understand a continuum R which is obtained from a surface S by the identification of a finite number of its points. If γ is the number of the identified points, then R is said to be an *γ-pinched surface*. Hence the 1-pinched surfaces are the same as surfaces. The point of R obtained by the identification of γ points of S is said to be the *peak* of R. It is uniquely defined only for γ-pinched surfaces with $\gamma > 1$. By (2.1), every pinched surface is a polyhedron.

Observe that the number γ is topologically determined by the γ-pinched surface R. In order to see this, denote for every point $x \in R$, by $n(x)$ the minimal integer such that for every sufficiently small neighborhood U (in R) of x the set $U \setminus (x)$ contains at least $n(x)$ components. It is clear that $\gamma = \operatorname{Sup}_{x \in R} n(x)$. Thus γ is determined by the topological type of R. We write $\gamma = \gamma(R)$.

One sees easily that

(5.1) $$p_1(R) = p_1(S) + \gamma - 1 \quad \text{and} \quad p_2(R) = p_2(S).$$

Since the topological type of S is determined by two integers:

$$\alpha = p_1(S) \quad \text{and} \quad \beta = p_2(S),$$

and since $p_1(S) = p_1(R) - \gamma(R) + 1$ and $p_2(S) = p_2(R)$, we infer that the topological type of R determines the topological type of the surface S. Moreover, the triplet $[\alpha, \beta, \gamma]$, where $\alpha = p_1(S)$, $\beta = p_2(S)$ and $\gamma = \gamma(R)$ determines the topological type of the pinched surface R. Observe that α is an arbitrary non-negative integer, β is 0 or 1 and γ is an arbitrary natural number. If $\beta = 0$, then the pinched surface R is said to be *non-orientable*, and if $\beta = 1$, then R is said to be *orientable*.

(5.2) LEMMA. *The pinched surfaces are the same as connected polyhedra $P \neq \emptyset$ for which all points, except at most one, have neighborhoods homeomorphic to the plane E^2.*

Proof. It suffices to show that if T is a triangulation of a connected polyhedron P and if there is a point $c \in P$ such that for every point $x \in P \setminus (c)$ there is a neighborhood of x in P homeomorphic to E^2, then P is a pinched surface.

Let $M(c)$ denote the union of all simplexes of T containing the point c and let $N(c)$ denote the union of all other simplexes of T. Setting $K(c) = M(c) \cap N(c)$, one sees easily — because each point $x\,K(c)$ has a neighborhood in P homeomorphic to E^2 — that every component of $K(c)$ is a simple closed curve. It follows that the set $M(c) \setminus (c)$ has a finite number of components $M_1, \ldots, M_\gamma$. By a compactification of M_i by one point c_i (where $c_i \neq c_j$ for $i \neq j$), one gets a continuum $\hat{M}_i$ such that every point of the set

$$\hat{P} = N(c) \cup \hat{M}_1 \cup \ldots \cup \hat{M}_\gamma$$

has a neighborhood homeomorphic to E^2. Hence $\hat{P}$ is a surface and it is clear that by the identification of points $c_1, \ldots, c_\gamma$ one gets from $\hat{P}$ a set homeomorphic to P. Hence P is a γ-pinched surface and the proof of Lemma (5.2) is finished.

The formula (5.1) implies that $\gamma \leqslant p_1(R) + 1$. It follows that

(5.3) *For every natural number n the collection of all topological types of pinched surfaces R satisfying the condition $p_1(R) \leqslant n$ is finite.*

6. Rosettes on pinched surfaces. Let c be the peak of a γ-pinched surface R. By a *rosette on R* we understand a bouquet $Z \subset R$ with center c and with k leaves $D_1, \ldots, D_k$ which are disks containing c on their boundaries. Setting $C_i = \dot{D}_i$, one obtains a bouquet consisting of k simple closed curves $C_1, \ldots, C_k$ lying on R. The point c is the center of this bouquet, which we call the *boundary of the rosette Z* and we denote it by $\dot{Z}$. The set $\mathring{Z} = Z \setminus \dot{Z}$ is said to be the *interior* of Z.

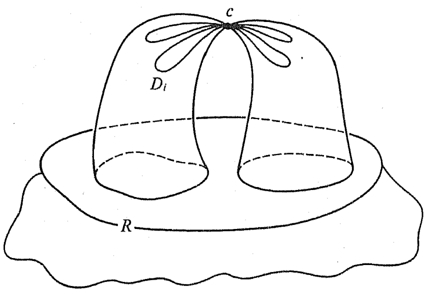
If R is obtained from a surface S by the identification of points $c_1, \ldots, c_\gamma \in S$, then to every leaf D_i of Z corresponds a disk $\hat{D}_i \subset S$ which passes onto D_i by this identification. It is clear that for every $\mu = 1, \ldots, \gamma$ the disks D_i containing the point c_μ constitute a rosette Z_μ on S with center c_μ; several of those rosettes may degenerate to singletons (c_μ). Moreover

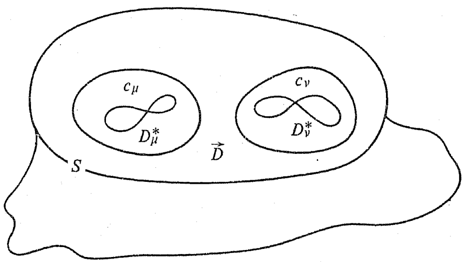
(6.1) $$Z_\mu \cap Z_\nu = \emptyset \quad \text{for} \quad \mu \neq \nu .$$

Using (4.1), we infer by (6.1) that there exist on S disks $D_1^*, \ldots, D_\gamma^*$ such that

(6.2) $$Z_\mu \text{ lies in the interior of } D_\mu^* \quad \text{for} \quad \mu = 1, \ldots, \gamma,$$

(6.3) $$D_\mu^* \cap D_\nu^* = \emptyset \quad \text{for} \quad \mu \neq \nu .$$





As a generalization of the notion of the oriented rosette lying on a surface, let us assign to a rosette Z with center c lying on a pinched surface R (with peak c) an oriented rosette $\vec{Z}$ on R, defined as follows:

The pinched surface R is obtained from a surface S (which is topologically determined by R) by the identification (with the peak c of R) of γ points $c_1, \ldots, c_\gamma \in S$. If $D_1, \ldots, D_k$ are leaves of Z given in a fixed order, then the corresponding disks $\hat{D}_1, \ldots, \hat{D}_k$ on S are given also in a fixed order and this order induces (for every $\mu = 1, \ldots, \gamma$) an order of the leaves of the rosette $Z_\mu \subset S$ with center c_μ (we preserve here the previous notations).

Consider now a local orientation of the surface S at the point c_μ. This local orientation induces an orientation of every disk lying on S and containing c_μ in its interior. In particular, it induces and orientation $\vec{D}_\mu^*$ of the disk $D_\mu^* \subset S$ (defined above) containing Z_μ in its interior, and consequently it induces an orientation of all leaves of the rosette Z_μ. If the local orientation of S at c_μ is given for every $\mu = 1, \ldots, \gamma$, then we obtain in this way orientations of all leaves $\hat{D}_i$, hence also orientations of the leaves D_i of the rosette Z.

Now let us distinguish two cases:

If R (hence also S) is non-orientable, then we understand by an oriented rosette $\vec{Z}$ the rosette Z together with a fixed order of its leaves and their orientations induced by arbitrarily given local orientations of S at the points $c_1, \ldots, c_\gamma$. Observe that the so oriented and ordered leaves $\vec{D}_i$ belonging to Z_μ constitute an oriented rosette $\vec{Z}_\mu$ on S with center c_μ.

If R (hence also S) is orientable, then the definition of $\vec{Z}$ is similar, only we assume that the local orientations of S at $c_1, \ldots, c_\gamma$ are all induced by a fixed orientation of the whole surface S.

By (3.3) in both cases, there exists on S an oriented disk $\vec{D}$ containing all disks D_μ^* in its interior, which is oriented in the same sense as the disks $\vec{D}_\mu^*$ for $\mu = 1, \ldots, \gamma$.

It is clear, that in the case $\gamma = 1$ (i.e. when $R = S$ is a surface), the definition of the oriented rosette $\vec{Z}$ is the same as the previous definition of the oriented rosette on a surface.

Consider now another oriented rosette $\vec{Z}'$ with center c' and with the same number k of oriented leaves given in a fixed order $\vec{D}_1', \ldots, \vec{D}_k'$, lying on another pinched surface R', which is homeomorphic to R. Then R' is obtained from another surface S' homeomorphic to S, by the identification (with c') of points $c_1', \ldots, c_\gamma' \in S'$. Similarly as to $\vec{Z}$, corresponds to $\vec{Z}'$ a system of γ oriented rosettes Z_μ' $(\mu = 1, \ldots, \gamma)$ on S' with centers c_μ'. Let us say that the *position of* $\vec{Z}'$ *on* R' *is the same as the position of* $\vec{Z}$ *on* R if, for every $\mu = 1, \ldots, \gamma$, the position of the oriented rosette $\vec{Z}_\mu'$ on S' is the same as the position of $\vec{Z}_\mu$ on S.

Instead to say that two oriented rosettes $\vec{Z}$ and $\vec{Z}'$ have the same positions on R and on R', we say also that $\vec{Z}$ and $\vec{Z}'$ belong to the same *class*. If k_μ denotes the number of leaves of the oriented rosette $\vec{Z}_\mu$ on S, then $\sigma(k_\mu)$ (defined in Section 4) is the number of classes of oriented rosettes with k_μ leaves lying on arbitrary surfaces. We infer that the number of classes of oriented rosettes with k leaves lying on a γ-pinched surface is finite and it depends only on k and on γ. Let us denote it by $\sigma(k, \gamma)$. Thus the position of an oriented rosette with k leaves lying on an γ-pinched surface may be determined by a coefficient ω with value equal to one of the numbers $1, \ldots, \sigma(k, \gamma)$.

Consider now a homeomorphism $h\colon Z \to Z'$ such that $h(\vec{D}_i) = \vec{D}_i'$ for $i = 1, \ldots, k$. Since R is obtained from S by the identification with c of all points $c_1, \ldots, c_\gamma$, and R' is obtained from S' by the identification with c' of all points $c_1', \ldots, c_\gamma'$, we infer that there exist two maps

$$\varphi\colon S \to R \quad \text{and} \quad \varphi'\colon S' \to R'$$

such that $\varphi(c_\mu) = c$, $\varphi'(c_\mu') = c'$ for $\mu = 1, \ldots, \gamma$ and that φ maps topologically

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$S \setminus (c_1, \ldots, c_\gamma)$ onto $R \setminus (c)$ and φ' maps topologically $S' \setminus (c'_1, \ldots, c'_\gamma)$ onto $R' \setminus (c')$. Consequently there exist two maps:

$$\psi\colon R \setminus (c) \to S \setminus (c_1, \ldots, c_\gamma); \quad \psi'\colon R' \setminus (c') \to S' \setminus (c'_1, \ldots, c'_\gamma)$$

such that

$$\varphi\psi(x) = x \quad \text{for every } x \in R \setminus (c),$$
$$\varphi'\psi'(x) = x \quad \text{for every } x \in R' \setminus (c').$$

Setting

$$h_\mu(c_\mu) = c'_\mu \quad \text{and} \quad h_\mu(x) = \psi' h \varphi(x) \quad \text{for} \quad x \in Z_\mu,$$

we get for every $\mu = 1, \ldots, \gamma$ a homeomorphism $h_\mu\colon Z_\mu \to Z'_\mu$ which maps the oriented leaves of $\vec{Z}_\mu$ onto the corresponding oriented leaves of $\vec{Z}'_\mu$. It follows by (4.2) that there exists a homeomorphism $\check{h}_\mu$ of $\vec{D}^*_\mu$ onto $\vec{D}'^*_\mu$ such that

$$\check{h}_\mu(x) = h_\mu(x) \quad \text{for every point } x \in Z_\mu.$$

But the orientation of the disks $\vec{D}^*_\mu$ (for $\mu = 1, \ldots, \gamma$) is the same as the orientation of a disk $\vec{D} \subset S$ containing all disks D^*_μ in its interior, and the orientation of $\vec{D}'^*_\mu$ is the same as the orientation of a disk $\vec{D}' \subset S'$ containing all D'^*_μ in its interior. One infers easily that there exists a homeomorphism $\check{h}\colon \vec{D} \to \vec{D}'$ such that

$$\check{h}(x) = \check{h}_\mu(x) \quad \text{for every point } x \in Z_\mu, \quad \mu = 1, \ldots, \gamma.$$

Since S and S' are homeomorphic one to the other, we infer by (3.2) that there exists a homeomorphism $\hat{h}\colon S \to S'$ such that

$$\hat{h}(x) = \check{h}(x) \quad \text{for every point } x \in \vec{D}.$$

Setting:

$$h^*(c) = c',$$
$$h^*(x) = \varphi'\hat{h}\psi(x) \quad \text{for every } x \in R \setminus (c),$$

we get a homeomorphism $h^*\colon R \to R'$ such that for every point $x \in Z \setminus (c)$ there is an index μ such that $\psi(x) \in Z_\mu$. Hence

$$h^*(x) = \varphi'\hat{h}\psi(x) = \varphi'\check{h}_\mu\psi(x) = \varphi' h_\mu \psi(x) = \varphi'\psi' h\varphi\psi(x) = h(x).$$

Thus we have established the following proposition:

(6.4) *Let $\vec{Z}$ be an oriented rosette with ordered system $\vec{D}_1, \ldots, \vec{D}_k$ of oriented leaves lying on a pinched surface R, and let $\vec{Z}'$ be another oriented rosette with ordered system $\vec{D}'_1, \ldots, \vec{D}'_k$ of oriented leaves lying on another pinched surface R', which is homeomorphic to R. If the position of $\vec{Z}$ on R is the same as the position of $\vec{Z}'$ on R', then for every homeomorphism h mapping $\vec{Z}$ onto $\vec{Z}'$, so that $h(\vec{D}_i) = \vec{D}'_i$ for $i = 1, \ldots, k$ there exists a homeomorphism $h^*\colon R \to R'$ such that $h^*(x) = h(x)$ for every $x \in Z$.*

7. Reduced and standard triangulations. By a *reduced* triangulation of a connected polyhedron P we understand any triangulation T of P satisfying two following conditions:

(7.1) *No edge of T is sharp.*

(7.2) *No edge of T is adjacent to more than 3 triangles of T.*

A triangulation T is said to be *standard* if it is reduced and if the following condition is satisfied:

(7.3) *The union of all free and of all 3-edges of T is a bouquet of circles.*

If a standard triangulation T contains free and also 3-edges, then the center c of the bouquet C consisting of all free and of all edges of ramification is the union of two bouquets: The bouquet A (called *free bouquet* of P) which is the union of all free edges of T, and the bouquet B (called the *bouquet of ramification* of P) which is the union of all 3-edges of T.

If T does not contain any free and any 3-edge, then P is a surface and as its center, we can select any point $c \in P$. Then we set $A = (c)$ and $B = (c)$. If T does not contain any free edge, but it contains 3-edges, then we set $A = (c)$, where c is the center of B. If T does not contain any 3-edge, but it contains free edges, then we set $B = (c)$, where c is the center of A.

(7.4) Remark. Observe that if a connected polyhedron P has a standard triangulation, then all its triangulations are standard — because all conditions (7.1)-(7.3) are topologically invariant. The same concerns reduced triangulations. Each connected polyhedron with reduced (or with standard) triangulations will be said to be a *reduced* (or a *standard*, respectively) *polyhedron*.

(7.5) Theorem. *Every connected polyhedron P is homotopy equivalent to a standard polyhedron.*

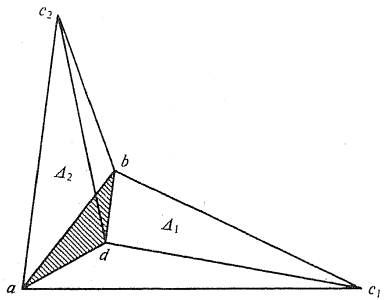
Proof. By the well known operation of collapsing, we can eliminate all sharp edges, without changing the homotopy type. Thus we can assume that already the given triangulation T of P satisfies condition (7.1).

Let $L_1, \ldots, L_k$ be all edges of T such that L_i is an n_i-edge with $n_i > 3$ for $i = 1, \ldots, k$. Let us set

$$\Lambda(T) = n_1 + n_2 + \ldots + n_k.$$

In order to obtain a polyhedron belonging to $\mathrm{Sh}(P)$ and satisfying both conditions (7.1) and (7.2), it suffices to show that if $\Lambda(T) > 0$, then there exists a polyhedron $P' \in \mathrm{Sh}(P)$ with a triangulation T' satisfying (7.1) and such that $\Lambda(T') < \Lambda(T)$. Consider two triangles $\Delta_1 \neq \Delta_2$ of T adjacent to L_k. Let a, b denote the ends of L_k and let c_ν (for $\nu = 1, 2$) denote the vertex of Δ_ν opposite to L_k. Consider a point d lying in the interior of Δ_1. One sees easily that if we remove from P both triangles Δ_1 and Δ_2 and if we add five (curvilinear) triangles with vertices abd, ac_1d, bc_1d, ac_2d and bc_2d, with interiors which are disjoint one to the other and disjoint also to the set $P \setminus (\Delta_1 \cup \Delta_2)$, then we get from P a polyhedron P' with the triangulation T'

consisting of all simplexes of T different from Δ_1, Δ_2 and of five new triangles and their edges and vertices. It is clear that T' has no sharp edges and that the edges L_i with $i = 1, \ldots, k-1$ remain still n_i-edges in T', but L_k is an (n_k-1)-edge in T'. Since every other edge L of T' is an n-edge with $n \leqslant 3$, we infer that $\Lambda(T') < \Lambda(T)$.



Moreover $P' \in \mathrm{Sh}(P)$, because if we decompose the triangle Δ_0 with vertices abd into a family F of parallel segments (two of them reduce to singletons) with one endpoint on the edge L_k and the other — on the union of two other edges of Δ_0, then the space of the decomposition of P' into segments belonging to F and into singletons is homeomorphic to P. But it is well known (see [5]), p. 86) that the shape of P' is the same as the shape of the decomposition space, hence it is the same as the shape of P, i.e. P and P' have the same homotopy type.

By iterating this proceeding, one gets finally a polyhedron $P_1 \in \mathrm{Sh}(P)$ with a triangulation T_1 satisfying both conditions (7.1) and (7.2). Thus we have shown that

(7.6) *Every connected polyhedron P is homotopy equivalent to a polyhedron P_1 with a reduced triangulation T_1.*

Let us add that if one replaces the reduced triangulation T by its barycentric subdivision, then one gets another reduced triangulation which satisfies the following condition

(7.7) *Every triangle contains at most one edge of ramification.*

Since P_1 is connected, there exists in T_1 a simple graph J which contains all vertices of T_1. Then $P = P_1/J$ is a polyhedron homotopy equivalent to P. By our construction, to every edge L of T_1 corresponds in P either the point c obtained by the identification of all points of J (if $L \subset J$), or a simple closed curve obtained from L by the identification of its endpoints. It follows that all simple closed curves obtained in this way from edges of T_1 constitute a bouquet with center c. Consequently every triangulation T of P satisfies condition (7.3), hence P is a standard polyhedron and the point c is its center. Thus the proof of Theorem (7.5) is finished.

(7.8) Remark (due to S. Nowak). Using theorem on extension of a simple homotopy (see [3], p. 19), one can show that all operations used in the proof of Theorem (7.5) do not change the simple homotopy type. Consequently Theorem (7.5) can be reformulated as follows:

For every connected polyhedron (of dimension $\leqslant 2$) there is a standard polyhedron of the same simple homotopy type.

Let us add that the problem if two polyhedra of dimension $\leqslant 2$ with the same shape have always the same simple homotopy type remains open. See [3], p. 81.

8. Wings and their carriers. Let T be a reduced triangulation of a connected polyhedron P. Assume that this triangulation satisfies condition (7.7). We know already that if J is a simple graph in T containing all vertices of T, then the polyhedron $\hat{P} = P/J$ is a standard representative of $\mathrm{Sh}(P)$. Let c denote the point of $\hat{P}$ obtained by the identification of all points of J, and let A be the free bouquet (with center c) obtained by this identification from the union of all free edges of T. Then $\hat{P}$ is the union of A and of a connected polyhedron P' which is the union of all sets obtained by this identification from triangles belonging to T. Let $l = l(\hat{P})$ denote the number of leaves of the bouquet A. It is clear that

(8.1) $l(\hat{P}) \leqslant p_1(P)$ *for every standard representative* $\hat{P}$ *of* $\mathrm{Sh}(P)$.

We know already that all edges of ramification of the triangulation T are 3-edges and that their union is a bouquet B with center c. Recall that the leaves of B are obtained by the considered identification from the edges of ramification of the reduced triangulation T. Let $m = m(\hat{P})$ denote the number of leaves of B. Then

$$B = B_1 \cup \ldots \cup B_m,$$

where B_i is a simple closed curve obtained from an edge L_i of T by the identification of its endpoints a_i and b_i. The edge L_i is adjacent to exactly three triangles $\Delta_i^1, \Delta_i^2, \Delta_i^3$ of T. By (7.7), all sides of the triangle Δ_i^ν ($\nu = 1, 2, 3$), different from L_i, are smooth edges of T.

Let us replace the edge L_i by three arcs L_i^1, L_i^2 and L_i^3 with common endpoints a_i, b_i and with disjoint interiors. We can assign to every $\nu = 1, 2, 3$ a curvilinear triangle $\hat{\Delta}_i^\nu$ with the boundary consisting of the arc L_i^ν and of two sides of $\hat{\Delta}_i^\nu$ different from L_i^ν. Assume that the interiors of all triangles $\hat{\Delta}_i^\nu$ (where $i = 1, \ldots, m$ and $\nu = 1, 2, 3$) are disjoint one to another and also disjoint to the set $P \setminus \bigcup_{\nu=1}^{3} \bigcup_{i=1}^{m} \Delta_i^\nu$. By the identification of a_i with b_i (recall that $a_i, b_i \in J$), the arcs L_i^1, L_i^2 and L_i^3 become simple closed curves B_i^1, B_i^2 and B_i^3 and that all curves B_i obtained in this way constitute a bouquet B^* with center c. We may consider B^* as the result of a tripling of the bouquet B.

Since B_i^ν is obtained from the segment L_i^ν by the identification of its endpoints a_i and b_i, there is a homeomorphism

$$\varphi_i^\nu \colon B_i^\nu \to B_i$$

induced by the homeomorphism of L_i^ν onto L_i by which the points a_i and b_i remain fixed. Let us say that φ_i^ν is a *natural homeomorphism* of B_i^ν onto B_i.

Denote by P^* the polyhedron obtained from the polyhedron P' by replacing in it of each triangle Δ_i^ν by the triangle $\hat{\Delta}_i^\nu$. From the intuitive point of view, the polyhedron P^* may be considered as to be obtained from P' by the cut along the bouquet B, with preserving its center c. It is clear that every edge of any triangulation of P^* is either smooth or sharp and that the union of all sharp edges is the bouquet B^*.

Consider a component G of the set $P^*\setminus B^*$. Then the closure $W = \bar{G}$ of G is a polyhedron (by (2.2)) for which the union of all sharp edges (for any triangulation of W) is the union of some leaves $B_1(G), \ldots, B_{k(G)}(G)$ of the bouquet B^*. Let us call the polyhedron W obtained in this way the *wing* of P corresponding to the component G of the set $P^*\setminus B^*$.

Let us span on the curve $B_\mu(G)$ (for $\mu = 1, \ldots, k(G)$) a disk $D_\mu(G)$ so that the interiors of all such disks are disjoint one to another and also disjoint to W. It is clear that the set

$$R(G) = G \cup \bigcup_{\mu=1}^{k(G)} D_\mu(G)$$

is a 2-dimensional polyhedron such that for every point $x \in R(G)\setminus(c)$ there is a neighborhood of x in $R(G)$ homeomorphic to the plane E^2. It follows by Lemma (5.1) that $R(G)$ is a pinched surface with peak c. The disks $D_1(G), \ldots, D_{k(G)}(G)$ constitute a rosette $Z(G)$ on $R(G)$ with the boundary $\dot{Z}(G) = B_1(G) \cup \ldots \cup B_{k(G)}(G)$. This pinched surface $R(G)$ will be said to be the *carrier* of the wing $W(G)$ and the set $\dot{Z}(G)$, which we denote also by $\dot{W}(G)$, will be said to be the *boundary* of the wing $W(G)$. Observe that one gets the wing $W = W(G)$ of P corresponding to G, if one removes from $R(G)$ the interior $\mathring{Z}(G)$ of the rosette $Z(G)$.

Let $n = n(P)$ denote the number of all components of the set $P^*\setminus B^*$. Denote those components by $G_1, \ldots, G_n$ and observe that $W_j = \bar{G}_j$ is the difference between the pinched surface $R_j = R(G_j)$ and the interior of the rosette $Z_j = Z(G_j)$ with center c. The boundary of the wing W_j is the same as the boundary of Z_j and it is the union of $k_j = k(G_j) \geqslant 0$ simple closed curves which are the same as the leaves of the bouquet B^* which lie on R_j. Observe that

(8.2) $$n(\hat{P}) = 0 \quad \text{only if} \quad P^* = (c)\,.$$

In this way we have assigned to a given standard polyhedron $\hat{P}$ three non-negative integers:

$$l = l(\hat{P}), \quad m = m(\hat{P}) \quad \text{and} \quad n = n(\hat{P})\,.$$

9. Scheme of a standard polyhedron. For a standard polyhedron P we have defined:

1° The free bouquet A with $l = l(P)$ leaves $A_1, \ldots, A_l$.

2° The bouquet of ramification B with $m = m(P)$ leaves $B_1, \ldots, B_m$.

Both bouquets A and B have the same center c, called *center* of P.

3° The bouquet B^* with center c consisting of $3m$ leaves B_i^ν, where $i = 1, \ldots, m$ and $\nu = 1, 2, 3$.

4° The natural homeomorphisms $\varphi_i^\nu\colon B_i^\nu \to B_i$ for $i = 1, \ldots, m$ and $\nu = 1, 2, 3$.

5° The pinched surfaces $R_1, \ldots, R_n$ with common center c and disjoint sets $R_1\setminus(c), \ldots, R_n\setminus(c)$. The pinched surface R_j is obtained from a surface S_j by the identification of γ_j points $c_{j,1}, \ldots, c_{j,\gamma_j}$ of S_j.

6° The numbers $\alpha_j = \alpha_j(P) = p_1(S_j)$, $\beta_j = \beta_j(P) = p_2(S_j)$ and $\gamma_j = \gamma_j(P)$ for $j = 1, \ldots, n$.

Now let us fix:

7° An order in the set of all leaves B_i of the bouquet B, and also an order in the set of all leaves B_i^ν of the bouquet B^*.

8° An orientation $\vec{B}_i$ of B_i for $i = 1, \ldots, m$.

9° A local orientation of S_j at $c_{j,\mu}$ for $j = 1, \ldots, n$ and $\mu = 1, \ldots, \gamma_j$. If S_j is orientable, then we select those local orientations as induced by a given orientation of S_j. If S_j is non-orientable, then we select those local orientations arbitrarily.

We know that every curve B_i^ν lies on exactly one of the pinched surfaces $R_1, \ldots, R_n$. The fixed local orientations of S_j at $c_{j,\mu}$ for $j = 1, \ldots, n$ and $\mu = 1, \ldots, \gamma_j$ induce orientations $\vec{B}_i^\nu$ of all leaves of the bouquet B^*.

10° For every $i = 1, \ldots, m; j = 1, \ldots, n$ and $\nu = 1, 2, 3$ we denote by $\delta_{i,j}^\nu = \delta_{i,j}^\nu(P)$ an integer defined as follows:

If B_i^ν does not lie on R_j, then $\delta_{i,j}^\nu = 0$.

If B_i^ν lies on R_j, then $\delta_{i,j}^\nu$ is equal to the degree of the natural homeomorphism $\varphi_i^\nu\colon \vec{B}_i^\nu \to \vec{B}_i$.

Let $k(j)$ denote the number of curves B_i^ν lying on R_j. Hence $k(j)$ is equal to the number of coefficients $\delta_{i,j}^\mu$ which, for a given j, do not vanish. If $\delta_{i,j}^\nu \neq 0$, then the γ_j-pinched surface R_j has been constructed so that B_i^ν is the boundary of a disk lying on R_j and all such disks constitute a rosette Z_j with center c lying on R_j. The order (fixed in 7°) of curves B_i^α and their orientations $\vec{B}_i^\nu$ induced by local orientations of the surfaces S_j (fixed in 9°) give to the rosette Z_j the character of the oriented rosette $\vec{Z}_j$. By Section 6, the position of $\vec{Z}_j$ on R_j is determined by

11° A coefficient $\omega_j = \omega_j(P)$, which is equal to one of the numbers

$$1, \ldots, \sigma(k(j), \gamma_j)\,.$$

Now we define the *scheme* $\mathfrak{S} = \mathfrak{S}(P)$ of a standard polyhedron P as the system consisting of numbers: $l(P)$, $m(P)$, $n(P)$, $\alpha_j(P)$, $\beta_j(P)$, $\gamma_j(P)$, $\omega_j(P)$ for $j = 1, \ldots, n(P)$ and $\delta_{i,j}^\nu(P)$ for $i = 1, \ldots, m(P)$; $j = 1, \ldots, n(P)$ and $\nu = 1, 2, 3$.

In this system the numbers $l(P)$, $m(P)$ are arbitrary non-negative integers, the number $n(P)$ is a non-negative integer, the number $\beta_j(P)$ is 0 or 1, γ_j is a natural number, the numbers $\delta_{i,j}^\nu(P)$ have values 0, 1 or -1 and they satisfy the condition

(9.1) *For every $i = 1, \ldots, m(P)$ and $\nu = 1, 2, 3$ there exists exactly one index j such that $\delta_{i,j}^\nu \neq 0$.*

If $k(j)$ denotes the number of coefficients $\delta^{\nu}_{i,j}$ which (for a fixed j) do not vanish, then the coefficient $\omega_j(P)$ has the value equal to one of the numbers

$$1, \ldots, \sigma(k(j), \gamma_j(P)).$$

Let us observe that the triplet $[\alpha_j(P), \beta_j(P), \gamma_j(P)]$ determines topologically the pinched surface R_j.

Now let us prove the following

(9.2) THEOREM. *For every given scheme $\mathfrak{S}$ there exists a standard polyhedron P which has $\mathfrak{S}$ as its scheme. If the scheme $\mathfrak{S}$ is given, then the bouquets of simple closed curves A with leaves $A_1, \ldots, A_l$ and B with leaves $B_1, \ldots, B_m$ and also the bouquet of pinched surfaces $R_1, \ldots, R_n$ with center c are topologically determined. Let S_j denote a surface such that R_j is obtained from S_j by the identification with c of a system of points $c_{j,1}, \ldots, c_{j,\gamma_j} \in S_j$. If we fix the orientations of curves B_i and the local orientations of surfaces S_j at $c_{j,\mu}$ $(\mu = 1, \ldots, \gamma_j)$ satisfying 9°, then P is topologically determined by $\mathfrak{S}$.*

Proof. Let $\mathfrak{S}$ be a scheme consisting of numbers: $l, m, n, \alpha_j, \beta_j, \gamma_j, \omega_j, \delta^{\nu}_{i,j}$ where $i = 1, \ldots, m$; $j = 1, \ldots, n$ and $\nu = 1, 2, 3$. Then there exist two bouquets: A with l leaves $A_1, \ldots, A_l$ and B with m leaves $B_1, \ldots, B_m$ such that all curves $A_1, \ldots, A_l, B_1, \ldots, B_m$ are different one from another and they constitute a bouquet. Let c be the center of this bouquet. Moreover, for every $j = 1, \ldots, n$ there exists a surface S_j such that $p_1(S_j) = \alpha_j$ and $p_2(S_j) = \beta_j$ and this surface is topologically determined by the numbers α_j and β_j. Consider a system of γ_j points $c_{j,1}, \ldots, c_{j,\gamma_j} \in S_j$. By identification of all those points with a point c_j, one gets from S_j a γ_j-pinched surface R_j with peak c_j and this pinched surface is topologically determined by the triplet $[\alpha_j, \beta_j, \gamma_j]$. Moreover we may assume that all pinched surfaces R_j are different one from another and that all sets $A_1, \ldots, A_l, B_1, \ldots, B_m, R_1, \ldots, R_n$ constitute a bouquet with center c.

Now let us assign to every curve B_i an orientation $\vec{B}_i$ and to every surface S_j a local orientation at the point $c_{j,\mu}$ and assume that in the case when S_j is orientable all such local orientations of S_j are induced by an arbitrarily given orientation of S_j.

Consider a bouquet B^* with center c consisting of $3m$ simple closed curves B^{ν}_i $(i = 1, \ldots, m$ and $\nu = 1, 2, 3)$ given in a fixed order and assume that B^{ν}_i lies on R_j if and only if $\delta^{\nu}_{i,j} \neq 0$ and that all curves B^{ν}_i lying on R_j constitute the boundary $\dot{Z}_j$ of a rosette $Z_j \subset R_j$ with center c_j. Since R_j is obtained from S_j by the identification with c_j of points $c_{j,1}, \ldots, c_{j,\gamma_j}$, we infer that to the rosette Z_j corresponds a system of γ_j rosettes $Z_{j,1}, \ldots, Z_{j,\gamma_j}$ lying on S_j with centers $c_{j,1}, \ldots, c_{j,\gamma_j}$. The fixed local orientation of S_j at $c_{j,\mu}$ induces orientations of all leaves of $Z_{j,\mu}$, consequently also an orientation $\vec{Z}_j$ of the rosette Z_j. The coefficient ω_j determines its position on the γ_j-pinched surface R_j.

Let $G_j = R_j \setminus Z_j$. Then $W = \bar{G}_j$ is the polyhedron obtained from R_j by removing the interior of the rosette Z_j. Let us assign to every curve B^{ν}_i a homeomorphism

$$\varphi^{\nu}_i\colon B^{\nu}_i \to B_i$$

such that $\varphi^{\nu}_i(c) = c$ and that, if B^{ν}_i lies on R_j, then φ^{ν}_i maps the oriented curve $\vec{B}^{\nu}_i$ onto the oriented curve $\vec{B}_i$ with the degree $\delta^{\nu}_{i,j}$. Setting

$$\varphi_j(x) = \varphi^{\nu}_i(x) \quad \text{for every point } x \in B^{\nu}_i \subset R_j,$$

we get a map φ_j of the boundary of the polyhedron W_j onto a subbouquet of the bouquet B, such that $\varphi_j(c) = c$.

One sees easily that the map φ_j can be extended to a map $\hat{\varphi}_j$ of W_j onto a polyhedron P_j so that $\hat{\varphi}_j/G_j$ is a homeomorphism and that $\hat{\varphi}_j(G_j) \cap B = \emptyset$. Moreover we can assume that the polyhedra P_j (where $j = 1, \ldots, n$) are constructed so that

$$P_j \cap P_{j'} \subset B \qquad \text{for} \quad j \neq j'.$$

Then the set $P' = P_1 \cup \ldots \cup P_n$ is a polyhedron.

Observe now that if a leaf B^{ν}_i of the bouquet B^* lies on R_j, then B^{ν}_i is the union of some sharp edges of any triangulation T_j of the polyhedron W_j and — because of (9.1) — there exist exactly three indices j_1, j_2, j_3 (not necessarily different) such that

$$|\delta^1_{i,j_1}| = |\delta^2_{i,j_2}| = |\delta^3_{i,j_3}| = 1.$$

Then the curve B^{ν}_i is the union of some sharp edges of W_{j_ν}, for $\nu = 1, 2, 3$. We infer that B_i is the union of some 3-edges of P'. Moreover it is clear that all other edges of P' are smooth.

Since the components $\hat{G}_j$ of the set $P' \setminus B$ are the same as sets $\hat{\varphi}_j(G_j)$, we infer that the wings of P' are homeomorphic to polyhedra W_j lying on pinched surfaces R_j, where $j = 1, \ldots, n$.

Adding to P' the bouquet A consisting of l simple closed curves (with center c), which has with P' only the point c in common, we get a connected standard polyhedron P. If we confront the construction of P with the procedure leading from polyhedron P to its scheme, we infer that $\mathfrak{S}$ is the scheme assigned to P. Thus the first part of Theorem (9.2) is proved.

Passing to the second part, consider two connected standard polyhedra ${}_1P$ and ${}_2P$ with the same scheme $\mathfrak{S}$. First let us consider the case when ${}_1P$ and ${}_2P$ do not have free edges, that is when $l = 0$. We may assume that ${}_1P$ and ${}_2P$ have the same bouquet B of ramification with center c and let us fix an orientation for every leaf B_i (where $i = 1, \ldots, m$) of B.

By the operation of cutting the polyhedron ${}_1P$ along B (preserving the point c), one gets from each leaf B_i of B three simple closed curves ${}_1B^1_i$, ${}_1B^2_i$, ${}_1B^3_i$ and all curves ${}_1B^{\nu}_i$ so obtained constitute a bouquet ${}_1B^*$ with center c. Denote by ${}_1\varphi^{\nu}_i$ the natural homeomorphism of ${}_1B^{\nu}_i$ onto B_i. Similarly the cutting of ${}_2P$ along B gives for every curve B_i three simple closed curves ${}_2B^1_i$, ${}_2B^2_i$, ${}_2B^3_i$ and all curves ${}_2B^{\nu}_i$ so obtained constitute a bouquet ${}_2B^*$ with center c. Let ${}_2\varphi^{\nu}_i$ be the natural homeomorphism of ${}_2B^{\nu}_i$ onto B_i.

The polyhedron ${}_1P^*$ is the union of n wings ${}_1W_1, \ldots, {}_1W_n$ and the polyhedron ${}_2P^*$ is the union of n corresponding wings ${}_2W_1, \ldots, {}_2W_n$. The wing ${}_1W_j$ is a polyhedron obtained from a γ_j-pinched surface ${}_1R_j$ by removing from it the interior of a rosette ${}_1Z_j$ with center c, consisting of $k(j)$ disks ${}_1D_1, \ldots, {}_1D_{k(j)}$ which are spanned on those curves ${}_1B_i^\nu$ for which $\delta_{i,j}^\nu \neq 0$. Similarly, the wing ${}_2W_j$ is a polyhedron obtained from a γ_j-pinched surface ${}_2R_j$ by removing from it the interior of a rosette ${}_2Z_j$ with center c, consisting of $k(j)$ disks ${}_2D_1, \ldots, {}_2D_{k(j)}$ spanned on those curves ${}_2B_i$ for which $\delta_{i,j}^\nu \neq 0$.

Both pinched surfaces ${}_1R_j$ and ${}_2R_j$ are homeomorphic, because they are determined by the same numbers α_j, β_j, γ_j. The γ_j-pinched surfaces ${}_1R_j$ and ${}_2R_j$ are obtained from two surfaces, ${}_1S_j$ and ${}_2S_j$ (which are homeomorphic) by the identification with the point c of γ_j points ${}_1c_1, \ldots, {}_1c_{\gamma_j}$ lying on ${}_1S_j$, and of γ_j points ${}_2c_1, \ldots, {}_2c_{\gamma_j}$ lying on ${}_2S_j$. Now let us fix the order of disks ${}_1D_1, \ldots, {}_1D_{k(j)}$ and the local orientations of ${}_1S_j$ at points ${}_1c_1, \ldots, {}_1c_{\gamma_j}$, assuming that those orientations are arbitrary if ${}_1S_j$ is non-orientable, and are induced by an arbitrarily given orientation of ${}_1S_j$, if ${}_1S_j$ is orientable. As we know, those local orientations induce orientations of disks ${}_1D_1, \ldots, {}_1D_{k(j)}$, and in this way we obtain from ${}_1Z_j$ anoriented rosette ${}_1\vec{Z}_j$ lying on ${}_1R_j$. Similarly we get from ${}_2Z_j$ an oriented rosette ${}_2\vec{Z}_j$ lying on ${}_2R_j$. By our scheme γ the position of ${}_1\vec{Z}_j$ on ${}_1R_j$ and the position of ${}_2\vec{Z}_j$ on ${}_2R_j$ are determined by the same coefficient ω_j and consequently they are the same.

Let ψ_i^ν denote the homeomorphism of B_i onto ${}_2B_i^\nu$ inverse to the natural homeomorphism ${}_2\varphi_i^\nu\colon {}_2B_i^\nu \to B_i$. Setting

$$\chi_i^\nu = \psi_i^\nu {}_1\varphi_i^\nu ,$$

we get a homemorphism $\chi_i^\nu\colon {}_1B_i^\nu \to {}_2B_i^\nu$ and it is clear that there exists a homeomorphism $\vartheta_i^\alpha\colon {}_1D_i \to {}_2D_i$ such that $\vartheta_i^\nu(x) = \chi_i^\nu(x)$ for every point $x \in {}_1B_i^\nu$. Setting

$$\vartheta_j(x) = \vartheta_i^\nu(x) \quad \text{for every point } x \in {}_1D_i \subset {}_1S_j ,$$

we get a homeomorphism ϑ_j which maps the rosette ${}_1Z_j$ onto the rosette ${}_2Z_j$. Moreover the degree of the map ${}_1\varphi_i^\nu$ (where ${}_1B_i^\nu \subset {}_1R_j$) is $\delta_{i,j}^\nu = \pm 1$ and it is equal to the degree of the map ${}_2\varphi_i^\nu$, hence also to the degree of the map ψ_i^ν. Consequently the degree of the homeomorphism χ_i^ν is 1, and we infer that this map, hence also the map ϑ_j preserves the orientation. Moreover, to the leaves ${}_1B_i^\nu$ of ${}_1\vec{Z}_j$, correspond by the map ϑ_j, in the same order, the leaves ${}_2B_i^\nu$ of ${}_2\vec{Z}_j$. Thus the homeomorphism ϑ_j maps the oriented rosette ${}_1\vec{Z}_j$ onto the oriented rosette ${}_2\vec{Z}_j$. Finally the position of ${}_1\vec{Z}_j$ on ${}_1R_j$ is the same as the position of ${}_2\vec{Z}_j$ on ${}_2R_j$, because those positions are determined by the same coefficient ω_j.

Using Proposition (6.4), we infer that there exists a homeomorphism $\hat{\vartheta}_j\colon {}_1R_j \to {}_2R_j$ such that $\hat{\vartheta}_j(x) = \vartheta_j(x)$ for every $x \in {}_1Z_j$.

Observe that $\hat{\vartheta}_j$ maps the wing ${}_1W_j$ onto the wing ${}_2W_j$. The natural homeomorphism ${}_1\varphi_i^\nu\colon {}_1B_i^\nu \to B_i$ induces a matching of ${}_1W_j$ with B, by which the point $x \in {}_1B_i^\nu \subset {}_1R_j$ is identified with the point ${}_1\varphi_i^\nu(x)$. By the homeomorphism $\hat{\vartheta}_j$, to the point x corresponds the point

$$\hat{\vartheta}_j(x) = \psi_{i1}^\nu \varphi_i^\nu(x) \in {}_2B_i^\nu$$

and the map ${}_2\varphi_i^\nu$ assigns to $\hat{\vartheta}_j(x)$ the point

$${}_2\varphi_i^\nu \hat{\vartheta}_j(x) = {}_2\varphi_i^\nu \psi_{i1}^\nu \varphi_i^\nu(x) = {}_1\varphi_i^\nu(x) .$$

Consequently the result of the matching of ${}_1W_j$ with B, induced by the natural homeomorphism ${}_1\varphi_i^\nu$ is homeomorphic with the result of the matching of ${}_2W_j$ with B induced by ${}_2\varphi_i^\nu$. It follows that the polyhedron ${}_1P$ is homeomorphic to the polyhedron ${}_2P$.

Thus in the case when ${}_1P$ and ${}_2P$ do not have free edges (i.e. in the case $l = 0$) the proof is finished. But in the case $l > 0$, the polyhedron ${}_\nu P$ is the union of a bouquet ${}_\nu A$ consisting of l curves with center c and of a standard polyhedron ${}_\nu P'$ which does not contain free edges and for which the bouquet of ramification has the point c as its center. Moreover ${}_\nu A \cap {}_\nu P' = (c)$ and we infer from the homeomorphism of $({}_1P', c)$ with $({}_2P', c)$ that ${}_1P$ is homeomorphic with ${}_2P$.

Let us add that the topological structure of ${}_1P$ does not depend on the choice of the natural homeomorphism ${}_1\varphi_i^\nu\colon {}_1B_i^\nu \to B_i$ (with the degree given by $\delta_{i,j}^\nu$), because if we assume that ${}_1R_j = {}_2R_j$ and ${}_1B_i^\nu = {}_2B_i^\nu$ for $i = 1, \ldots, m$ and $\nu = 1, 2, 3$, then ${}_2P$ differs from ${}_1P$ only by replacing the natural homeomorphism ${}_1\varphi_i^\nu$ by another natural homeomorphism ${}_2\varphi_i^\nu\colon {}_1B_i^\nu \to B_i$.

10. Special polyhedra. The following remark shows that the numbers l, m, n appearing in the scheme are not shape invariant:

Let P_1 denote the union of the 2-dimensional sphere S^2 with a disk D such that the set $D \cap S^2$ coincides with the boundary $\dot{D}$ of D. Then P_1 is a standard polyhedron, for which $l = 0$, $m = 1$ and $n = 3$. But P_1 has the same shape as the polyhedron P_2 which is the union of two 2-dimensional spheres with only one point in common. Also P_2 is standard, but for it all numbers l, m, n vanish.

The question how to reinforce the conditions characterising standard polyhedra in order to guarantee the uniqueness of the numbers l, m, n is rather hard. In an attempt to approach this aim, let us introduce the notion of special polyhedra.

Consider a point c of a connected space X. Let us say that c *locally separates* X if there exists a connected neighborhood U of c such that $U \setminus (c)$ is not connected.

Let P be a standard polyhedron with center c and let A denote the free bouquet of P, and B — the bouquet of ramification of P. Let $W_1, \ldots, W_n$ be the wings of P. We say that the polyhedron P is a *special polyhedron* if it satisfies additionally the following conditions:

(10.1) *No wing of P is a disk.*

(10.2) *The closure of no component of $P \setminus A$ is locally separated by c.*

Now let us prove the following

(10.3) THEOREM. *Every connected polyhedron is homotopy equivalent to a special polyhedron.*

Proof. Assume that P is a standard representative of the shape of the given polyhedron and let c denote the center of P. If a wing W of P is a disk, then $\hat{P} = P/W$ is a standard representative of $\mathrm{Sh}(P)$ such that:

$$l(\hat{P}) = l(P), \quad m(\hat{P}) = m(P) - 1 \quad \text{and} \quad n(\hat{P}) < n(P) .$$

By iterating this proceeding, one gets finally a standard polyhedron $P^1 \in \mathrm{Sh}(P)$ with center c which satisfies the condition (10.1) and for which

$$l(P^1) = l(P), \quad m(P^1) \leqslant m(P) \quad \text{and} \quad n(P^1) \leqslant n(P).$$

Assume now that G is a component of $P^1 \setminus A$ such that $\bar{G}$ is locally separated by c. Let T be a triangulation of P^1 such that, with

$$U = \mathrm{St}(c) \cap \bar{G},$$

$U \setminus (c)$ is not connected. One sees easily that there exists a compactification $\tilde{G}$ of G by λ different points $c_1, \ldots, c_\lambda$, where λ is the number of components of $U \setminus (c)$. There exist two trees L_1 and L_2 containing the points $c_1, \ldots, c_\lambda$ such that $L_1 \cap \tilde{G} = (c_1, \ldots, c_\lambda)$, where $c_1, \ldots, c_\lambda$ are all vertices of L_1, and $L_2 \setminus (c_1, \ldots, c_\lambda) \subset G$.

Since $\bar{G} \underset{\text{top}}{=} (\tilde{G} \cup L_1)/L_1$, we infer that

$$\mathrm{Sh}(\bar{G}, c) = \mathrm{Sh}(\tilde{G} \cup L_1, c_1) = \mathrm{Sh}(\tilde{G} \cup L_1/L_2, c_1/L_2)$$

and thus that

$$\mathrm{Sh}(P) = \mathrm{Sh}\big((P \setminus G, c) \vee ((\tilde{G} \cup L_1)/L_2, c_1/L_2)\big),$$

where $\vee$ denotes the union of pointed spaces at their base points.

By iterating this proceeding, one gets finally a standard representative P_0 of $\mathrm{Sh}(P)$ such that

$$l(P_0) \geqslant l(P), \quad m(P_0) \leqslant m(P), \quad n(P_0) \leqslant n(P)$$

and that P_0 satisfies both conditions (10.1) and (10.2). Hence P_0 is a special polyhedron and the proof of Theorem (10.3) is finished.

Similarly as in the case of Theorem (7.5) (see Remark (7.8)) one can reformulate Theorem (10.3) as follows:

For every connected polyhedron (*of dimension* $\leqslant 2$) *there is a special polyhedron of the same simple homotopy type.*

Theorem (10.3) implies the following

(10.4) COROLLARY. *If P is a special polyhedron with $m(P) = 0$, then P si a bouquet for which each leaf is either a simple closed curve or a surface.*

In fact, then the bouquet of ramification B of P reduces to the singleton (c). If A is the free bouquet of P, then the closure of every component of $P \setminus A$ is a pinched surface R with the peak c. Since c does not locally separates it, R is a surface.

Let us add that, by a theorem of A. Kadlof if two bouquets, consisting of leaves which are either simple closed curves or surfaces, are of the same shape, then they are homeomorphic. Consequently, there exists for a connected polyhedron P with $m(P) = 0$ only one special polyhedron, which is a representative of $\mathrm{Sh}(P)$.

11. Coefficients $l_0(P)$, $m_0(P)$ and $n_0(P)$. For every connected polyhedron P (of dimension $\leqslant 2$) let us denote by $\Gamma(P)$ the collection of all standard polyhedra $P' \in \mathrm{Sh}(P)$. Observe that for every $P' \in \Gamma(P)$ the inequality $l(P') \leqslant p_1(P)$ holds true. Setting:

$$l_0(P) = \operatorname*{Max}_{P' \in \Gamma(P)} l(P'); \quad m_0(P) = \operatorname*{Min}_{P' \in \Gamma(P)} m(P'); \quad n_0(P) = \operatorname*{Min}_{P' \in \Gamma(P)} n(P'),$$

we get three non-negative integers $l_0(P)$, $m_0(P)$, $n_0(P)$ and it is clear that those integers are shape invariants of P. However, the problem how to compute their values for a given polyhedron P remains open and it seems to be difficult. It is clear that if P is a bouquet consisting of l simple closed curves, then $l_0(P) = l$. However the problem if for every given non-negative number m (or n) there exists a connected polyhedron P with $m_0(P) = m$ (or with $n_0(P) = n$) remains open. The role of the coefficient $n_0(P)$ is illustrated by the following

(11.1) THEOREM. *For every natural number k there exists only a finite number of different shapes of connected polyhedra P satisfying both conditions: $p_1(P) \leqslant k$ and $n_0(P) \leqslant k$.*

Proof. First let us observe that

(11.2) If T is a triangulation of a polyhedron P and if $\mathring{\Delta}$ is the interior of a triangle $\Delta \in T$, then

$$p_1(P) \leqslant p_1(P \setminus \mathring{\Delta}) \leqslant p_1(P) + 1.$$

Assume now that P is a standard polyhedron with $n(P) = n_0(P) \leqslant k$. If B is the bouquet of ramification of P and W is a wing of P, then if T_W is a triangulation of W and $\mathring{\Delta}_W$ is the interior of a triangle $\Delta_W \in T_W$, then the polyhedron $W \setminus \mathring{\Delta}_W$ can be transformed by a collapsing onto the union of the boundary $\mathring{W}$ of the wing W and of a polyhedron of dimension $\leqslant 1$. It follows that there exists a retraction of the set $W \setminus \mathring{\Delta}_W$ to $\mathring{W}$. Consequently

$$p_1(\mathring{W}) \leqslant p_1(W \setminus \mathring{\Delta}_W) \leqslant p_1(W) + 1. \tag{11.3}$$

The polyhedron P is obtained by the matching of all wings W of P with the bouquet B by the natural homeomorphisms of all curves B_i^{ν} onto the curves B_i. If we replace every wing W of P by the set $W \setminus \mathring{\Delta}_W$, then we get, instead of P a polyhedron $\hat{P}$. Using (11.3), one sees easily that this last polyhedron satisfies the condition

$$p_1(\hat{P}) \leqslant p_1(P) + n(P).$$

Moreover, since $\mathring{W}$ is a retract of $W \setminus \mathring{\Delta}_W$, one sees easily that B is a retract of $\hat{P}$ and we infer that

$$m = p_1(B) \leqslant p_1(P) + n(P).$$

Since both numbers $p_1(P)$ and $n(P)$ are $\leqslant k$, we infer that $m \leqslant 2k$. Then for every wing W of P the following inequality holds true:

$$p_1(\mathring{W}) \leqslant p_1(B) \leqslant 2k.$$

It follows that the carrier c_W of W is a pinched surface which satisfies the condition

$$p_1(C_W) \leqslant p_1(W).$$

Hence $p_1(C_W) \leqslant 2k$. We infer by (5.3) that the collection of all topological types of carriers C_W is finite. Since also the number $l = l(P) \leqslant p_1(P) \leqslant k$ is finite, we infer that there exists only a finite number of shapes of polyhedra P with $p_1(P) \leqslant k$ and $n_0(P) \leqslant k$. Thus the proof of Theorem (11.1) is finished.

The limitation of the values of $p_1(P)$ and of $p_2(P)$ does not suffice for the finity of the collection of shapes of P. In fact, consider the 3-dimensional Poincaré sphere M (i.e. a polyhedron which is a closed 3-manifold with $p_1(M) = p_2(M) = 0$ and $p_3(M) = 1$, but with a non-trivial fundamental group). Let Δ be a 3-dimensional simplex of a triangulation of M. Then $N = \overline{M \setminus \Delta}$ is an acyclic 3-dimensional polyhedron with a non-trivial fundamental group. Using the operation of collapsing, one gets from N a 2-dimensional acyclic polyhedron P (of dimension $\leqslant 2$) with a non-trivial shape.

Consider now a system $P_1, \ldots, P_k$ of polyhedra homeomorphic to P and constituting a bouquet with center c. Setting $P_k^* = P_1 \cup \ldots \cup P_k$, one gets for every $k = 1, 2, \ldots$ an acyclic polyhedron P_k^* and one sees easily that $\operatorname{Sh}(P_k^*) \neq \operatorname{Sh}(P_{k'}^*)$ for $k \neq k'$.

Let us add that the values of $m_0(P)$ and of $n_0(P)$ remain unknown. The following problem remains open:

(11.4) PROBLEM. *Does there exist a connected polyhedron such that for every standard representative of its shape not all carriers of wings are surfaces?*

12. Non connected polyhedra. If $P_1, \ldots, P_k$ are components of a polyhedron P, then every set P_i is a connected polyhedron and we can assign to every standard representative of it its scheme $\mathfrak{S}_i$. The system consisting of those schemes is a finite numerical system which can be considered as the scheme of the polyhedron P. It is clear that Theorem (9.2) implies that this scheme determines the homotopy type of P.

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On a question of H. H. Corson and some related problems

by

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Abstract. In this paper we investigate a property of a Banach space defined by Corson [7] which is a convex counterpart to the Lindelöf property of weak topology.

1. Introduction. H. H. Corson defined in [7] the following property of a Banach space E (which we shall call "*the property* (C)"): *every collection of closed convex subsets of E with empty intersection contains a countable subcollection with empty intersection.*

If a Banach space E is Lindelöf in the weak topology then it has the property (C) (as the closed convex sets are the same in the norm or weak topology) and Corson asked [7, Remark (1), p. 7], whether the converse is true. It turns out that many familiar function spaces $C(K)$ have the property (C) while their weak topology fails to have the Lindelöf property (¹), for example, this is the case, when K is the lexicographic square. In fact, we show that the Banach spaces with the property (C) form a rather wide class, closed under some standard operations.

We prove that the property (C) of a Banach space E is equivalent to a property of the unit ball in the dual space E' endowed with the weak-star topology, which is a convex analogue to the countable tightness (²). We show further that for a compact scattered space K the property (C) of the function space $C(K)$ is equivalent to the countable tightness of K; in general, the property (C) of the function space $C(K)$ being, as stated above, related to a kind of the countable tightness in the space of Radon measures on K, seems essentially stronger than the countable tightness of K — however, we do not know a correspondent example. Another result about function spaces is that if K is an Eberlein compact and E has the property (C), then so does the space $C(K, E)$, but we do not know, for example, if $C(S \times S)$ has the property (C), provided that $C(S)$ does. We discuss these, and related, questions in the last paragraph.

(¹) Thus the counterpart to the classical characterization of compactness for the weak topology [18, Theorem 11.2 (c)] fails for the Lindelöf property.

(²) The terminology is explained in the next paragraph.