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Collectionwise normality and extensions of locally finite coverings

by

Teodor C. Przymusiński * (Warszawa)
and **Michael L. Wage** ** (New Haven, Conn.)

Abstract. We study the following properties of a space X:

(1) X is collectionwise normal and countably paracompact.

(2) X is normal and every locally finite open covering of a closed subspace F of X can be extended to a locally finite open covering of X.

(3) X is normal and every locally finite functionally open covering of a closed subspace F of X can be extended to a locally finite open (or — equivalently — functionally open) covering of X.

(4) X is collectionwise normal.

Katětov proved in 1958 that (1) → (2) → (3) → (4) and raised the problem of the validity of inverse implications. We present three examples showing that none of the implications above can be reversed. We also prove the following.

THEOREM. *A T_1-space X is collectionwise normal* (*resp. satisfies* (3)) *if and only if every locally finite partition of unity on a closed subspace F of X can be extended to a partition* (*resp. locally finite partition*) *of unity on X.*

§ 1. **Introduction.** In his 1958 paper [6], M. Katětov studied extensions of locally finite coverings and raised several problems that have long remained open. Consider the following properties of a space X:

(1) X is collectionwise normal and countably paracompact.

(2) X is normal and every locally finite open covering of a closed subspace F of X can be extended to a locally finite open covering of X.

(3) X is normal and every locally finite functionally open covering of a closed subspace F of X can be extended to a locally finite open (or — equivalently — functionally open) covering of X.

(4) X is collectionwise normal.

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We define a space X to be *Katětov* if it satisfies (2) and to be *functionally Katětov* if it satisfies (3). Katětov proved that (1) implies (2) implies (3) implies (4). We record his result, using our terminology, in the diagram below.

collectionwise normal
and countably paracompact
$\downarrow$
Katětov
$\downarrow$
functionally Katětov
$\downarrow$
collectionwise normal

Katětov asked whether every Katětov space is countably paracompact ([6]; p. 243) and whether every collectionwise normal space is Katětov ([6]; p. 244).

The main purpose of this paper is to provide negative answers to Katětov's questions. The following three examples show that none of the implications in the above diagram can be reversed. Example 1 is constructed under the assumption of the Gödel Axiom of Constructibility, V = L (see e.g. [4]) and also seems to be the first example of a hereditarily (collectionwise) normal Dowker space.

Example 1. (V = L) A hereditarily normal, hereditarily separable, first countable, locally countable and locally compact Katětov space, which is not countably paracompact.

Example 2. The Dowker space constructed by M. E. Rudin in [11] is a functionally Katětov space which is not Katětov.

Example 3. A collectionwise normal space which is not functionally Katětov.

We also prove the following two theorems characterizing collectionwise normal and functionally Katětov spaces in terms of extensions of partitions of unity and sheding some light on the relationship between these two classes of spaces.

Theorem 1. *A T_1-space X is collectionwise normal if and only if every locally finite partition of unity on a closed subspace F of X can be extended to a partition of unity on X.*

Theorem 2. *A T_1-space X is functionally Katětov if and only if every locally finite partition of unity on a closed subspace F of X can be extended to a locally finite partition of unity on X.*

We say that a space X is *countably Katětov* (resp. *countably functionally Katětov*) if it satisfies the condition (2) (resp. (3)) with coverings assumed to be countable. Katětov's proofs show, in fact, that the following implications hold.

normal
and countably paracompact
$\downarrow$
countably Katětov
$\downarrow$
countably functionally Katětov
$\downarrow$
normal

Examples 1–3 show in effect that none of the implications in the above diagram can be reversed.

The following counterparts of Theorems 1 and 2 hold.

Theorem 3. *A T_1-space X is normal if and only if every countable locally finite partition of unity on a closed subspace F of X can be extended to a partition of unity on X.*

Theorem 4. *A T_1-space X is countably functionally Katětov if and only if every countable locally finite partition of unity on a closed subspace F of X can be extended to a locally finite partition of unity on X.*

Though hereditarily normal spaces are not necessarily countably paracompact (cf. Example 1), nevertheless we have the following

Theorem 5. *Hereditarily normal spaces are countably Katětov.*

Theorems 1–5 are proved in Section 2 and Examples 1–3 are constructed in Section 3. We conclude the paper with a few open questions.

Notation and definitions. Throughout this paper a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. By $\mathrm{cf}(\lambda)$ we mean the cofinality of λ. By N, R and I we denote, respectively, the set of natural numbers, the real line and the unit interval $[0, 1]$.

A T_2-space is called *countably paracompact* if every countable open covering has a locally finite refinement. In a normal space, this is equivalent to saying that whenever $\{F_n\}_{n\in N}$ is a decreasing sequence of closed sets having empty intersection, then there exists a sequence, $\{U_n\}_{n\in N}$, of open sets having empty intersection such that for each $n \in N$, $F_n \subset U_n$. A *Dowker space* is a normal space that is not countably paracompact. If $K \subset X$ and $\{V_s\}_{s\in S}$ is a family of subsets of K, then we say that, $\{U_s\}_{s\in S}$, a family of subsets of X, *extends* $\{V_s\}_{s\in S}$ if $U_s \cap K = V_s$ for all $s \in S$ A subset of a space X is called *functionally open* (= cozero) if it can be represented as $\{x \in X \mid f(x) \neq 0\}$ for some continuous function $f\colon X \to R$. In a normal space, a set is functionally open if and only if it is an open F_σ. A collection, $\mathscr{H}$, of subsets of X is called *discrete* if each point in X has a neighborhood that meets at most one member of $\mathscr{H}$. A T_1-space is *collectionwise normal* if every discrete collection of closed subsets of the space can be separated by pairwise disjoint open sets. A family $\mathscr{F} = \{f_s\}_{s\in S}$ of continuous functions $f_s\colon F \to I$ is a *partition of unity* on F if $\sum_{s\in S} f_s(x) = 1$ for all $x \in F$. This partition is *locally finite* if the family $\{f_s^{-1}(0,1])\}_{s\in S}$ is locally finite in F. A partition $\mathscr{G} = \{g_s\}_{s\in S}$ on $X \supset F$ is an *extension* of $\mathscr{F}$ if $g_s|_F = f_s$ for all $s \in S$. A covering $\mathscr{U}$ of X is *uniformly locally finite* if there exists a locally finite open covering $\mathscr{V}$ of X such that each $V \in \mathscr{V}$ intersects only finitely many elements of $\mathscr{U}$. For the undefined notions and symbols, the reader is refered to [3].

Remark 1. Notice that in the definitions above of Katětov and functionally Katětov spaces, we can replace the term "covering" by "family" and obtain equivalent definitions. A family $\{A_s\}_{s\in S}$ of subsets of F can be extended in the specified manner if and only if the covering $\{F\} \cup \{A_s\}_{s\in S}$ can be extended. We will use this equivalence freely in our proofs and constructions. ■

Remark 2. The implications in the first diagram are consequences of the following two, interesting in themselves, results:

THEOREM A ([2], [6]). *A normal space X is collectionwise normal and countably paracompact if and only if for every locally finite covering $\{A_s\}_{s\in S}$ of a closed subspace F of X there exists a locally finite open covering $\{V_s\}_{s\in S}$ of X such that $A_s \subset V_s$, for $s \in S$.*

THEOREM B [1]. *A normal space X is collectionwise normal if and only if every locally finite open covering of a closed subspace F of X has a refinement that can be extended to a locally finite open covering of X.*

Example 1 shows that Theorem A becomes false if the sets A_s are additionally assumed to be open in F and Examples 2 and 3 show that the sequence ... "has a refinement that" ... cannot be omitted in Theorem B.

Another interesting characterization of collectionwise normality has been obtained by Katětov.

THEOREM C [6]. *A normal space X is collectionwise normal if and only if every uniformly locally finite open covering of a closed subspace F of X can be extended to a locally finite open covering of X.*

Examples 2 and 3 show that the assumption of uniformity in Theorem C is essential. A simple example of a collectionwise normal space with a countable functionally open locally finite covering which is not uniformly locally finite is given in [10]. On the other hand we have

THEOREM D [6]. *Every locally finite covering of a collectionwise normal and countably paracompact space is uniformly locally finite.* ■

Remark 3. Extra set-theoretic assumptions can be used to strengthen our results. Assuming V = L one can construct a locally countable, locally compact and hereditarily separable functionally Katětov space which is not Katětov and a locally countable and hereditarily separable collectionwise normal space which is not functionally Katětov (cf. [13]). ■

§ 2. Proofs of the theorems.

Proof of Theorem 1. $\Leftarrow$ Let $\{F_s\}_{s\in S}$ be a discrete collection of closed subsets of X, $F = \bigcup_{s\in S} F_s$ and let $f_s\colon F \to I = [0,1]$ be defined for $s \in S$ by

$$f_s(x) = \begin{cases} 1, & \text{if } x \in F_s, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly the family $\mathscr{F} = \{f_s\}_{s\in S}$ is a locally finite partition of unity on F and hence we can find a partition of unity $\mathscr{G} = \{g_s\}_{s\in S}$ on X extending F. The open sets $U_s = \{x \in X\colon g_s(x) > \frac{1}{2}\}$ are clearly disjoint and $F_s \subset U_s$, for $s \in S$.

$\Rightarrow$ Let $\mathscr{F} = \{f_s\}_{s\in S}$ be a locally finite partition of unity on a closed subspace F of a collectionwise normal space X and let $B = l_1(S)$ be the Banach space of all sequences $z = \{z_s\}_{s\in S}$ of real numbers such that $\sum_{s\in S} |z_s| < \infty$, with the norm $\|z\| = \sum_{s\in S} |z_s|$.

Denote by $p_s\colon B \to R$ the continuous projections of B onto the real line R defined by $p_s(z) = z_s$ and consider the mapping $\Psi\colon F \to B$, where $\Psi(x) = \{f_s(x)\}_{s\in S}$ for $x \in F$. Let us observe that $f_s = p_s \circ \Psi$ and that the mapping Ψ is continuous. Indeed, for every $x_0 \in F$ and $\varepsilon > 0$ there exists a neighborhood U of x_0 in F and a finite subset $S_0 \subset S$ such that for $x \in U$

$$f_s(x) = 0, \quad \text{if} \quad s \in S \setminus S_0,$$

$$|f_s(x_0) - f_s(x)| > \frac{\varepsilon}{|S_0|} \quad \text{for} \quad s \in S_0.$$

Therefore, for $x \in U$ we have

$$\|\Psi(x_0) - \Psi(x)\| = \sum_{s\in S} |f_s(x_0) - f_s(x)| = \sum_{s\in S_0} |f_s(x_0) - f_s(x)| < |S_0| \frac{\varepsilon}{|S_0|} = \varepsilon.$$

The set $K = \{z \in B\colon \|z\| = 1\} \cap \bigcap_{s\in S} \{z \in B\colon p_s(z) \geqslant 0\}$ is a closed convex subset of B and clearly $\Psi(F) \subset K$. Since X is collectionwise normal there exists a continuous extension $\varphi\colon X \to K \subset B$ of Ψ onto X (see e.g. [7]). Let us put $g_s = p_s \circ \varphi\colon X \to I$. One easily sees that $g_s|F = f_s$ and $\sum_{s\in S} g_s(x) = \sum_{s\in S} p_s(\varphi(x)) = \|\varphi(x)\| = 1$, hence $\{g_s\}_{s\in S}$ is a partition of unity on X and extends F. ■

Remark 4. A similar proof shows that the following — more general — result holds (see e.g. [9] for the definition of a P-embedded subset):

THEOREM 1*. *A subset A of a space X is P-embedded if and only if every locally finite partition of unity on A can be extended to a partition of unity on X.*

Analogously, P^λ-embedded subsets can be characterized. ■

Proof of Theorem 3 is completely analogous. ■

Proof of Theorem 2. $\Leftarrow$ It follows from Theorem 1 that X is normal. Let $\mathscr{U} = \{U_s\}_{s\in S}$ be a locally finite functionally open covering of a closed subspace F

of X. For each $s\in S$ choose a continuous function $h_s\colon F\to I$ such that $h_s^{-1}((0,1]) = U_s$ and let $h\colon F\to R$ be defined by $h(x) = \sum_{s\in S} h_s(x)$. Clearly h is continuous and $h(x)>0$ for $x\in F$. The family $\mathscr{F} = \{f_s\}_{s\in S}$, where $f_s = h_s/h$, is a locally finite partition of unity on F and $f_s^{-1}((0,1]) = U_s$. Let $\mathscr{G} = \{g_s\}_{s\in S}$ be a locally finite partition of unity on X extending F. Clearly the locally finite open covering $\mathscr{V} = \{g_s^{-1}((0,1])\}_{s\in S}$ of X is an extension of $\mathscr{U}$.

$\Rightarrow$ Let $\mathscr{F} = \{f_s\}_{s\in S}$ be a locally finite partition of unity on a closed subspace F of a functionally Katětov space X and let $\mathscr{V} = \{V_s\}_{s\in S}$ be a locally finite open covering of X such that $f_s^{-1}((0,1]) = V_s\cap F$, for $s\in S$. Let $h_s^*\colon F\cup(X\setminus V_s)\to I$ be a continuous function defined by

$$h_s^*(x) = \begin{cases} 0, & \text{if } x\in X\setminus V_s,\\ f_s(x), & \text{if } x\in F.\end{cases}$$

Since X is normal, the functions h_s^* can be continuously extended onto X. Let $h_s\colon X\to I$ be an extension of h_s^*. Obviously, $h_s^{-1}((0,1])\subset V_s$ and therefore the function $h\colon X\to R$ defined by $h(x) = \sum_{s\in S} h_s(x)$ is continuous. Choose $s_0\in S$ and put

$$\varphi_s(x) = \begin{cases} h_s(x), & \text{if } s\neq s_0,\\ h_s(x)+|1-h(x)|, & \text{if } s = s_0.\end{cases}$$

Let $\varphi\colon X\to R$ be defined by $\varphi(x) = \sum_{s\in S}\varphi_s(x)$. Clearly $\varphi>0$ and the family $\mathscr{G} = \{g_s\}_{s\in S}$, where $g_s = \varphi_s/\varphi$ is a locally finite partition of unity on X extending F. ■

Proof of Theorem 4 is analogous. ■

Proof of Theorem 5. Let $\mathscr{U} = \{U_n\}_{n<\omega}$ be a locally finite open covering of a closed subspace F of a hereditarily normal space X. Let $G_n = \{x\in F\colon$ there exists a neighbourhood G_x of x in F such that $G_x\cap U_k = \emptyset$, for $k\geqslant n\}$. Clearly G_n's are open in F, $G_n\cap U_k = \emptyset$, for $k\geqslant n$ and $F = \bigcup_{n<\omega} G_n$. Since X is hereditarily normal we can find disjoint open subsets G_n^*, H_n^* of X such that $G_n^*\cap F = G_n$ and $H_n^*\cap F = \bigcup_{k\geqslant n} U_k$. The closed set $K = \bigcap_{n<\omega}(X\setminus G_n^*) = X\setminus\bigcup_{n<\omega} G_n^*$ is disjoint from F and we can find a neighbourhood H of F such that $\overline{H}\cap K = \emptyset$. Let $V_0 = U_0\cup X\setminus F$ and for each $n\geqslant 1$ choose an open subset V_n of X such that $V_n\cap F = U_n$ and $V_n\subset H\cap\bigcap_{i\leqslant n} H_i^*$. The open covering $\mathscr{V} = \{V_n\}_{n<\omega}$ of X extends $\mathscr{U}$ and is locally finite. Indeed, $X\setminus\overline{H}\supset K$ can intersect only V_0 and if $x\in X\setminus K$ then there exists an n such that $x\in G_n^*$. But

$$G_n^*\cap V_k\subset G_n^*\cap\bigcap_{i\leqslant k} H_i^*\subset G_n^*\setminus\bigcup_{i\leqslant k} G_i^* = \emptyset, \quad \text{if} \quad k\geqslant n+1. \ ■$$

§ 3. Construction of the examples. Since every locally finite open covering of a separable space is countable, we infer from Theorem 5 that every hereditarily normal and hereditarily separable space is Katětov.

In [5] a normal and hereditarily separable non-countably paracompact space Z is constructed assuming Continuum Hypothesis (Z can be made locally compact assuming $V = L$). The authors do not know, however, whether Z is hereditarily normal.

EXAMPLE 1 ($V = L$). A hereditarily normal, hereditarily separable, first countable, locally countable and locally compact space X which is not countably paracompact.

Since our construction is a modification of the example Z from [5], we omit many of the details. Let us recall, that $V = L$ implies $\omega_1 = 2^\omega$.

Let $X = \omega_1$ be the set of all countable ordinals and let $\Lambda\subset\omega_1$ denote the set of all limit ordinals $<\omega_1$. Find disjoint subsets L_n, $n<\omega$, of X such that

(1) $X = \bigcup_{n<\omega} L_n$, and

(2) for every $\lambda\in\Lambda$ and $n<\omega$ the set $\{k\colon \lambda+k\in L_n\}$ is infinite.

Sets L_n are schematically shown on the picture below.

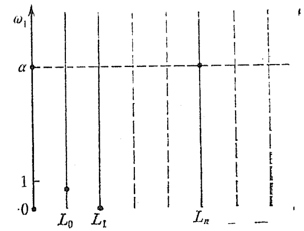

Fig. 1.

For each $\alpha\leqslant\omega_1$ and $n<\omega$ put $X_{\alpha,n} = \alpha\cup\bigcup_{i<n} L_i\subset X$ and let $\{A_{\alpha,n,\beta}\}_{\beta<\omega_1}$ be the family of all countable subsets of $X_{\alpha,n}$. One easily finds a function φ from Λ *onto* $(\omega_1+1)\times\omega\times\omega_1$ such that

(3) $A_{\varphi(\lambda)}\subset\lambda^*<\lambda$, where $\lambda^*\in\Lambda\cup\{0\}$.

The assumption of $V = L$ implies that there exists (see [8] or [12]) a family $\{S_\lambda\}_{\lambda\in\Lambda}$ of subsets of ω_1 such that

(4) S_λ is cofinal in λ,

(5) if $S\subset\omega_1$ is uncountable, then there exists a $\lambda\in\Lambda$ such that $S_\lambda\subset S$, and obviously we can assume additionally that

(6) for each $\lambda\in\Lambda$ there exists an $m(\lambda)<\omega$ such that $S_\lambda\subset L_{m(\lambda)}$.

By induction on Λ we construct for each $\mu\in\Lambda$ a topology τ_μ on μ and for each $\alpha<\mu$ a family $\mathscr{B}_\alpha = \{B_k(\alpha)\}_{k<\omega}$ of subsets of X satisfying the following conditions (let us notice that sets $B_k(\alpha)$ do not depend on μ):

$(7)_\mu$ τ_μ is metrizable and for each $\alpha<\mu$ the family $\mathscr{B}_\alpha$ is a base of compact neighborhoods of α in τ_μ,

$(8)_\mu$ $B_k(\alpha)\subset(\alpha+1)\cap\bigcup_{i\leqslant n}L_i$ for $k<\omega$ and $\alpha\in L_n$,

$(9)_\mu$ for each limit $\lambda<\mu$ and α such that $\lambda\leqslant\alpha<\mu$, if $\alpha\in L_n$ and $n\geqslant m(\lambda)$, then α belongs to the closure of S_λ in τ_μ,

$(10)_\mu$ for each limit $\lambda<\mu$, if $\varphi(\lambda)=(\alpha,n,\beta)$ and $A_\lambda=A_{\alpha,n,\beta}$ is closed in $\lambda\cap X_{\alpha,n}$, then there exists a set V_λ (independent of μ) such that $A_\lambda\subset V_\lambda\subset\lambda^*$ and V_λ is a closed-and-open subset of $X_{\alpha,n}\cap\mu$ (both $\lambda\cap X_{\alpha,n}$ and $\mu\cap X_{\alpha,n}$ are considered as subspaces of the space (μ,τ_μ)).

For $\mu=\omega$ and $m,k<\omega$ we put $B_k(m)=\{m\}$ and the conditions $(7)_\omega$-$(10)_\omega$ are clearly satisfied.

Assume that for each limit $\lambda<\mu\in A$ the topology τ_λ satisfying conditions $(7)_\lambda$-$(10)_\lambda$ has been constructed. If μ is a limit of limit ordinals, then the topology τ_μ induced on μ by the bases $\mathscr{B}_\alpha$ of points $\alpha<\mu$ also satisfies conditions $(7)_\mu$-$(10)_\mu$.

Otherwise, $\mu=\lambda+\omega$ for some limit λ and it is sufficient to construct bases $\mathscr{B}_{\lambda+m}=\{B_k(\lambda+m)\}_{k<\omega}$ of points $\lambda+m$, for $m<\omega$ in such a way that the conditions $(7)_\mu$-$(10)_\mu$ are satisfied.

If $\varphi(\lambda)=(\alpha,n,\beta)$ and the set $A_\lambda=A_{\alpha,n,\beta}$ is closed in $\lambda\cap X_{\alpha,n}$, then since λ is countable and metrizable, λ^* is open in λ and $A_\lambda\subset\lambda^*<\lambda$, there exists an open-and-closed subset V_λ of $\lambda\cap X_{\alpha,n}$ such that $A_\lambda\subset V_\lambda\subset\lambda^*$. If A_λ is not closed we skip this step. Let us enumerate by $\varkappa_1,\varkappa_2,\ldots$ all limit ordinals $\varkappa\leqslant\lambda$ for which the sets $V_\varkappa$ have been constructed (in this and preceding steps).

By (2) and (4) there exists an increasing sequence $T=\{\alpha_s\}_{s<\omega}$ of ordinals converging to λ such that

(i) $T\cap L_n$ is infinite for every $n<\omega$;

(ii) $T\cap S_\lambda$ is infinite.

Clearly, we can decompose T into countably many subsequences T_m, $m<\omega$, each of which satisfies counterparts of (i) and (ii).

Since the set T is closed and discrete in λ there exists a discrete in λ collection $\{G_s\}_{s<\omega}$ of basic neighborhoods of points α_s in λ and we can assume that

(iii) $G_s\cap V_{\varkappa_m}=\emptyset$, for every $m\leqslant s$ such that $\alpha_s\notin\mathrm{cl}_\lambda V_{\varkappa_m}$.

Let $k,m<\omega$ and let $\lambda+m\in L_n$. We define sets $B_k(\lambda+m)$ by putting:

$$B_k(\lambda+m)=\{\lambda+m\}\cup\bigcup\{G_s\colon s\geqslant k\text{ and }\alpha_s\in T_m\cap\bigcup_{i\leqslant n}L_i\}.$$

One easily sees that conditions $(7)_\mu$-$(10)_\mu$ are satisfied.

The space X with the topology determined by bases $\mathscr{B}_\alpha$ of neighborhoods of points $\alpha<\omega_1$ is locally countable, locally compact, first countable and has the following properties:

(11) For each uncountable $S\subset L_n$ there exists a $\lambda\in A$ such that
(i) $S_\lambda\subset S$,
(ii) $S\setminus\lambda\subset X\setminus(\lambda\cup\bigcup_{i<n}L_i)=X\setminus X_{\lambda,n}\subset\bar{S}_\lambda$ (use (5) and (9)).

(12) X is hereditarily separable.

(13) The sets $F_n=\bigcup_{k>n}L_k$ are closed and $\bigcap_{n<\omega}F_n=\emptyset$ (use (8)).

(14) For each open U containing F_n the set $L_1\setminus U$ is countable (use (11)).

(15) X is not countably paracompact.

(16) For each $\alpha\leqslant\omega_1$, $n<\omega$ and a countable closed subset A of $X_{\alpha,n}$ there exists a countable closed-and-open subset V of $X_{\alpha,n}$ containing A (use (10)).

(17) For each $Y\subset X$ and two disjoint closed subsets A and B of Y such that $\mathrm{cl}_X A$ and $\mathrm{cl}_X B$ are uncountable there exist $\lambda<\omega_1$ and $n<\omega$ so that $Y\subset Z=X_{\lambda,n}$ and either $\mathrm{cl}_Z A$ or $\mathrm{cl}_Z B$ is countable.

Indeed, let n_1 be the first natural number such that $\mathrm{cl}_X A\cap L_{n_1}$ is uncountable and let n_2 be the first natural number such that $\mathrm{cl}_X B\cap L_{n_2}$ is uncountable. By (11) there exist $\lambda_1,\lambda_2\in A$ such that $X\setminus X_{\lambda_1,n_1}\subset\mathrm{cl}_X A$ and $X\setminus X_{\lambda_2,n_2}\subset\mathrm{cl}_X B$. Let $n=\max\{n_1,n_2\}$ and $\lambda=\max\{\lambda_1,\lambda_2\}$. Clearly $X\setminus X_{\lambda,n}\subset\mathrm{cl}_X A\cap\mathrm{cl}_X B$ and since A and B are closed and disjoint in Y, it follows that $Y\subset X_{\lambda,n}$.

(18) X is hereditarily normal.

Let A and B be closed disjoint subsets of a subspace Y of X. By (17) there exist $\alpha\leqslant\omega_1$ and $n<\omega$ such that $Y\subset Z=X_{\alpha,n}$ and either $\mathrm{cl}_Z A$ or $\mathrm{cl}_Z B$ is countable. Without loss of generality we can assume that $\mathrm{cl}_Z A$ is countable. By (16) there exists a closed-and-open countable subset V of Z containing $\mathrm{cl}_Z A$. Since V is metrizable, there exist disjoint open subsets U and G of Y such that $V\supset U\supset A$ and $G\supset V\cap B$. Clearly $B\subset G\cup Y\setminus V$ and $(G\cup Y\setminus V)\cap U=\emptyset$.

This completes the proof of the properties of Example 1. ■

Example 2. M. E. Rudin's Dowker space X [11] is a functionally Katětov space which is not Katětov.

Proof. We recall first the definition of X. Let $F=\{f\colon N\to\omega_\omega\colon f(n)\leqslant\omega_n$, for all $n\in N\}$ and let $X=\{f\in F\colon$ there exists an $i\in N$ such that $\omega<\mathrm{cf}(f(n))<\omega_i$ for all $n\in N\}$. Suppose f and g belong to F. If $f(n)<g(n)$ for all $n\in N$, we write $f<g$ and if $f(n)\leqslant g(n)$, for all $n\in N$ then we write $f\leqslant g$. The family of all sets

$$U_{f,g}=\{h\in X\colon f<h\leqslant g\},$$

where $f,g\in F$ is a basis for topology on X.

Since every F_σ subset of X is closed ([11], Lemma 4) every functionally open subset of a closed subspace K of X is closed-and-open in K, which easily implies that X is functionally Katětov.

Indeed, let $\{U_s\}_{s\in S}$ be a locally finite clopen covering of K. For every $x\in K$ let $V(x)=\bigcap\{U_s\colon x\in U_s\}\setminus\bigcup\{U_s\colon x\notin U_s\}$. The family $\{V(x)\}$ decomposes K into a disjoint collection of clopen subsets. Since X is collectionwise normal there exists a discrete in X collection $\{G(x)\}$ of open subsets of X such that $V(x)\subset G(x)$. The locally finite family $\{U_s^*\}_{s\in S}$, where $U_s^*=\bigcup\{G(x)\colon x\in U_s\}$, is the required extension.

We prove that X is not Katětov.

Let $K = \{f \in X \mid f(n) = \omega_n$ for some $n > 1\}$, and for each $n > 1$ set $V_n = \{f \in X \mid f(i) = \omega_i$ if and only if $i = n$ or $i = 1\}$. Clearly K is closed in X and each V_n is an open subset of K. The collection $\{V_n \mid n > 1\}$ is locally finite since if $f \in X$, then $f(i) < \omega_i$ for all i sufficiently large, so $\{g \in X \mid g \leqslant f\}$ is an open set containing f that intersects only a finite number of elements of $\{V_n \mid n > 1\}$. Fix $n > 1$ and assume that U_n is open in X and $V_n \subset U_n$. We claim there exists $g_n \in X - K$ such that $\{f \in X - K \mid g_n < f\} \subset U_n$. Suppose the claim is false. Then we can inductively choose a set $\{f_\alpha \mid \alpha < \omega_n\} \subset X - (K \cup U_n)$ such that $f_\beta(i) < f_\alpha(i)$ whenever $i \geqslant n$ and $\beta < \alpha < \omega_n$. (The induction proceeds as follows. Having chosen $\{f_\alpha \mid \alpha < \varkappa\}$ for some $\varkappa < \omega_n$, define g by $g(i) = \omega_1$ when $i < n$ and $g(i) = \sup_{\alpha < \varkappa} f_\alpha(i)$ for $i \geqslant n$. The set $\{f \in X - K \mid g < f\}$ is guaranteed to be nonempty by the assumption that the claim is false. Let $f_\varkappa$ be any element of $\{f \in X - K \mid g < f\}$.) Since there are only ω_{n-1} functions from n into ω_{n-1}, there exist $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that $\{\alpha \mid f_\alpha(i) = \alpha_i$ for all $i < n\}$ has cardinality ω_n. Define $f \in X$ by $f(i) = \alpha_i$ for each $i < n$, and $f(i) = \sup\{f_\alpha(i) \mid \alpha < \omega_n\}$ for $i > n$. Notice that $f \in V_n \subset U_n$, but f is in the closure of $\{f_\alpha \mid \alpha < \omega_n\}$. This contradicts the assumption that U_n is open, and hence proves the claim. Now suppose that $\{U_n \mid n > 1\}$ is an open extension of $\{V_n \mid n > 1\}$ to X. For each $n > 1$, let g_n be the function guaranteed by the claim above. Define $g \in F$ by $g(i) = \sup\{g_n(i) \mid n > 1\}$. Then $\emptyset \neq \{f \in X - K \mid f > g\} \subset \bigcap \{U_n \mid n > 1\}$, so $\{U_n \mid n > 1\}$ is not locally finite and it follows that X is not Katětov. ■

Example 3. A collectionwise normal space that is not functionally Katětov. For all $n > 0$, define

$$W_n = \{f \in X \mid f(i) = \omega_i \text{ iff } i \leqslant n \text{ and } \forall i(\mathrm{cf}(f(i)) \leqslant \omega_n)\}.$$

Let $W = \bigcup \{W_n\}_{n>0}$ and give W the subspace topology from X. We will use W to construct a collectionwise normal space, Z, that is not functionally Katětov.

We assume the reader is familiar with Rudin's proof that X is a Dowker space. We need a few lemmas concerning W and its subspaces.

Lemma A. *Let I be a subset of positive integers and set $W^* = \bigcup \{W_m:\ m \in I\}$. Then W^* is a collectionwise normal subspace of W.*

Proof. Let $\mathscr{H}$ be a discrete collection of pairwise disjoint closed subsets of W^*. Since W^* is a subspace of X, if we can separate the collection $\mathscr{H}$ in X we can certainly separate it in W^*. Rudin's proof that X is collectionwise normal shows that some collections called $\mathscr{H}$ can be separated, so we will try it on this $\mathscr{H}$. Follow Rudin's argument and notice that everything works perfectly except for the phrase "since $g \in X$", on line 13 of the last page of the proof of Lemma 6. She uses the fact that $g \in X$ and $\mathscr{H}$ is discrete in X to conclude that there exists a neighbourhood of g that intersects at most one element of $\mathscr{H}$. For each n we must either find a different proof of Lemma 6 or show that $g \in W^*$. For each n such that $U_n \neq U_{n-1}$ (for $n = 1$ this should be interpreted to mean $U_n \neq \emptyset$) it is indeed true that $g \in W^*$. To see this, note that $U_n \neq U_{n-1}$ implies $W_n \subset W^*$ and, for all $i \leqslant n$, $g(i) = t(i) = \omega_i$ (since $W_n \cap U_n \neq \emptyset$ and for every $f \in W_n$ and every $i \leqslant n$ $f(i) = \omega_i$), so $g \in W_n \subset W^*$.

Now we can prove Rudin's Lemma 6 holds for our collection $\mathscr{H}$ by induction. Assume the desired function g exists for all integers less than some fixed n. By the comments above, if $U_n \neq U_{n-1}$ Rudin's proof shows that the nth desired function also exists. If $U_n = U_{n-1}$, then we can simply let the function g of step n be the same as the function g defined at the step $n-1$ (or if $n = 1$, $g \equiv 0$). ■

Lemma B. *For each $m > 1$, $W_1 \cup W_m$ is countably paracompact.*

Proof. Fix $m > 1$ and assume that $\{F_k\}_{k \in N}$ is a decreasing sequence of closed subsets of $W_1 \cup W_m$ having empty intersection. We modify Rudin's proof that X is collectionwise normal by constructing for each $\alpha < \omega_1$, a cover $\mathscr{T}_\alpha$ of F_1 consisting of disjoint open sets such that the following conditions hold:

If $\beta < \alpha < \omega_1$ and $V \in \mathscr{T}_\alpha$, then there is a $U \in \mathscr{T}_\beta$ such that:

(1) $V \subset U$,

(2) if V intersects infinitely many F_k, then $t_V \neq t_U$, and

(3) if U intersects finitely many F_k, then $U = V$.

Proceed as in Rudin's construction, replacing statements such as "intersects more than one term of $\mathscr{H}$" by "intersects infinitely many F_k". The only part of the construction that needs modification is the proof of Lemma 6, which is handled as in the proof of Lemma A above. Thus the sequence $\{\mathscr{T}_\alpha\}_{\alpha < \omega_1}$ exists.

Let $U_{\alpha f}$ be defined from $\{\mathscr{T}_\alpha\}$ as in [11]. Set $U_k = \bigcup \{U_{\alpha f}:\ f \in F_k\}$. Then $F_k \subset U_k$ for each k and $\bigcap U_k = \emptyset$. Thus $W_1 \cup W_m$ is countably paracompact. ■

Note. Let $m > 1$ and assume $\{B_k\}_{k \in N}$ is a locally finite collection of closed subsets of the space $W_1 \cup W_m$. Then, the family $\{\mathrm{cl}_W B_k\}_{k \in N}$ is locally finite in W. To prove this, let $t \in W \setminus (W_1 \cup W_m)$ and set $U = \{f \in W:\ f \leqslant t\}$ and $F_k = B_k \cap U$. Then we can apply the proof of Lemma B to $\{F_k\}_{k \in N}$. In the construction of $\mathscr{T}_1$ it is shown that there exists a $g < t$ such that $U_{g,t} = \{f \in W:\ g < f \leqslant t\}$ intersects only finitely many F_k and $U_{g,t}$ is open in W. ■

Lemma C. *Fix $n > 1$ and let U be open in the subspace $W_1 \cup W_n$. If $W_n \subset U$, then there exists $f \in W_1$ such that $\{g \in W_1:\ f < g\} \subset U$.*

Proof. We modify the proof of Lemma 6 of [11]. Assume the lemma is false, that is, for each $f \in W_1$ there exists an $h \in W_1$ such that $f < h$ and $h \notin U$. Define $s_{j,r_\lambda(i)}$ as in Lemma 6 of [11] in the case where for all $i \in N$ $t(i) = \omega_i$ and n is the n in the statement of our lemma.

For each ordinal $\lambda < \omega_n$ we inductively select an $f_\lambda \in F$ and $h_\lambda \in W_1 \setminus U$ as follows. Define f_0 by letting $f_0(j) = s_{j,r_0(j)}$ for $j \leqslant n$ and $f_0(j) = 0$, for $j > n$. Then choose $h_0 \in W_1 \setminus U$ such that $f_0 < h_0$. Fix $\lambda < \omega_n$ and assume $h_\gamma \in W_1 \setminus U$ has been chosen for each $\gamma < \lambda$. Define f_λ by letting $f_\lambda(j) = s_{j,r_\lambda(j)}$, for $j \leqslant n$ and $f_\lambda(j) = \sup\{h_\gamma(j)\}_{\gamma < \lambda}$ for $j > n$. Then $f_\lambda \in F$ and there is $g_\lambda \in W_1$ with $f_\lambda \leqslant g_\lambda$, so we can fix a function $h_\lambda \in W_1 \setminus U$ with $h_\lambda > g_\lambda \geqslant f_\lambda$.

Define $g \in F$ by letting $g(j) = \omega_j$ for $j \leqslant n$ and $g(j) = \sup\{h_\lambda(j)\}_{\lambda < \omega_n}$ for $j > n$. Then $g \in W_n$ yet g is in the closure of $\{h_\lambda\}_{\lambda < \omega_n} \subset W_1 \setminus U$. Hence U is not an open neighbourhood of W_n. This contradiction proves the lemma. ■

We now modify W to obtain the desired example, Z. For $n, m \in N$ and $A \subset W$, let $A^{n,m}$ denote $A \times \{(n, m)\}$ and A^n denote $A \times \{n\}$. Define

$$Z = W \cup \bigcup \{W_1^{n,m}\colon n, m \in N, m > 1\} \cup \bigcup \{W_k^m\colon k > 1, m \in N\}.$$

We generate a base for Z from the open sets in W. For each U, open in W, and each $n, m \in N$ with $m > 1$ and sequence of integers $\{k_j\}_{j \in N}$, the following two sets are declared to be basic open subsets of Z:

$$(U \cap W_1)^{n,m} \cup (U \cap W_m)^n$$

and

$$U \cup \bigcup \{(U \cap W_1)^{i,j}\colon i > k_j\} \cup \bigcup \{(U \cap W_j)^i\colon i > k_j\}.$$

Z is not functionally Katětov. Let $K = Z \setminus (W_1 \cup \bigcup \{W_1^{n,m}\}_{n,m \in N})$ and for each $m > 1$, let $V_m = \bigcup \{W_m^n\}_{n \in N}$. Then $\{V_m\}_{m > 1}$ is a locally finite functionally open family in K. Yet suppose that for each $m > 1$, U_m is open in Z and $U_m \cap K = V_m$. Then, by Lemma C, for each n and m there exists $f_{n,m} \in W_1$ such that $\{g \in W_1 : g > f_{n,m}\}^{n,m} \subset U_m$. But then if f is defined by $f(i) = \sup\{f_{n,m}(i)\}_{n,m \in N}$, for each n and m, $\{g \in W_1 : g > f\}^{n,m} \subset U_m$, and hence $\{g \in W_1\colon g > f\} \subset \mathrm{cl}_Z U_m$ for each m. Thus $\{U_m\}_{m > 1}$ is not locally finite and Z is not functionally Katětov.

Z is collectionwise normal. Let $\{H_\alpha\}_{\alpha < \lambda}$ be a discrete collection of closed subsets of Z. If $\bigcup \{H_\alpha\}_{\alpha > \lambda} \subset W$, then a separation in W gives rise to a separation in Z. If $\bigcup \{H_\alpha\}_{\alpha < \lambda} \subset Z \setminus W$, then the H_α's can be separated by disjoint open sets since $Z \setminus W$ is the disjoint union of the collectionwise normal (by Lemma A) and clopen sets $W_1^{n,m} \cup W_m^n$. It is easy to show that we will have proved the collectionwise normality of Z once we know that if $H \subset Z \setminus W$ and H is closed, then there exists an open set, U, containing W such that $\overline{U} \cap H = \emptyset$. For each $m > 1$, let

$$A_m = W_1 \cup W_m \cup \bigcup \{W_1^{n,m} \cup W_m^n\}_{n \in N}.$$

Then A_m is normal since it is homeomorphic to $(W_1 \cup W_m) \times (\omega + 1)$ and $W_1 \cup W_m$ is normal and countably paracompact. Hence for each $m > 1$, there is a U_m, open in A_m such that $\overline{U}_m \cap H = \emptyset$ and $W_1 \cup W_m \subset U_m$. Let $U = \bigcup \{U_m\}_{m > 1}$.

The collection $\{U_m\}_{m > 1}$ is locally finite at points in $Z \setminus W$, and hence $H \cap \overline{U} = \emptyset$. Since $W \subset U$ by construction, all we need do is show that U is a neighbourhood of W.

Let $f \in W$. We will show that f is in the interior of U. Set

$$B_m^n = (W_m^n \setminus U) \cup (W_1^{n,m} \setminus U).$$

Let $\pi(B_m^n)$ be the projection of B_m^n into W. Then since U is a neighbourhood of $W_1 \cup W_m$ in A_m, the family $\{\pi(B_m^n)\}_{n \in N}$ is locally finite in $W_1 \cup W_m$. Thus, by the note after Lemma B, the family $\{\mathrm{cl}_W \pi(B_m^n)\}_{n \in N}$ is also locally finite. This implies that $f \notin \mathrm{cl}_Z(\bigcup \{B_m^n\}_{n \in N})$, so there exists an open neighbourhood V_m of f in W and $k_m \in N$ such that $V_m \times \{i\}_{i > k_m}$ is disjoint from $\bigcup \{B_m^n\}_{n \in N}$, i.e. $V_m \times \{i\}_{i > k_m} \subset U$. Let $V = \bigcup \{V_m\}_{m > 1}$. Then V is open in W (by Lemma 4 of [11]) so

$$V \cup \bigcup \{(V \cap W_1)^{i,j}\colon i > k_j\} \cup \bigcup \{(U \cap W_j)^i\colon i > k_j\}$$

is a basic open subset of Z that contains f and is contained in U. Thus U is open in Z. ■

QUESTIONS

1. Is there an absolute example of a Katětov space that is not countably paracompact? Is the space W constructed above an example of such a space?

2. Is every hereditarily collectionwise normal space Katětov?

3. Is every collectionwise normal countably Katětov space Katětov?

It follows from Theorem 5 that a positive answer to 3 would imply a positive answer to 2. Let us notice that Bing's Example (see [3], Example 5.1.23) is hereditarily normal but not collectionwise normal, hence it is not Katětov.

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
Warszawa
YALE UNIVERSITY
New Haven, Connecticut
and
THE INSTITUTE FOR MEDICINE AND MATHEMATICS
OHIO UNIVERSITY
Athens, Ohio

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