Calculating self-referential statements

by

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Abstract. The successful application of self-reference in metamathematics has been somewhat overshadowed by the even greater success of recursion theory and the occasional application of the latter in metamathematics. This overshadowing has a two-fold negative effect on metamathematical investigations: First, the resulting de-emphasis of self-reference in expositions creates a vacuum in which every use of self-reference is met with surprise. Second, the illusion of the greater applicability and ease of application of recursion theoretic techniques to metamathematical problems is nurtured. The author attempts to set the record straight: Applications of self-reference usually require little imagination and, it appears, yield stronger results than those obtained by a comparable amount of recursion theoretic effort.

To establish his thesis that applications of self-reference require little imagination, the author offers a couple of explanations. The shallowest is the statistical evidence that most known applications rely on instances of the same self-referential formula — one first introduced by Shepherdson and termed the Shepherdson fixed point by the author. The explanation for this seems not to be the lack of imagination of the many users of the Shepherdson fixed point — a glance at the bibliographies of the various papers shows that most were unaware of Shepherdson's work — but in a very useful fact about this fixed point: It is a very simple matter to determine (the calculation of the title) exact conditions on the provability and refutability of its numerical instances and those of certain of its variants. To illustrate the importance of this fact for applications, the author systematically considers several variants of the Shepherdson fixed point and performs the corresponding calculations — thereby enabling him to routinely obtain a number of new results and refinements of old ones.

The author's contention that application of internal diagonalization (i.e. self-referential formulae) is superior to that of external diagonalization (i.e. recursion theory) is, if only temporarily, an established fact: He offers a number of examples (complete with fanfare) of results obtainable both via internal and external diagonalization for which the natural internal diagonalization yields immediate refinements which either require deeper recursion theoretic results for their proofs via external diagonalization or are not yet so obtainable.

0. Introduction. Initially, the application of diagonalization to metamathematics was limited to the simulation of various paradoxes to prove incompleteness results or other, philosophically oriented (or even philosophically pretentious), results. Since Bernays [1], the scene has changed. Self-referential formulae have been cleverly used in a variety of ways to yield a number of (occasionally surprising) results. In Part I [33], we promised to review and extend our knowledge of post-Bernays applications of diagonalization, and to explain away the cleverness behind them by means of "non-explicit" calculations of the self-referential formulae involved. This is actually quite an easy task: An examination of the literature reveals that most applications of diagonalization use minor variants of the same self-referential formula.

The self-referential formula in question first made its appearance in [32]. Briefly put, the reason for the popularity of Shepherdson's self-referential formula is the control one has over its provability and refutability. The verification of this control constitutes a non-explicit calculation of the self-referential formula, or fixed-point. Following some preliminaries in Section 1, we devote Sections 2-5 to the calculation and application of variants of Shepherdson's fixed-point. In Section 2, we calculate the basic Shepherdson fixed-point and give applications to the metamathematics of consistent r.e. extensions of Robinson's theory $\mathscr{R}$. In Section 3, we generalize the fixed-point, the calculation, and the applications to r.e. sequences of such theories. In Section 4, the basic Shepherdson fixed-point is modified for the purpose of giving non-uniform results. In Section 5, a functorially generalized Shepherdson fixed-point appears. Finally, in Section 6, we consider a few other closely related fixed-points. We should mention that it is possible to combine the features of the fixed-points of Sections 2-4 and 6 into two general fixed-points, which we might call left and right quasi-uniform modified partial Shepherdson fixed-points, and give only the two calculations. For the sake of exposition, however, we take a more concrete approach.

The types of applications included are three-fold: i. basic incompleteness results (Rosser [29], Mostowski [21], this paper); ii. results on semi-representability in r.e. theories (Shepherdson [32], this paper); and iii. characterizations of Σ_1-soundness among r.e. theories (Friedman [6], Jensen and Ehrenfeucht [12], Guaspari [9], this paper). Not all of the results presented nor their proofs are new: If we are to convince the reader that non-explicit calculations offer a usable guideline to the applications of fixed-points, we must demonstrate that they routinely: i. account for many known applications of fixed-points; ii. yield new proofs of results originally proven by other means; and iii. allow us to obtain new results.

The "other means" referred to are recursion-theoretic diagonalizations external to the formal theory. Such recursion-theoretic proofs are often prima facie simpler than their formal counterparts. There seems to be a reason for this: The simple proofs only yield special cases of the results obtainable. (Compare the recursion-theoretic results of Ehrenfeucht and Feferman [3], Putnam and Smullyan [26], and Hájková and Hájek [10] with those obtained by formal diagonalization in Shepherdson [32].) With respect to self-referential formulae, the special results cannot be considered that special — for, we have routinely obtained improvements not previously accessible to recursion-theoretic techniques. (Compare: a. nothing in the literature with the uniform semi-representability result of Section 3; b. the relevant partial results of Di Paola [24] and [25] with those of Section 4; and c. the result of Ritchie and Young [27] with the sharp uniform result of Section 5.) We say "not previously accessible" because we have, via a devious trick, been able to extend the recursion-theoretic proofs to obtain some of the improved results — cf. Smoryński [34]. Some results still have no recursion-theoretic proofs.

With respect to this dichotomy of methods, we should note that P. Päppinghaus has devised a scheme for transforming the recursion-theoretic diagonalizations into formal ones.

The new results of this paper (and, indeed, its general orientation) grew out of our background research on a book on the Metamathematics of Arithmetic for Springer-Verlag. More detailed proofs of some results are deferred to this book.

We offer our thanks to: Dave Guaspari for his inspiring notation and informative correspondence; Peter Päppinghaus for an informative discussion of diagonalization; and to Gert H. Müller for having provided us with an environment conducive to work.

1. Preliminaries. This section is devoted to notation and a review of basic results of which we will make heavy use in the sequel.

Language. For convenience, we consider only the usual arithmetical languages — each including a numeral $\bar{x}$ for each natural number x. Arithmetical formulae are denoted by the lower case Greek letters φ, ψ, χ, θ. Special classes of arithmetical formulae are singled out: The class of Δ_0 formulae is the smallest class of arithmetical formulae containing the atomic formulae and closed under application of propositional connectives and bounded quantification. The class of Σ formulae is the smallest class of formulae containing the Δ_0 formulae and closed under conjunction, disjunction, bounded quantification, and existential quantification. The class of Σ_1 formulae is the subclass of the class of Σ formulae consisting of the formulae of the form $\exists v\varphi$, $\varphi \in \Delta_0$.

Theories. Depending on the desired extent of the arithmetization of metamathematics, we can appeal to any of a number of basic formal theories in which to work. The basic theory for applications of the sorts i (incompleteness) and ii (semi-representability), as outlined in the Introduction, is Robinson's $\mathscr{R}$. Results of sort iii (characterization of Σ_1-soundness) require some induction. For this purpose, we use the wastefully powerful Peano arithmetic, $\mathscr{PA}$. When we need to formalize an argument, we work in a definitional extension of $\mathscr{PA}$ by constants naming primitive recursive functions.

Σ-Completeness. One of the two reasons behind the popularity of $\mathscr{R}$ as a basic theory is its ability to prove every true Σ sentence. This result is the following theorem, which we cite with two important corollaries:

Σ-COMPLETENESS THEOREM. *Let φ be a Σ sentence. Then $N \models \varphi \Rightarrow \mathscr{R} \vdash \varphi$.*

π-SOUNDNESS THEOREM (Significance of consistency). *Let $\mathscr{T}$ be a consistent extension of $\mathscr{R}$ and let φ be a π sentence (i.e. $\varphi = \neg\psi$ for some $\psi \in \Sigma$). Then, $\mathscr{T} \vdash \varphi \Rightarrow N \models \varphi$.*

Δ_0-COMPLETENESS THEOREM. *Let $\mathscr{T}$ be a consistent extension of $\mathscr{R}$ and let φ be a Δ_0 sentence. Then*

i. $N \models \varphi \Leftrightarrow \mathscr{T} \vdash \varphi$,

ii. $N \models \neg\varphi \Leftrightarrow \mathscr{T} \vdash \neg\varphi$.

Comparison of witnesses. Any number x satisfying $\varphi\bar{x}$ is said to be a *witness* to $\exists v_0\,\varphi v_0$. The method of comparison of witnesses is as useful in metamathematical arguments as in recursion-theoretic ones and, for this reason, we incorporate the notation of Guaspari [9]. If $\varphi = \exists v_0\,\theta v_0$, $\psi = \exists v_0\,\chi v_0$, we define

$$\varphi \preccurlyeq \psi\colon\ \exists v_0\,[\theta v_0 \wedge \forall v_1 < v_0\, \neg\chi v_1]\,,$$
$$\varphi \prec \psi\colon\ \exists v_0\,[\theta v_0 \wedge \forall v_1 \leqslant v_0\, \neg\chi v_1]\,.$$

$\varphi\preccurlyeq\psi$ asserts that φ is witnessed and, moreover, it is witnessed at least as early as ψ is witnessed. $\varphi\prec\psi$ asserts that φ is witnessed and, in fact, it is witnessed earlier than ψ is witnessed. (N. B. By the existential import of the notation, $\preccurlyeq$, we do not necessarily have $\varphi\preccurlyeq\varphi$ — this sentence is true only when φ is. When restricted to true existentially quantified sentences, however, $\preccurlyeq$ becomes a prewellordering — if not provably so in weak theories. (We should mention that the need to assert comparability, $(\varphi\preccurlyeq\psi)\vee(\psi\prec\varphi)$, on assumption of φ or ψ, is one of the reasons for occasionally restricting our attention to extensions of $\mathscr{PA}$: Note that, for $\varphi = \exists v_0\,\theta v_0$, the least number principle for θ is just $\varphi \to .\varphi\preccurlyeq\varphi$.)) We shall be especially fond of applying this notation in the Σ_1 case: Note that, if φ, ψ are Σ_1, then so are $\varphi\preccurlyeq\psi$, $\varphi\prec\psi$. Finally, an occurrence of $\varphi\vee\psi$ in a context, $(\varphi\vee\psi)\preccurlyeq\tau$ or $\tau\preccurlyeq(\varphi\vee\psi)$, is assumed rewritten with only one outer existential quantifier via the equivalence $\varphi\vee\psi \leftrightarrow \exists v_0(\theta v_0 \vee \chi v_0)$.

Reduction theorem. The second reason for the popularity of $\mathscr{R}$ is the formalization within $\mathscr{R}$ of a syntactic version of the Reduction Theorem.

SYNTACTIC REDUCTION THEOREM. *Let* $\varphi v_0 \ldots v_{n-1}, \psi v_0 \ldots v_{n-1} \in \Sigma_1$. *Then, for all* $x_0, \ldots, x_{n-1}$,

i.a. $N \models (\varphi\preccurlyeq\psi)\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash (\varphi\preccurlyeq\psi)\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash \varphi\bar{x}_0 \ldots \bar{x}_{n-1}$,

b. $N \models (\psi\prec\varphi)\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash (\psi\prec\varphi)\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash \psi\bar{x}_0 \ldots \bar{x}_{n-1}$,

ii.a. $N \models (\varphi\preccurlyeq\psi)\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash \neg(\psi\prec\varphi)\bar{x}_0 \ldots \bar{x}_{n-1}$,

b. $N \models (\psi\prec\varphi)\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash \neg(\varphi\preccurlyeq\psi)\bar{x}_0 \ldots \bar{x}_{n-1}$,

iii. $N \models \varphi\bar{x}_0 \ldots \bar{x}_{n-1}$ *or* $N \models \psi\bar{x}_0 \ldots \bar{x}_{n-1} \Rightarrow \mathscr{R} \vdash (\varphi\preccurlyeq\psi)\bar{x}_0 \ldots \bar{x}_{n-1}$ *or* $\mathscr{R} \vdash (\psi\prec\varphi)\bar{x}_0 \ldots \bar{x}_{n-1}$.

Semi-representability. Let $\mathscr{T}$ be an arithmetical theory. A formula $\varphi v_0 \ldots v_{n-1}$ semi-represents a relation $R \subseteq N^n$ if, for all $x_0, \ldots, x_{n-1} \in N$, we have

$$\mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_{n-1} \Leftrightarrow Rx_0 \ldots x_{n-1}\,.$$

φ represents R if φ, $\neg\varphi$ semi-represent R, $\neg R$, respectively. The representation of a function $f\colon N^n \to N$ by a formula $\varphi v_0 \ldots v_0$ requires that i. φ represents the graph of f, and ii. φ satisfies a weak functionality condition: For all $x_0, \ldots, x_n$, if $x_n = fx_0 \ldots x_{n-1}$, $\mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_{n-1}v \to v = \bar{x}_n$. The Σ_1-soundness of $\mathscr{R}$ together with Σ_1 completeness yields the semi-representability of all r.e. relations in $\mathscr{R}$. In the presence of the Syntactic Reduction Theorem, this yields the representability of all recursive relations and functions in $\mathscr{R}$ and all of its consistent extensions.

Codes. We ambiguously denote the code and the numeral designating the code of a syntactic object α (a term, formula, or derivation) by $\ulcorner\alpha\urcorner$. In particular, $\ulcorner\varphi v_0\urcorner$ is the code of a formula with a free variable. For the function,

$$x, \ulcorner\varphi v_0\urcorner \to \ulcorner\varphi\bar{x}\urcorner,$$

we follow Feferman [4] and write $\ulcorner\varphi\dot{v}_0\urcorner$. An occurrence of this function in a context $\psi(\ulcorner\varphi\dot{v}_0\urcorner)$ is an abbreviation for $\exists v[\sigma(\ulcorner\varphi v_0\urcorner, v_0, v) \wedge \psi v]$, where $\sigma v_0 v_1 v_2$ is a formula representing the given function.

Diagonalization. The representability of the substitution function rather quickly yields the Diagonalization Theorem — a result which one can almost describe as the best known and least known of all results on the metamathematics of arithmetic. Implicit in the pioneering papers of the 1930's (Gödel [7], Gödel [8], and Rosser [29]), the Diagonalization Theorem seems first to have been singled out as a lemma worthy of mention in an expository paper (Rosser [30]) the year following the appearance of the Recursion Theorem (Kleene [14]). Moreover, the first explicit statement of the definitive version the author could find in the literature is from Montague [20]:

DIAGONALIZATION THEOREM. *Let* $\psi v_0 \ldots v_n$ *be an arithmetical formula with only* $v_0, \ldots, v_n$ *free. There is a formula* $\varphi v_0 \ldots v_{n-1}$ *with only* $v_0, \ldots, v_{n-1}$ *free and such that*

$$\mathscr{R} \vdash \varphi v_0 \ldots v_{n-1} \leftrightarrow \psi(v_0, \ldots, v_{n-1}, \ulcorner\varphi v_0 \ldots v_{n-1}\urcorner)\,.$$

Proof predicates. For each $n \geqslant 0$, we let $\mathrm{Thm}^n_{\mathscr{T}}$, $\mathrm{Ref}^n_{\mathscr{T}}$ denote arbitrary Σ_1 formulae defining over N the respective relations:

$$T^n_{\mathscr{T}}(x_0, \ldots, x_n)\colon\ \exists\psi\,[\psi \text{ has exactly } v_0, \ldots, v_{n-1} \text{ free } \& \ \& \ x_n = \ulcorner\psi v_0 \ldots v_{n-1}\urcorner \ \& \ \mathscr{T} \vdash \psi\bar{x}_0 \ldots \bar{x}_{n-1}]\,;$$

$$R^n_{\mathscr{T}}(x_0, \ldots, x_n)\colon\ \exists\psi\,[\psi \text{ has exactly } v_0, \ldots, v_{n-1} \text{ free } \& \ \& \ x_n = \ulcorner\psi v_0 \ldots v_{n-1}\urcorner \ \& \ \mathscr{T} \vdash \neg\psi\bar{x}_0 \ldots \bar{x}_{n-1}]\,.$$

For results that require formalization, we must be particular about the choice of $\mathrm{Thm}^n_{\mathscr{T}}$. For $\mathscr{T} \supseteq \mathscr{PA}$, there is a Σ_1 formula $\mathrm{Pr}_{\mathscr{T}}(v_0) = \exists v_1\,\mathrm{Prov}_{\mathscr{T}}(v_1, v_0)$ for which the most familiar properties of the set of theorems of $\mathscr{T}$ are derivable (e.g. Σ-Completeness: For $\varphi v_0 \ldots v_{n-1} \in \Sigma$ with the free variables as shown, we have $\mathscr{T} \vdash \varphi v_0 \ldots v_{n-1} \to \mathrm{Pr}_{\mathscr{T}}(\ulcorner\varphi\dot{v}_0 \ldots \dot{v}_{n-1}\urcorner)$.). When formalizing results, we use

$$\mathrm{Thm}^n_{\mathscr{T}}(v_0, \ldots, v_{n-1}, \ulcorner\varphi v_0 \ldots v_{n-1}\urcorner)\colon\ \mathrm{Pr}_{\mathscr{T}}(\ulcorner\varphi\dot{v}_0 \ldots \dot{v}_{n-1}\urcorner)\,;$$
$$\mathrm{Ref}^n_{\mathscr{T}}(v_0, \ldots, v_{n-1}, \ulcorner\varphi v_0 \ldots v_{n-1}\urcorner)\colon\ \mathrm{Pr}_{\mathscr{T}}(\mathrm{neg}(\ulcorner\varphi\dot{v}_0 \ldots \dot{v}_{n-1}\urcorner))\,,$$

where neg is a function constant for a similarly natural representation of negation. We finally define $\mathrm{Con}_{\mathscr{T}} = \neg\mathrm{Pr}_{\mathscr{T}}(\ulcorner 0 = 1\urcorner)$.

2. The unadorned Shepherdson fixed-point. Throughout this section, $\mathscr{T}$ will be a fixed consistent r.e. extension of Robinson's $\mathscr{R}$. For all Σ_1 formulae $\psi v_0 \ldots v_{n-1}$, $\chi v_0 \ldots v_{n-1}$, with only $v_0, \ldots, v_{n-1}$ free, define

$$\Phi^n(v_0, \ldots, v_n)\colon\ (\mathrm{Ref}^n_{\mathscr{T}}(v_0, \ldots, v_n) \vee \psi v_0 \ldots v_{n-1}) .\preccurlyeq. (\mathrm{Thm}^n_{\mathscr{T}}(v_0, \ldots, v_n) \vee \chi v_0 \ldots v_{n-1}) .$$

FIXED-POINT CALCULATION. *Let $\varphi v_0 \ldots v_{n-1}$ be Shepherdson's fixed-point:*

$$\mathscr{T} \vdash \varphi v_0 \ldots v_{n-1} \leftrightarrow \Phi^n(v_0, \ldots, v_{n-1}, \ulcorner \varphi v_0 \ldots v_{n-1} \urcorner) .$$

Then, for all $x_0, \ldots, x_{n-1} \in N$,

i. $\mathscr{T} \vdash \varphi \bar{x}_0 \ldots \bar{x}_{n-1} \Leftrightarrow N \models (\psi \preccurlyeq \chi) \bar{x}_0 \ldots \bar{x}_{n-1}$,

ii. $\mathscr{T} \vdash \neg \varphi \bar{x}_0 \ldots \bar{x}_{n-1} \Leftrightarrow N \models (\chi \prec \psi) \bar{x}_0 \ldots \bar{x}_{n-1}$.

Proof. For notational convenience we assume $n = 0$. Write

$$\begin{aligned} \mathrm{Thm}^0_{\mathscr{T}}(v_0) &= \exists v_1 \tau v_0 v_1 , \\ \mathrm{Ref}^0_{\mathscr{T}}(v_0) &= \exists v_1 \varrho v_0 v_1 , \\ \psi &= \exists v_1 \psi' v_1 , \\ \chi &= \exists v_1 \chi' v_1 . \end{aligned}$$

Note that any of the conditions, $\mathscr{T} \vdash \varphi$, $\mathscr{T} \vdash \neg\varphi$, $N \models \psi$, $N \models \chi$, and hence any of the conditions, $\mathscr{T} \vdash \varphi$, $\mathscr{T} \vdash \neg\varphi$, $N \models \psi \preccurlyeq \chi$, $N \models \chi \prec \psi$, will make $\Phi^0(\ulcorner \varphi \urcorner)$ (and hence φ) provably Δ_0. But

$$\begin{aligned} \mathscr{T} \vdash \varphi &\Rightarrow \exists x \mathscr{T} \vdash \exists v_1 \leqslant \bar{x} \big[(\varrho(\ulcorner \varphi \urcorner, v_1) \vee \psi' v_1) \wedge \forall v_2 < v_1 \neg (\tau(\ulcorner \varphi \urcorner, v_2) \vee \chi' v_2) \big] \\ &\Rightarrow \exists x N \models \exists v_1 \leqslant \bar{x} \big[(\varrho(\ulcorner \varphi \urcorner, v_1) \vee \psi' v_1) \wedge \forall v_2 < v_1 \neg (\tau(\ulcorner \varphi \urcorner, v_2) \vee \chi' v_2) \big] \\ &\Rightarrow N \models \exists v_1 (\psi' v_1 \wedge \forall v_2 < v_1 \neg \chi' v_2), \text{ by the consistency of } \mathscr{T} \\ &\Rightarrow N \models \psi \preccurlyeq \chi . \end{aligned}$$

Similarly,

$$\mathscr{T} \vdash \neg \varphi \Rightarrow N \models \chi \prec \psi .$$

But also,

$$\begin{aligned} N \models \psi \preccurlyeq \chi &\Rightarrow \mathscr{T} \vdash \varphi \text{ or } \mathscr{T} \vdash \neg\varphi, \text{ since } \varphi \text{ becomes } \Delta_0 \\ &\Rightarrow \mathscr{T} \vdash \varphi \text{ or } N \models \chi \prec \psi \\ &\Rightarrow \mathscr{T} \vdash \varphi . \end{aligned}$$

Similarly,

$$N \models \chi \prec \psi \Rightarrow \mathscr{T} \vdash \neg \varphi . \ ■$$

From the calculation, we can read off a number of applications. The simplest such application is Rosser's incompleteness theorem.

APPLICATION 1 (Rosser's theorem). *Let* $\mathscr{T} \vdash \varphi \leftrightarrow \mathrm{Ref}^0_{\mathscr{T}}(\ulcorner \varphi \urcorner) \preccurlyeq \mathrm{Thm}^0_{\mathscr{T}}(\ulcorner \varphi \urcorner)$. *Then*

i. $\mathscr{T}$ *not* $\vdash \varphi$,

ii. $\mathscr{T}$ *not* $\vdash \neg\varphi$.

Proof. Observe $\mathscr{T} \vdash \varphi \leftrightarrow \Phi^0(\ulcorner \varphi \urcorner)$, where ψ, χ are (say) $\exists v_1 (v_1 \neq v_1)$. ■

Rosser's Theorem formalizes: If $\mathscr{T} \supseteq \mathscr{P}\!\mathscr{A}$, and φ is the sentence of Application 1, then

i. $\mathscr{T} + \mathrm{Con}_{\mathscr{T}} \vdash \neg \mathrm{Pr}_{\mathscr{T}}(\ulcorner \varphi \urcorner)$,

ii. $\mathscr{T} + \mathrm{Con}_{\mathscr{T}} \vdash \neg \mathrm{Pr}_{\mathscr{T}}(\ulcorner \neg\varphi \urcorner)$.

Our next application is due to P. Päppinghaus. It gives an independence result for a theory $\mathscr{T}_0$ not formalizable in another theory $\mathscr{T}_1$.

APPLICATION 2. *Let $\mathscr{T}_0$, $\mathscr{T}_1$ be consistent r.e. extension of $\mathscr{R}$. Let*

$$\begin{aligned} \mathscr{R} \vdash \varphi \leftrightarrow \colon & [\mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \vee \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner)] .\preccurlyeq. \\ & \preccurlyeq. [\mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \vee \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner)] . \end{aligned}$$

Then

i.a. $\mathscr{T}_0$ *not* $\vdash \varphi$,

b. $\mathscr{T}_0$ *not* $\vdash \neg\varphi$,

ii.a. $\mathscr{T}_1$ *not* $\vdash \neg \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner)$,

b. $\mathscr{T}_1$ *not* $\vdash \neg \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner)$.

Proof. i. By the Fixed-point calculation,

$$\begin{aligned} \mathscr{T}_0 \vdash \varphi &\Rightarrow N \models \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) \preccurlyeq \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) \\ &\Rightarrow N \models \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) \\ &\Rightarrow \mathscr{T}_1 \vdash \neg \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) . \end{aligned}$$

But

$$\begin{aligned} \mathscr{T}_0 \vdash \varphi &\Rightarrow \mathscr{R} \vdash \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \\ &\Rightarrow \mathscr{T}_1 \text{ is inconsistent} . \end{aligned}$$

Thus $\mathscr{T}_0$ not $\vdash \varphi$. Similarly, $\mathscr{T}_0$ not $\vdash \neg\varphi$.

ii. From $\mathscr{T}_1 \vdash \neg \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner)$ or $\mathscr{T}_1 \vdash \neg \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner)$ we conclude

$$N \models \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) \quad \text{or} \quad N \models \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) .$$

Reduction yields

$$N \models \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) \preccurlyeq \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner)$$

or

$$N \models \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Ref}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) \prec \mathrm{Ref}^0_{\mathscr{T}_1}(\ulcorner \mathrm{Thm}^0_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \urcorner) .$$

From these and the Fixed-point calculation, we conclude $\mathscr{T}_0 \vdash \varphi$ or $\mathscr{T}_0 \vdash \neg\varphi$, which we have just demonstrated to be false. ■

As we said in the Introduction, after incompleteness results there are other types of applications of diagonalization. Shepherdson proved the following

APPLICATION 3 (Dual semi-representability theorem). *Let R, S be disjoint n-ary r.e. relations. There is a Σ_1 formula $\varphi v_0 \ldots v_{n-1}$ with the free variables shown such that*

i. φ *semi-represents* R *in* $\mathcal{T}$.

ii. $\neg\varphi$ *semi-represents* S *in* $\mathcal{T}$.

Proof. Let ψ, χ be Σ_1 definitions of R, S, respectively, and notice that, if we define

$$\mathcal{T} \vdash \varphi v_0 \dots v_{n-1} \leftrightarrow \Phi^n(v_0, \dots, v_{n-1}, \ulcorner \varphi v_0 \dots v_{n-1} \urcorner),$$

then φ, $\neg\varphi$ semi-represent R, S, respectively in $\mathcal{T}$. In $\mathcal{R}$, φ will not quite be Σ_1 (having an extra existential quantifier), but the equivalent formula

$$\Phi^n(v_0, \dots, v_{n-1}, \ulcorner \varphi v_0 \dots v_{n-1} \urcorner) \text{ is } \Sigma_1. \blacksquare$$

The Σ_1 semi-representation of the relation R has a curious feature: It is correct. That is (provided we assume the diagonalization to be within $\mathcal{R}$ or any other sound theory), φ defines R in N as well as semi-represents R in $\mathcal{T}$: For all $x_0, \dots, x_{n-1} \in N$,

$$\mathcal{T} \vdash \varphi \bar{x}_0 \dots \bar{x}_{n-1} \Leftrightarrow R x_0 \dots x_{n-1}$$
$$\Leftrightarrow N \vDash \varphi \bar{x}_0 \dots \bar{x}_{n-1}.$$

Traditionally, the recursion-theoretic constructions of semi-representations either have not had this correctness property (Ehrenfeucht and Feferman [3], Putnam and Smullyan [26]) or have not resulted in Σ_1 formulae (Hájková and Hájek [10]). In Smoryński [34], we achieve a correct Σ_1 semi-representation recursion-theoretically only by mimicking Shepherdson's fixed-point to derive a recursion-theoretic lemma.

The third type of application we offer has been described in the Introduction as characterizing Σ_1-soundness among r.e. theories. While such a characterization is not the intent of such applications, it is the most obvious common feature of these applications. Our first application of this type is due independently to Jensen and Ehrenfeucht [12] and Guaspari [9], and is closely related to a result of Friedman [6]. We present Guaspari's proof.

Definition. Let $\mathcal{T}$ be given.

1. $\mathcal{T}$ is Σ_1-*sound* iff, for all Σ_1 sentences φ, $\mathcal{T} \vdash \varphi \Rightarrow N \vDash \varphi$.

2. $\mathcal{T}$ has the Σ_1-*disjunction property* iff, for all Σ_1 sentences φ, ψ, $\mathcal{T} \vdash \varphi \vee \psi \Rightarrow \mathcal{T} \vdash \varphi$ or $\mathcal{T} \vdash \psi$.

3. A Σ_1 sentence φ is $\mathcal{T}$-*provably* Δ_1 iff there is a Σ_1 sentence ψ such that $\mathcal{T} \vdash \neg\varphi \leftrightarrow \psi$.

4. $\mathcal{T}$ *decides* a sentence φ iff $\mathcal{T} \vdash \varphi$ or $\mathcal{T} \vdash \neg\varphi$.

Application 4. *Let* $\mathcal{T} \supseteq \mathcal{PA}$. *The following are equivalent*:

i. $\mathcal{T}$ *is* Σ_1-*sound.*

ii. $\mathcal{T}$ *has the* Σ_1-*disjunction property.*

iii. $\mathcal{T}$ *decides all* $\mathcal{T}$-*provably* Δ_1 *sentences.*

Proof. The implications i $\Rightarrow$ ii $\Rightarrow$ iii are trivial. We prove iii $\Rightarrow$ i contrapositively. Let ψ be a false Σ_1 sentence such that $\mathcal{T} \vdash \psi$ and define φ by

$$\mathcal{T} \vdash \varphi \leftrightarrow .(\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner) \vee \psi) \preccurlyeq \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner).$$

Note that, for any $\theta, \chi \in \Sigma_1$,

a. $\mathcal{PA} \vdash \neg[(\theta \preccurlyeq \chi) \wedge (\chi \prec \theta)]$,

b. $\mathcal{PA} \vdash \theta \rightarrow .(\theta \preccurlyeq \chi) \vee (\chi \prec \theta)$,

whence

$$\mathcal{PA} \vdash \theta \rightarrow [(\chi \prec \theta) \leftrightarrow \neg(\theta \preccurlyeq \chi)].$$

Letting $\theta = \mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner) \vee \psi$, $\chi = \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner)$, we see that

$$\mathcal{T} \vdash \neg\varphi \leftrightarrow .\mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner) \prec (\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner) \vee \psi).$$

Hence φ is $\mathcal{T}$-provably Δ_1. By the Fixed-point calculation, $\mathcal{T}$ does not decide φ. $\blacksquare$

Jensen and Ehrenfeucht proved the theorem by means of a simultaneous diagonalization which we comment on in Section 6. The related result of Friedman is an intuitionistic generalization of the equivalence i $\Leftrightarrow$ ii. His self-referential sentence differs from the one we use only in that he does not assume ψ to be Σ_1. The underlying calculation is slightly different and we discuss this also in Section 6.

The final application of this section is new.

Definition. Let Γ be a set of sentences. A sentence φ is Γ-*conservative* over $\mathcal{T}$ iff, for all $\psi \in \Gamma$, we have $\mathcal{T} + \varphi \vdash \psi \Rightarrow \mathcal{T} \vdash \psi$.

As first observed in Kreisel [15] (but cf. also Macintyre and Simmons [18]), for $\mathcal{T} \supseteq \mathcal{PA}$, $\neg\mathrm{Con}_{\mathcal{T}}$ is always π_1-conservative (though not necessarily independent). For $\mathrm{Con}_{\mathcal{T}}$, we have the following

Application 5. *Let* $\mathcal{T} \supseteq \mathcal{PA}$. *The following are equivalent*:

i. $\mathcal{T}$ *is* Σ_1-*sound.*

ii. *Every* $\mathcal{T}$-*independent* π_1 *sentence is* Σ_1-*conservative over* $\mathcal{T}$.

iii. $\mathrm{Con}_{\mathcal{T}}$ *is* Σ_1-*conservative over* $\mathcal{T}$.

Proof. The implications i $\Rightarrow$ ii $\Rightarrow$ iii are trivial (as is, indeed, the equivalence i $\Leftrightarrow$ ii). We will prove iii $\Rightarrow$ i by formalizing the Fixed-point calculation.

Let ψ be an as yet unspecified Σ_1 sentence and define

$$(*) \qquad \mathcal{T} \vdash \varphi \leftrightarrow .(\mathrm{Pr}_{\mathcal{T}}(\ulcorner \neg\varphi \urcorner) \vee \psi) \preccurlyeq \mathrm{Pr}_{\mathcal{T}}(\ulcorner \varphi \urcorner).$$

The Fixed-point calculation assumed only the consistency of $\mathcal{T}$. Thus, formalization yields

$$(1) \qquad \mathcal{T} + \mathrm{Con}_{\mathcal{T}} \vdash \mathrm{Pr}_{\mathcal{T}}(\ulcorner \varphi \urcorner) \leftrightarrow \psi.$$

Further, the proof that φ is Δ_0 under the assumption $\mathrm{Pr}_{\mathcal{T}}(\ulcorner \varphi \urcorner)$, together with the significance of consistency, yields upon formalization:

$$(2) \qquad \mathcal{T} + \mathrm{Con}_{\mathcal{T}} \vdash \mathrm{Pr}_{\mathcal{T}}(\ulcorner \varphi \urcorner) \rightarrow \varphi.$$

Combining (1) and (2) we get

$$(**) \qquad \mathcal{T} + \mathrm{Con}_{\mathcal{T}} \vdash \psi \rightarrow \varphi.$$

Now suppose $\mathcal{T}$ is not Σ_1-sound. Then there is a false $\psi \in \Sigma_1$ such that $\mathcal{T} \vdash \psi$. Defining φ by (*), the Fixed-point calculation yields $\mathcal{T}$ not $\vdash \varphi$, but (**) yields $\mathcal{T} + \mathrm{Con}_{\mathcal{T}} \vdash \varphi$, whence $\mathrm{Con}_{\mathcal{T}}$ is not Σ_1-conservative over $\mathcal{T}$. $\blacksquare$

3. The Mostowski–Shepherdson fixed-point. By generalizing Rosser's self-referential sentence to an r.e. sequence of consistent extensions of $\mathcal{R}$, Mostowski [21] obtained a generalization of Rosser's Theorem. In this section, we similarly generalize Shepherdon's fixed-point.

Throughout this section, $\mathcal{T}_0, \mathcal{T}_1, \ldots$, will be a fixed r.e. sequence of consistent extensions of $\mathcal{R}$. For $n \geqslant 0$, let $\mathrm{Thm}^n(v_0, \ldots, v_n)$, $\mathrm{Ref}^n(v_0, \ldots, v_n)$ be arbitrary Σ_1 definitions of the respective relations

$$T^n(x_0, \ldots, x_n)\colon \exists y\psi[\psi \text{ has exactly } v_0, \ldots, v_{n-1} \text{ free } \& \\ \& \; x_n = \ulcorner \psi v_0 \ldots v_{n-1} \urcorner \,\&\, \mathcal{T}_y \vdash \psi \bar{x}_0 \ldots \bar{x}_{n-1}]\,;$$

$$R^n(x_0, \ldots, x_n)\colon \exists y\psi[\psi \text{ has exactly } v_0, \ldots, v_{n-1} \text{ free } \& \\ \& \; x_n = \ulcorner \psi v_0 \ldots v_{n-1} \urcorner \,\&\, \mathcal{T}_y \vdash \neg\psi \bar{x}_0 \ldots \bar{x}_{n-1}]\,.$$

In analogy to Section 2, define for $\psi, \chi \in \Sigma_1$,

$$\Phi^n(v_0, \ldots, v_n)\colon (\mathrm{Ref}^n(v_0, \ldots, v_n) \vee \psi v_0 \ldots v_{n-1}) . \!\preccurlyeq\! . (\mathrm{Thm}^n(v_0, \ldots, v_n) \vee \chi v_0 \ldots v_{n-1})\,.$$

Uniform fixed-point calculation. *Let $\varphi v_0 \ldots v_{n-1}$ satisfy*:

$$\mathcal{R} \vdash \varphi v_0 \ldots v_{n-1} \leftrightarrow \Phi^n(v_0, \ldots, v_{n-1}, \ulcorner \varphi v_0 \ldots v_{n-1} \urcorner)\,.$$

Then, for all $x_0, \ldots, x_{n-1} \in N$,

i. $\exists y \mathcal{T}_y \vdash \varphi \bar{x}_0 \ldots \bar{x}_{n-1} \Leftrightarrow N \models (\psi \preccurlyeq \chi)\bar{x}_0 \ldots \bar{x}_{n-1}$
$\Leftrightarrow \forall y \mathcal{T}_y \vdash \varphi \bar{x}_0 \ldots \bar{x}_{n-1}$,

ii. $\exists y \mathcal{T}_y \vdash \neg\varphi \bar{x}_0 \ldots \bar{x}_{n-1} \Leftrightarrow N \models (\chi \prec \psi)\bar{x}_0 \ldots \bar{x}_{n-1}$
$\Leftrightarrow \forall y \mathcal{T}_y \vdash \neg\varphi \bar{x}_0 \ldots \bar{x}_{n-1}$.

Proof. As before, any condition, $\mathcal{T}_y \vdash \varphi$, $\mathcal{T}_z \vdash \neg\varphi$, $N \models \psi \preccurlyeq \chi$, $N \models \chi \prec \psi$, makes $\Phi(\ulcorner \varphi \urcorner)$ and hence φ probably Δ_0. Thus, for all y,

$$\mathcal{T}_y \vdash \varphi \Leftrightarrow N \models \varphi\,,$$
$$\mathcal{T}_y \vdash \neg\varphi \Leftrightarrow N \models \neg\varphi\,.$$

Thus the provability or refutability of φ is uniform in all $\mathcal{T}_y$ and we can simply repeat the proof of the non-uniform result of Section 2. ■

A first application is Mostowski's generalized incompleteness theorem:

Application 1 (Uniform Rosser theorem). *Let* $\mathcal{R} \vdash \varphi \leftrightarrow .\mathrm{Ref}^0(\ulcorner \varphi \urcorner) \preccurlyeq \mathrm{Thm}^0(\ulcorner \varphi \urcorner)$. *Then, for all $y \in N$,*

i. $\mathcal{T}_y$ *not* $\vdash \varphi$,

ii. $\mathcal{T}_y$ *not* $\vdash \neg\varphi$.

Definition. A sequence $\varphi_0, \varphi_1, \ldots$, of sentences is *absolutely independent* over a theory $\mathcal{T}$ iff every theory $\mathcal{T} + \{\varphi_0^{e_0}, \varphi_1^{e_1}, \ldots\}$ is consistent, where $e_i \in \{0, 1\}$ and $\varphi^0 = \varphi$, $\varphi^1 = \neg\varphi$.

A corollary to Application 1 is this: There is a sequence $\varphi_0, \varphi_1, \ldots$ of Σ_1 sentences such that, for each y, the sequence is absolutely independent over $\mathcal{T}_y$. (Proof. Use Application 1 to find successively sentences φ_0 independent every each $\mathcal{T}_y$; φ_1 independent over each $\mathcal{T}_y + \varphi_0$ and $\mathcal{T}_y + \neg\varphi_0$; etc. ■) We shall shortly improve on this. First, however, we shall give some simpler direct applications.

The results of Päppinghaus and Shepherdson can also be generalized via the Uniform fixed-point calculation. The latter reads:

Application 2 (Uniform dual semi-representability theorem). *Let R, S be disjoint n-ary r.e. relations. There is a Σ_1 formula $\varphi v_0 \ldots v_{n-1}$, with the free variables shown, such that, for all y,*

i. φ *semi-represents R in $\mathcal{T}_y$,*

ii. $\neg\varphi$ *semi-represents S in $\mathcal{T}_y$.*

As in the non-uniform case, the Σ_1 semi-representation is correct. In the non-uniform case, however, this is no revelation: Letting $\mathcal{T}_{-1}$ be $\mathcal{R}$ (or any other Σ_1-sound theory), any uniform Σ_1 semi-representation of R in $\mathcal{T}_{-1}, \mathcal{T}_0, \ldots$ must be correct.

The Uniform dual semi-representability Theorem has a minor corollary — the dual semi-representability theorem for $\mathcal{T} = \bigcap_y \mathcal{T}_y$.

Application 3. *Let each $\mathcal{T}_y \supseteq \mathcal{PA}$. The following are equivalent*:

i. *Each $\mathcal{T}_y$ is Σ_1-sound.*

ii. *For all Σ_1 sentences φ, ψ, and all y,*

$$\mathcal{T}_y \vdash \varphi \vee \psi \Rightarrow \exists z(\mathcal{T}_z \vdash \varphi \text{ or } \mathcal{T}_z \vdash \psi)\,.$$

The less direct application promised is another result from Mostowski [21].

Definition. A formula φv_0 is *absolutely independent* over $\mathcal{T}$ iff the sequence $\varphi\bar{0}, \varphi\bar{1}, \ldots$ is absolutely independent over $\mathcal{T}$.

Application 4. *There is a Σ_1 formula φv_0 which is absolutely independent over each $\mathcal{T}_y$.*

Proof. To prove this, we shall concoct a uniform fixed-point that simulates the Iterated Application 1 used (parenthetically) above to establish the existence of absolutely independent sentences.

First, some notation: We assume a primitive recursive 1-1 correspondence between natural numbers and finite *binary* sequences. We write $\mathrm{lh}(z)$ for the length n of the sequence, say $(z_0, \ldots, z_{n-1})$, encoded by z. For $y < \mathrm{lh}(z)$, $(z)_y$ denotes the yth element of this sequence, z_y. If φv_0 has only v_0 free, we define, for all $z \in N$,

$$\tilde{\varphi}(z) = \bigwedge\!\!\bigwedge_{y < \mathrm{lh}(x)} (\varphi\bar{y})^{(z)_y}\,,$$

where $\varphi^0 = \varphi$, $\varphi^1 = \neg\varphi$. (The sentence $\tilde{\varphi}(z)$ asserts that the course-of-truth-values of φ up to $\mathrm{lh}(z) - 1$ is just the sequence encoded by z.)

Let Thm^*, Ref^* be Σ_1 definitions of the following respective relations,

$$T^*(x_0, x_1, x_2, x_3)\colon \exists zy\varphi\psi[\varphi, \psi \text{ have only } v_0 \text{ free } \& \; x_2 = \ulcorner \psi \urcorner \; \& \\ \& \; x_3 = \ulcorner \varphi \urcorner \,\&\, \mathrm{lh}(z) = x_1 \,\&\, \mathcal{T}_y \vdash \tilde{\varphi}(z) \to \psi \bar{x}_0]\,;$$

$$R^*(x_0, x_1, x_2, x_3)\colon\ \exists zy\varphi\psi\,[\varphi,\ \psi \text{ have only } v_0 \text{ free } \&\ x_2 = \ulcorner\psi\urcorner\ \&$$
$$\&\ x_3 = \ulcorner\varphi\urcorner\ \&\ \mathrm{lh}(z) = x_1\ \&\ \mathscr{T}_y \vdash \tilde{\varphi}(z) \to \neg\psi\bar{x}_0]\,.$$

Then choose the fixed-point φ:

$$\mathscr{R} \vdash \varphi v_0 \leftrightarrow .\mathrm{Ref}^*(v_0, v_0, \ulcorner\varphi v_0\urcorner, \ulcorner\varphi v_0\urcorner) \preccurlyeq \mathrm{Thm}^*(v_0, v_0, \ulcorner\varphi v_0\urcorner, \ulcorner\varphi v_0\urcorner)\,.$$

Note that, for fixed x,

$$\mathrm{Ref}^*(v_0, \bar{x}, v_1, \ulcorner\varphi v_0\urcorner) = \mathrm{Ref}^1(v_0, v_1)\,,$$
$$\mathrm{Thm}^*(v_0, \bar{x}, v_1, \ulcorner\varphi v_0\urcorner) = \mathrm{Thm}^1(v_0, v_1)$$

for the sequence of theories $\mathscr{T}_y + \tilde{\varphi}(z)$, all y and all z with $\mathrm{lh}(z) = x$. Thus,

$$\mathscr{R} \vdash \varphi\bar{x} \leftrightarrow .\mathrm{Ref}^1(\bar{x}, \ulcorner\varphi v_0\urcorner) \preccurlyeq \mathrm{Thm}^1(\bar{x}, \ulcorner\varphi v_0\urcorner)\,,$$

which is not quite equal to, but is sufficiently similar to

$$\mathscr{R} \vdash \varphi\bar{x} \leftrightarrow .\mathrm{Ref}^0(\ulcorner\varphi\bar{x}\urcorner) \preccurlyeq \mathrm{Thm}^0(\ulcorner\varphi\bar{x}\urcorner)\,,$$

for us to conclude that we know how to prove $\varphi\bar{x}$ to be independent of every $\mathscr{T}_y + \tilde{\varphi}(z)$ ($\mathrm{lh}(z) = x$). Treating this last assumption as an inductive hypothesis completes the proof. For, we know every $\mathscr{T}_y + \tilde{\varphi}(z)$ to be consistent, whence $\varphi\bar{0}, \varphi\bar{1}, \ldots$ is absolutely independent. ∎

Notice that we have not added side-formulae ψ, χ to the above fixed-point. The reason is simple: Let $(\psi \preccurlyeq \chi)\bar{x}$ be true. Then we would have $\mathscr{T}_y \vdash \varphi\bar{x}$. But then we could not proceed to step $x+1$ because many of the theories $\mathscr{T}_y + \tilde{\varphi}(z)$ ($\mathrm{lh}(z) = x+1$) are inconsistent. To circumvent this obstacle, we would add conditions

$$N \models (\psi \preccurlyeq \chi)\,\bar{y} \Rightarrow (z)_y = 0\,,$$
$$N \models (\chi \prec \psi)\,\bar{y} \Rightarrow (z)_y = 1\,,$$

to our definitions of T^*, R^*. Unfortunately, these conditions are not Σ_1.

About recursion-theoretic proofs: Myhill [22] obtained Applications 1 and 4 by means of effectively inseparable sets. Applications 2 and 3 are new, if not novel, and hence have not previously been given recursion-theoretic proofs. In [34], we apply a result for a configuration of pairs of effectively inseparable r.e. sets to obtain Application 2 recursion-theoretically.

4. The modified Shepherdson fixed-point. We now wish to consider some non-uniform results. Let $\mathscr{T}$ be a fixed consistent r.e. extension of $\mathscr{R}$ and, for $\psi v_0 \ldots v_{n-1}, \chi v_0 \ldots v_{n-1} \in \Sigma_1$, define as usual

$$\Phi^n_{\mathscr{T}}(v_0, \ldots, v_n)\colon\ (\mathrm{Ref}^n_{\mathscr{T}}(v_0, \ldots, v_n) \vee \psi v_0 \ldots v_{n-1}) .\preccurlyeq. (\mathrm{Thm}^n_{\mathscr{T}}(v_0, \ldots, v_n) \vee \chi v_0 \ldots v_{n-1})\,.$$

We modify Shepherdson's fixed-point by attaching additional formulae to $\Phi^n_{\mathscr{T}}$.

MODIFIED FIXED-POINT CALCULATION. *Let $\theta_0 v_0 \ldots v_{n-1}, \theta_1 v_0 \ldots v_{n-1}$ have only the indicated free variables and define*

i. $\mathscr{T} \vdash \varphi_1 v_0 \ldots v_{n-1} \leftrightarrow .\theta_0 v_0 \ldots v_{n-1} \vee \theta_1 v_0 \ldots v_{n-1} \wedge \Phi^n_{\mathscr{T}}(v_0, \ldots, v_{n-1}, \ulcorner\varphi_1\urcorner)$,

ii. $\mathscr{T} \vdash \varphi_2 v_0 \ldots v_{n-1} \leftrightarrow .\theta_0 v_0 \ldots v_{n-1} \wedge [\theta_1 v_0 \ldots v_{n-1} \vee \Phi^n_{\mathscr{T}}(v_0, \ldots, v_{n-1}, \ulcorner\varphi_2\urcorner)]$,

iii. $\mathscr{T} \vdash \varphi_3 v_0 \ldots v_{n-1} \leftrightarrow .\theta_0 v_0 \ldots v_{n-1} \vee \Phi^n_{\mathscr{T}}(v_0, \ldots, v_{n-1}, \ulcorner\varphi_3\urcorner)$,

iv. $\mathscr{T} \vdash \varphi_4 v_0 \ldots v_{n-1} \leftrightarrow .\theta_0 v_0 \ldots v_{n-1} \wedge \Phi^n_{\mathscr{T}}(v_0, \ldots, v_{n-1}, \ulcorner\varphi_4\urcorner)$.

Then, for all $x_0, \ldots, x_{n-1} \in N$,

i.a. $\mathscr{T} \vdash \varphi_1 \Leftrightarrow \mathscr{T} \vdash \theta_0$ *or* $[\mathscr{T} \vdash \theta_0 \vee \theta_1\ \&\ N \models \psi \preccurlyeq \chi]$,
b. $\mathscr{T} \vdash \neg\varphi_1 \Leftrightarrow \mathscr{T} \vdash \neg\theta_0\ \&\ [\mathscr{T} \vdash \neg(\theta_0 \vee \theta_1)$ *or* $N \models \chi \prec \psi]$,

ii.a. $\mathscr{T} \vdash \varphi_2 \Leftrightarrow \mathscr{T} \vdash \theta_0 \wedge \theta_1$ *or* $[\mathscr{T} \vdash \theta_0\ \&\ N \models \psi \preccurlyeq \chi]$,
b. $\mathscr{T} \vdash \neg\varphi_2 \Leftrightarrow \mathscr{T} \vdash \neg(\theta_0 \wedge \theta_1)\ \&\ [\mathscr{T} \vdash \neg\theta_0$ *or* $N \models \chi \prec \psi]$,

iii.a. $\mathscr{T} \vdash \varphi_3 \Leftrightarrow \mathscr{T} \vdash \theta_0$ *or* $N \models \psi \preccurlyeq \chi$,
b. $\mathscr{T} \vdash \neg\varphi_3 \Leftrightarrow \mathscr{T} \vdash \neg\theta_0\ \&\ N \models \chi \prec \psi$,

iv.a. $\mathscr{T} \vdash \varphi_4 \Leftrightarrow \mathscr{T} \vdash \theta_0\ \&\ N \models \psi \preccurlyeq \chi$,
b. $\mathscr{T} \vdash \neg\varphi_4 \Leftrightarrow \mathscr{T} \vdash \neg\theta_0$ *or* $N \models \chi \prec \psi$,

(*where each sentence φ_i, θ_i, $\psi \preccurlyeq \chi$, $\chi \prec \psi$ abbreviates $\varphi_i \bar{x}_0 \ldots \bar{x}_{n-1}$, etc.*).

Proof. First observe that ii reduces to i by defining $\theta_0' = \theta_0 \wedge \theta_1$, $\theta_1' = \theta_0$ and noting $\theta_1' \leftrightarrow \theta_0' \vee \theta_1'$. Parts iii and iv reduce to parts i and ii, respectively, by taking θ_1 to be provable and refutable, respectively.

To prove i, note that, as usual, $\Phi^n_{\mathscr{T}}$ has a propensity to be Δ_0, hence decidable. To prove i.a, note

$$\mathscr{T} \vdash \varphi \Rightarrow \mathscr{T} \vdash \theta_0 \vee \theta_1 \wedge \Phi^n_{\mathscr{T}}\,.$$

If $N \models \neg(\psi \preccurlyeq \chi)$, then $\Phi^n_{\mathscr{T}}$ is false and $\mathscr{T} \vdash \theta_0$. If $N \models \psi \preccurlyeq \chi$, then $\Phi^n_{\mathscr{T}}$ is either true and $\mathscr{T} \vdash \theta_0 \vee \theta_1$ or false and $\mathscr{T} \vdash \theta_0$, whence $\mathscr{T} \vdash \theta_0 \vee \theta_1$. For the converse, suppose $\mathscr{T} \vdash \theta_0$ or $\mathscr{T} \vdash \theta_0 \vee \theta_1$ and $N \models \psi \preccurlyeq \chi$. In the first case, evidently $\mathscr{T} \vdash \varphi$. So assume $N \models \psi \preccurlyeq \chi$ and $\mathscr{T} \vdash \theta_0 \vee \theta_1$. If $\Phi^n_{\mathscr{T}}$ is true, $\mathscr{T} \vdash \theta_0 \vee \theta_1 \wedge \Phi^n_{\mathscr{T}}$, whence $\mathscr{T} \vdash \varphi$. If $\Phi^n_{\mathscr{T}}$ is false, the fact that $N \models \psi \preccurlyeq \chi$ means that $N \models \mathrm{Thm}^n_{\mathscr{T}}(\bar{x}_0, \ldots, \bar{x}_{n-1}, \ulcorner\varphi\urcorner)$, i.e. $\mathscr{T} \vdash \varphi$.

We omit the similar verification of i.b. ∎

While there are few things of life more appealing than the above calculation we must confess that our motivation for performing it was not purely aesthetic We have in mind applications to the problem of semi-representing distinct relations in distinct theories by a single formula.

APPLICATION 1. *Let $\mathscr{T}_0 \subsetneqq \mathscr{T}_1$ be consistent r.e. extension of $\mathscr{R}$ and let $R_0 \subseteq R_1$ be n-ary r.e. relations. There is a formula $\varphi v_0 \ldots v_{n-1}$ such that φ semi-represents R_i in $\mathscr{T}_i$. Moreover, if $\mathscr{T}_1$ proves a Σ_1 (π_1) sentence not provable in $\mathscr{T}_0$, then φ can be chosen Σ_1 (π_1).*

Proof. Let θ be a sentence such that $\mathscr{T}_1 \vdash \theta$, $\mathscr{T}_0$ not $\vdash \theta$. Let further $\psi_1 \in \Sigma_1$ semi-represent R_1 in $\mathscr{T}_0$, $\mathscr{T}_1$, and let $\psi_0 \in \Sigma_1$ define R_0. First define

$$\mathscr{R} \vdash \varphi_1 \leftrightarrow .\theta \vee [(\mathrm{Ref}^n_{\mathscr{T}_0}(\ulcorner\varphi_1\urcorner) \vee \psi_0) \preccurlyeq \mathrm{Thm}^n_{\mathscr{T}_0}(\ulcorner\varphi_1\urcorner)]\,,$$

where we suppress the variables for notational convenience. By the Calculation, for any $x_0, \ldots, x_{n-1} \in N$,

$$\begin{aligned}
\mathscr{T}_0 \vdash \varphi_1 \bar{x}_0 \ldots \bar{x}_{n-1} &\Leftrightarrow \mathscr{T}_0 \vdash \theta \text{ or } N \vDash \psi_0 \bar{x}_0 \ldots \bar{x}_{n-1}\\
&\Leftrightarrow N \vDash \psi_0 \bar{x}_0 \ldots \bar{x}_{n-1}\\
&\Leftrightarrow R x_0 \ldots x_{n-1}.\\
\mathscr{T}_1 \vdash \varphi_1 \bar{x}_0 \ldots \bar{x}_{n-1}, &\quad \text{since} \quad \mathscr{T}_1 \vdash \theta.
\end{aligned}$$

Thus φ_1 semi-represents R_0 in $\mathscr{T}_0$ and N^n in $\mathscr{T}_1$. Letting $\varphi = \varphi_1 \wedge \psi_1$, we see that φ semi-represents R_i in $\mathscr{T}_i$.

As to the logical form of φ, if $\theta \in \Sigma_1$, then prenexing φ will make it almost Σ_1 — φ has a few extra existential quantifiers. However, even in $\mathscr{R}$ the class of Σ_1 formulae is provably closed under existential quantification (cf. [2]). If $\theta \in \pi_1$, use a dual argument. ■

APPLICATION 2. *Let $\mathscr{T}_0, \mathscr{T}_1$ be incomparable r.e. extensions of $\mathscr{R}$ and let R_0, R_1 be n-ary r.e. relations. There is a formula $\varphi v_0 \ldots v_{n-1}$ such that φ semi-represents R_i in $\mathscr{T}_i$. Moreover, if each $\mathscr{T}_i$ proves a Σ_1 (π_1) sentence not provable in $\mathscr{T}_{1-i}$, then φ can be chosen Σ_1 (π_1).*

Proof. Let $\mathscr{T}_i \vdash \theta_i$, $\mathscr{T}_{1-i}$ not $\vdash \theta_i$ and let $\psi_i \in \Sigma_1$ define R_i. Letting

$$\Phi^n_{\mathscr{T}_i}(\ulcorner \varphi \urcorner) = (\mathrm{Ref}^n_{\mathscr{T}_i}(\ulcorner \varphi \urcorner) \vee \psi_i) \preccurlyeq \mathrm{Thm}^n_{\mathscr{T}_i}(\ulcorner \varphi \urcorner),$$

define

$$\mathscr{R} \vdash \varphi \leftrightarrow .\theta_0 \wedge \Phi^n_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \vee \theta_1 \wedge \Phi^n_{\mathscr{T}_1}(\ulcorner \varphi \urcorner),$$

again suppressing variables. For all $x_0, \ldots, x_{n-1} \in N$, we have

$$\begin{aligned}
\mathscr{T}_i \vdash \varphi \bar{x}_0 \ldots \bar{x}_{n-1} &\Leftrightarrow \mathscr{T}_i \vdash \theta_{1-i} \wedge \Phi^n_{\mathscr{T}_{1-i}} \text{ or}\\
&\qquad (\mathscr{T}_i \vdash \theta_{1-i} \wedge \Phi^n_{\mathscr{T}_{1-i}} \vee \theta_i \;\&\; N \vDash \psi_i \bar{x}_0 \ldots \bar{x}_{n-1})\\
&\Leftrightarrow N \vDash \psi_i \bar{x}_0 \ldots \bar{x}_{n-1}.
\end{aligned}$$

Hence φ semi-represents R_i in $\mathscr{T}_i$.

The complexity statement follows as before. ■

Application 2 trivializes in the case in which $\mathscr{T}_0, \mathscr{T}_1$ are mutually inconsistent. But, in that case, something more holds:

APPLICATION 3. *Let $\mathscr{T}_0, \mathscr{T}_1$ be consistent r.e. extensions of $\mathscr{R}$ and assume them mutually inconsistent. Let (R_0, S_0) and (R_1, S_1) be pairs of disjoint n-ary r.e. relations. There is a formula $\varphi v_0 \ldots v_{n-1}$ such that*

i. *φ semi-represents R_i in $\mathscr{T}_i$,*

ii. *$\neg\varphi$ semi-represents S_i in $\mathscr{T}_i$.*

Moreover, if each theory proves a Σ_1 sentence refuted by the other, φ can be chosen Σ_1.

Proof. Let $\mathscr{T}_i \vdash \theta_i \wedge \neg\theta_{1-i}$, let ψ_i, χ_i be Σ_1 definitions of R_i, S_i, respectively, and finally let

$$\Phi^n_{\mathscr{T}_i}(\ulcorner \varphi \urcorner) = (\mathrm{Ref}^n_{\mathscr{T}_i}(\ulcorner \varphi \urcorner) \vee \psi_i) \preccurlyeq (\mathrm{Thm}^n_{\mathscr{T}_i}(\ulcorner \varphi \urcorner) \vee \chi_i).$$

Defining

$$\mathscr{R} \vdash \varphi \leftrightarrow .\theta_0 \wedge \Phi^n_{\mathscr{T}_0}(\ulcorner \varphi \urcorner) \vee \theta_1 \wedge \Phi^n_{\mathscr{T}_1}(\ulcorner \varphi \urcorner),$$

the Calculation yields: For all $x_0, \ldots, x_{n-1} \in N$ (suppressing their mention),

$$\begin{aligned}
\mathscr{T}_i \vdash \varphi &\Leftrightarrow \mathscr{T}_i \vdash \theta_{1-i} \wedge \Phi^n_{\mathscr{T}_{1-i}} \text{ or } (\mathscr{T}_i \vdash \theta_{1-i} \wedge \Phi^n_{\mathscr{T}_{1-i}} \vee \theta_i \;\&\; N \vDash \psi_i \preccurlyeq \chi_i)\\
&\Leftrightarrow N \vDash \psi_i \preccurlyeq \chi_i\\
&\Leftrightarrow N \vDash \psi_i,\\
\mathscr{T}_i \vdash \neg\varphi &\Leftrightarrow \mathscr{T}_i \vdash \neg(\theta_{1-i} \wedge \Phi^n_{\mathscr{T}_{1-i}}) \;\&\; (\mathscr{T}_i \vdash \neg(\theta_{1-i} \wedge \Phi^n_{\mathscr{T}_{1-i}} \vee \theta_i) \text{ or } N \vDash \chi_i \prec \psi_i)\\
&\Leftrightarrow N \vDash \chi_i \prec \psi_i\\
&\Leftrightarrow N \vDash \chi_i. \;■
\end{aligned}$$

We leave to the reader the investigation of further applications to finite configurations of theories and relations, noting only that, by results of Guaspari [9], there are infinite r.e. chains of theories, $\mathscr{T}_0 \subseteq \mathscr{T}_1 \subseteq \ldots$ such that any formula φv_0 must semi-represent the same set from some point on in the sequence.

Applications 1-3 compare favourably with the results so far obtained recursion-theoretically. Di Paola ([24] and [25]) prove the special cases of Applications 1 and 2 in which R_0 and R_1 have a recursive interpolant and intersection, respectively. In [34], we show that Application 1 can be given a completely recursion-theoretic proof by a simple appeal to di Paola's partial result and our uniform semi-representability result. Applications 2 and 3 have yet to be obtained recursion-theoretically.

5. The functorial Shepherdson fixed-point. The Reduction theorem, which is a key component in the recursion-theoretic mechanism behind the Shepherdson fixed-point and its variants already discussed, is a special case of the Selection theorem. As the Reduction theorem has its syntactic counterpart, so has the Selection theorem:

SYNTACTIC SELECTION THEOREM. *Let $\varphi_1 v_0 \ldots v_n \in \Sigma_1$. There is another $\varphi v_0 \ldots v_n \in \Sigma_1$, which we denote by $\mathrm{Sel}[\varphi_1 v_0 \ldots v_n] v_0 \ldots v_n$ such that*

i. $\mathscr{R} \vdash \varphi v_0 \ldots v_{n-1} v \wedge \varphi v_0 \ldots v_{n-1} v' \to v = v'$,

ii. $\mathscr{R} \vdash \varphi v_0 \ldots v_n \to \varphi_1 v_0 \ldots v_n$,

iii. *For all $x_0, \ldots, x_{n-1} \in N$,*

$$N \vDash \exists v_n \varphi_1 \bar{x}_0 \ldots \bar{x}_{n-1} v_n \Rightarrow N \vDash \exists v_n \varphi \bar{x}_0 \ldots \bar{x}_{n-1} v_n.$$

Moreover, writing $\varphi v_0 \ldots v_n = \exists v \psi v v_0 \ldots v_n$, we can assume

iv.a. $\mathscr{R} \vdash \psi v v_0 \ldots v_n \to v_n \leqslant v \vee \neg v \leqslant v_n$,

b. *For all $x \in N$,*

$$\mathscr{R} \vdash \psi v v_0 \ldots v_{n-1} \bar{x} \to \bar{x} \leqslant v.$$

We shall not prove this here. Essentially, $\mathrm{Sel}[\varphi_1] v_0 \ldots v_n$ asserts, for the given $v_0, \ldots, v_{n-1}$, that v_n is the value first witnessed to satisfy $\varphi_1 v_0 \ldots v_n$. The proofs of the various parts of the theorem are unproblematic if we replace $\mathscr{R}$ by

$\mathscr{R}^+ = \mathscr{R}+\forall v_0 v_1(v_0 \leqslant v_1 \vee v_1 \leqslant v_0)$. Working in $\mathscr{R}$ requires a small trick and some minor bookkeeping.

We continue to assume $\mathscr{T}$ to be a consistent r.e. extension of $\mathscr{R}$.

FUNCTORIAL FIXED-POINT CALCULATION. *Let, for* $\psi v_0 \ldots v_n \in \Sigma_1$,

$$\mathscr{T} \vdash \varphi v_0 \ldots v_n \leftrightarrow \mathrm{Sel}[\mathrm{Ref}^{n+1}_{\mathscr{T}}(\ulcorner\varphi\urcorner) \vee \psi] v_0 \ldots v_n .$$

Then, for all $x_0, \ldots, x_{n-1}, x_n \in N$,

i. $\mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_n \Leftrightarrow N \models \mathrm{Sel}[\psi v_0 \ldots v_n]\bar{x}_0 \ldots \bar{x}_n$,

ii. $\mathscr{T} \vdash \neg\varphi\bar{x}_0 \ldots \bar{x}_n \Leftrightarrow \exists y \neq x_n \mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_{n-1}\,\bar{y}$
$\Leftrightarrow \exists y \neq x_n N \models \mathrm{Sel}[\psi v_0 \ldots v_n]\bar{x}_0 \ldots \bar{x}_{n-1}\,\bar{y}$.

The proof is fairly straightforward. First, one notes that conclusion i of the Syntactic selection theorem yields $\mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_n \to \neg\varphi\bar{x}_0 \ldots \bar{x}_{n-1}\,\bar{y}$ for any $y \neq x_n$. Thus, the condition $\mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_n$, in addition to the conditions $N \models \mathrm{Sel}[\psi]\bar{x}_0 \ldots \bar{x}_n$ and $\mathscr{T} \vdash \neg\varphi\bar{x}_0 \ldots \bar{x}_n$, makes φ provably Δ_0. The customary argument is now possible. We leave the details to the reader.

Our first application is a slight improvement of a result of Ritchie and Young [27] and Ritter [28].

APPLICATION 1. *Let f be an n-ary partial recursive function. There is a formula* $\varphi v_0 \ldots v_n \in \Sigma_1$ *such that, for all* $x_0, \ldots, x_n \in N$,

i. $\mathscr{T} \vdash \varphi\bar{x}_0 \ldots \bar{x}_n \Leftrightarrow f x_0 \ldots x_{n-1} = x_n$,

ii. $\mathscr{T} \vdash \neg\varphi\bar{x}_0 \ldots \bar{x}_n \Leftrightarrow \exists y \neq x_n (y = f x_0 \ldots x_{n-1})$,

iii. $\mathscr{T} \vdash \varphi v_0 \ldots v_{n-1} v \wedge \varphi v_0 \ldots v_{n-1} v' \to v = v'$.

Proof. Let $\psi v_0 \ldots v_n$ be a Σ_1 definition of the graph of f and define

$$\mathscr{T} \vdash \varphi v_0 \ldots v_n \leftrightarrow \mathrm{Sel}[\mathrm{Ref}^{n+1}_{\mathscr{T}}(\ulcorner\varphi\urcorner) \vee \psi] . \blacksquare$$

By Application 1, every partial recursive function is (correctly) semi-representable in $\mathscr{T}$ by a Σ_1 formula — where we say that a partial recursive function f is semi-represented by φ if, as in i, φ semi-represents the graph of f and, as in iii, some unicity condition is satisfied. By conclusion ii of the Application, we can even impose some control on the refutability of $\varphi\bar{x}_0 \ldots \bar{x}_n$ when $f x_0 \ldots x_{n-1} \neq x_n$. Without this additional control, the semi-representability of f is easily established as a corollary to the Syntactic selection theorem: If ψ is a correct Σ_1 semi-representation of the graph of f, then $\mathrm{Sel}[\psi]$ is a correct Σ_1 semi-representation of f.

One use of the additional condition ii is this: The Dual semi-representability theorem can be proven as a corollary to Application 1. For, let R, S be disjoint n-ary r.e. relations and define f by

$$f x_0 \ldots x_{n-1} = \begin{cases} 0, & R x_0 \ldots x_{n-1} , \\ 1, & S x_0 \ldots x_{n-1} . \end{cases}$$

If φ_1 is the semi-representation of f supplied by Application 1, then $\varphi v_0 \ldots v_{n-1} = \varphi_1 v_0 \ldots v_{n-1}\bar{0}$ offers the dual semi-representation.

Our second application is a proof via diagonalization of a theorem of Kripke [17].

APPLICATION 2. *Let* $\mathscr{T} \vdash \varphi v_0 \leftrightarrow \mathrm{Sel}[\mathrm{Ref}^1_{\mathscr{T}}(v_0, \ulcorner\varphi v_0\urcorner)]v_0$. *Then, for all* $x, y \in N$.

i. $\mathscr{T}$ *not* $\vdash \varphi\bar{x}$,

ii. $\mathscr{T}$ *not* $\vdash \neg\varphi\bar{x}$,

iii. *if* $y \neq x$, $\mathscr{T} \vdash \neg(\varphi\bar{x} \wedge \varphi\bar{y})$.

In words, φv_0 is a formula each of whose instances is independent and any pair of whose instances is inconsistent. In terms of the negations, it generalizes the failure of the π_1-Disjunction property: There are $\varphi, \psi \in \pi_1$ such that $\mathscr{T} \vdash \varphi \vee \psi$, but $\mathscr{T}$ not $\vdash \varphi$, $\mathscr{T}$ not $\vdash \psi$. As the Application is so simple, we omit the proof.

There is a uniform functorial Shepherdson fixed-point which allows us to extend Applications 1 and 2 to r.e. sequences of consistent theories. We leave the details to the reader.

We do not consider a modified functorial Shepherdson fixed-point since the extra sentences adjoined will usually destroy the unicity condition.

The semi-representability of partial recursive functions was originally proven recursion-theoretically in [27]. In [34] we supply an additional recursion-theoretic lemma allowing a recursion-theoretic proof of the slightly more powerful Application 1. This proof uniformizes. Kripke [17] proved Application 2 by appeal to the semi-representability of partial recursive functions and the Recursion theorem. A uniform proof via effectively inseparable sets ([23]) can also be given.

We should remark that the proof in [28] of the semi-representability of partial recursive functions used a self-referential formula which we would describe as a modified partial Shepherdson fixed-point (cf. the next section).

6. Other fixed-points. In the Introduction to Part I ([33]), we announced that these two papers would constitute a sort of survey of self-referential formulae with the present paper devoted to the more applicable ones. Our survey of applicable self-referential formulae has been a bit narrow — we have really considered only one such formula. In this final Section, we comment on a few other notable examples.

For convenience, we shall consider only sentences where possible.

We begin with the most Shepherdsonesque fixed-points.

Friedman's fixed-point. For intuitionistic theories $\mathscr{T}$ containing Heyting's arithmetic, Friedman [5] used the fixed-point,

$$\mathscr{T} \vdash \varphi \leftrightarrow .(\mathrm{Ref}^0_{\mathscr{T}}(\ulcorner\varphi\urcorner) \vee \psi) \preccurlyeq \mathrm{Thm}^0_{\mathscr{T}}(\ulcorner\varphi\urcorner) ,$$

where $\psi = \exists v_0 \psi' v_0$ is not necessarily Σ_1 (and so is not really a Shepherdson fixed-point). In order to apply φ when $\psi \notin \Sigma_1$, Friedman had to assume that $\mathscr{T}$ had the Disjunction property: For all sentences θ_0, θ_1, if $\mathscr{T} \vdash \theta_0 \vee \theta_1$, then $\mathscr{T} \vdash \theta_0$ or $\mathscr{T} \vdash \theta_1$. On this assumption, one can actually calculate φ. First, define a sentence $\exists v_0 \theta v_0$ to be *formally witnessed* in $\mathscr{T}$ by a number x if $\mathscr{T} \vdash \theta\bar{x}$. Then, for $\psi = \exists v_0 \psi'$, $\chi = \exists v_0 \chi'$, if we define

$$\mathscr{T} \vdash \varphi \leftrightarrow .(\mathrm{Ref}^0_{\mathscr{T}}(\ulcorner\varphi\urcorner) \vee \psi) \preccurlyeq (\mathrm{Thm}^0_{\mathscr{T}}(\ulcorner\varphi\urcorner) \vee \chi)$$

and assume $\mathcal{T}$ to have the Disjunction property, then

$$\mathcal{T} \vdash \varphi \Leftrightarrow \psi \preccurlyeq \chi \text{ is formally witnessed in } \mathcal{T}.$$

With respect to Friedman's fixed-point, this reads

$$\mathcal{T} \vdash \varphi \Leftrightarrow \psi \text{ is formally witnessed in } \mathcal{T}.$$

Since $\mathrm{Thm}^0_{\mathcal{T}} \in \Sigma_1$, evidently

$$\mathcal{T} \vdash \psi \to \varphi \vee \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner),$$

whence

$$\mathcal{T} \vdash \psi \Rightarrow \mathcal{T} \vdash \varphi \quad \text{or} \quad \mathcal{T} \vdash \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner).$$

But the Disjunction property implies the Σ_1-Disjunction property, which even intuitionistically implies Σ_1-soundness, whence $\mathcal{T} \vdash \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi \urcorner) \Rightarrow \mathcal{T} \vdash \varphi$, and

$$\begin{aligned} \mathcal{T} \vdash \psi &\Rightarrow \mathcal{T} \vdash \varphi \\ &\Rightarrow \psi \text{ is formally witnessed in } \mathcal{T}. \end{aligned}$$

This last implication is a result which surprised a number of proof-theorists (the author included). Theoretically, the present discussion removes some of the mystery behind it.

So much for our digression to intuitionistic metamathematics.

Shepherdson's partial-Shepherdson fixed-points. In addition to the basic Shepherdson fixed-point, Shepherdson [32] also considered the weaker fixed-points,

$$\mathcal{T} \vdash \varphi_1 \leftrightarrow .\psi \preccurlyeq \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner),$$
$$\mathcal{T} \vdash \varphi_2 \leftrightarrow .\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner) \preccurlyeq \chi,$$

and offered the partial computations:

$$\mathcal{T} \vdash \varphi_1 \Leftrightarrow N \models \psi,$$
$$\mathcal{T} \vdash \neg\varphi_2 \Leftrightarrow N \models \chi.$$

Shepherdon's original application of these fixed-points was, of course, non-dual semi-representability. These fixed-points and their mild generalizations,

$$\mathcal{T} \vdash \varphi_3 \leftrightarrow .\psi \preccurlyeq (\mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi_3 \urcorner) \vee \chi),$$
$$\mathcal{T} \vdash \varphi_4 \leftrightarrow .(\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_4 \urcorner) \vee \psi) \preccurlyeq \chi,$$

which calculate to,

$$\mathcal{T} \vdash \varphi_3 \Leftrightarrow N \models \psi \preccurlyeq \chi,$$
$$\mathcal{T} \vdash \neg\varphi_4 \Leftrightarrow N \models \chi \prec \psi,$$

are rather popular among those who do not need to control both provability and refutability in an application — either because they are not interested, or because the control is already there because the side formula (e.g. χ in φ_2) is implied by the extra disjunct (here, $\mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner)$).

We promised in Section 2 to comment on

Jensen and Ehrenfeucht's fixed-points. Jensen and Ehrenfeucht [12] proved the equivalence of Σ_1-soundness and the decidability of all provably Δ_1 sentences by means of the simultaneous diagonalization:

$$\mathcal{T} \vdash \varphi_1 \leftrightarrow .(\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner) \vee \psi) \preccurlyeq \mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner),$$
$$\mathcal{T} \vdash \varphi_2 \leftrightarrow .\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner) \prec (\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner) \vee \psi).$$

The claim is that, if ψ is a false Σ_1 theorem of $\mathcal{T}$, then $\mathcal{T} \vdash \varphi_1 \leftrightarrow \neg\varphi_2$ and $\mathcal{T}$ does not decide φ_1. Evidently, $\mathcal{T} \vdash \neg(\varphi_1 \wedge \varphi_2)$ and $\mathcal{T} \vdash \psi \to \varphi_1 \vee \varphi_2$. Thus, if $\mathcal{T} \vdash \psi$, we have $\mathcal{T} \vdash \varphi_1 \leftrightarrow \neg\varphi_2$. The partial calculation of the partial Shepherdson fixed-points yields ($\preccurlyeq$ and $\prec$ are interchangeable in these matters),

$$\mathcal{T} \vdash \neg\varphi_1 \Leftrightarrow N \models \mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner \prec \psi),$$
$$\mathcal{T} \vdash \neg\varphi_2 \Leftrightarrow N \models \mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner) \vee \psi.$$

Since N not $\models \psi$ we have e.g.

$$\mathcal{T} \vdash \neg\varphi_1 \Leftrightarrow N \models \mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner).$$

But this yields

$$\mathcal{T} \vdash \neg\varphi_1 \Leftrightarrow \mathcal{T} \vdash \neg\varphi_2$$

while the equivalence $\mathcal{T} \vdash \varphi_1 \leftrightarrow \neg\varphi_2$ yields

$$\mathcal{T} \vdash \varphi_1 \leftrightarrow \mathcal{T} \vdash \neg\varphi_2,$$

and φ_1 is not decided by $\mathcal{T}$. How does this compare with Guaspari's proof? We note that, if we make illegitimate substitutions based on the equivalence $\varphi_1 \leftrightarrow \neg\varphi_2$, we obtain

$$\varphi_1 \leftrightarrow .(\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner) \vee \psi) \preccurlyeq \mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner),$$
$$\varphi_2 \leftrightarrow .\mathrm{Thm}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner) \prec (\mathrm{Ref}^0_{\mathcal{T}}(\ulcorner \varphi_1 \urcorner) \vee \psi).$$

which are the formulae used in Guaspari's proof, presented in Section 2, of the result.

Open problem. Find other applications of simultaneous diagonalization. Occasionally, such an application ought to be more efficient than a corresponding application of a single diagonalization.

We should also comment on

Ritter's fixed-point. In the last section, we remarked that Ritter [28] proved the semi-representability of partial recursive functions by means of a modified partial Shepherdson fixed-point. In simplified form, the fixed-point is

$$\mathcal{T} \vdash \varphi v_0 v_1 \leftrightarrow \mathrm{Sel}[\psi v_0 v_1] \wedge [\exists v_2 \psi v_0 v_2 \preccurlyeq \mathrm{Thm}^2_{\mathcal{T}}(v_0, v_1, \ulcorner \varphi v_0 v_1 \urcorner)],$$

where

i. $\psi \in \Sigma_1$ defines the graph of a unary partial function f,

ii. We assume the two existential quantifiers of $\exists v_2 \psi$ to be contracted.

Assuming a modified partial calculation, we get, for all $x, y \in N$,

$$\begin{aligned}\mathcal{T} \vdash \varphi\bar{x}\bar{y} &\Leftrightarrow \mathcal{T} \vdash \mathrm{Sel}[\psi v_0 v_1]\bar{x}\bar{y} \,\&\, N \models \exists v_2 \psi\bar{x}v_2\\ &\Leftrightarrow N \models \psi\bar{x}\bar{y}\\ &\Leftrightarrow fx = y\,.\end{aligned}$$

To conclude the second equivalence, note that the unicity condition yields, for x in the domain of f,

$$\mathcal{T} \vdash \mathrm{Sel}[\psi v_0 v_1]\bar{x}\bar{y} \Leftrightarrow fx = y\,.$$

But the assertion $N \models \exists v_2 \psi \bar{x} v_2$ means that x is in this domain.

Guaspari's fixed-points. Guaspari [9] proves the existence of independent $\pi_1(\Sigma_1)$ sentences which are Σ_1-(π_1-) conservative over a given $\mathcal{T} \supseteq \mathcal{PA}$. The proofs require characterizations of such sentences and two fixed-points of the forms

$$\mathcal{T} \vdash \varphi_1 \leftrightarrow .\mathrm{Pr}_{\mathcal{T}}(\ulcorner \neg\varphi_1 \urcorner) \preccurlyeq \chi(\ulcorner \varphi_1 \urcorner), \qquad \chi \in \Sigma_2\,,$$
$$\mathcal{T} \vdash \varphi_2 \leftrightarrow \exists v_0 [\psi(v_0, \ulcorner \varphi_2 \urcorner) \preccurlyeq \mathrm{Pr}_{\mathcal{T}}(\ulcorner \varphi_2 \urcorner)], \qquad \psi \in \Sigma_1\,.$$

OPEN PROBLEM. Give a good accounting for the success of Guaspari's fixed-points. How general can a Guaspari fixed-point be? It may be that some conditions have to be imposed on ψ, χ for a smooth calculation. How much uniformity can be built into Guaspari fixed-points? The unboundable existential quantifiers of φ_1, φ_2 preclude an easy uniformization.

Hierarchically generalized fixed-points. The literature abounds with hierarchical generalizations of the fixed-points we have been discussing. Cf. e.g. [5], [11], [12], [13], [16], [19], and [31]. While the applications can be novel (especially Kreisel and Levy [16]), the fixed-points usually are not. One works in $\mathcal{T} \supseteq \mathcal{PA}$ where truth definitions, $\mathrm{Tr}_n(\cdot)$ for sentences of bounded complexity are available (with occasionally useful provable closure properties) and replace $\mathrm{Pr}_{\mathcal{T}}(v_0)$ by

$$\exists v_1 [\mathrm{Tr}_n(v_1) \wedge \mathrm{Pr}_{\mathcal{T}}(\mathrm{imp}(v_1, v_0))]\,,$$

which asserts that the formula with code v_0 is derivable from some true sentence of complexity at most n. In any non-standard model of $\mathcal{PA}$, Σ_{n+1} formulae behave like Σ_1 formulae relative to the true sentences of complexity at most n. Thus, in any non-standard model of $\mathcal{T}$, the underlying calculation can be made. Hence, if the application in mind is sufficiently invariant, it can be made. With these considerations, we can cavalierly dismiss the hierarchically generalized fixed-points as being unproblematic.

There are, of course, other fixed-points worth discussing. Probably, though, we have considered sufficiently many to have demonstrated the usefulness of the program of fixed-point calculations. So we shall stop here.

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On order locally finite and closure-preserving covers

by

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Abstract. The present paper deals with structural properties of the covers and contains the order locally finite sum theorem and the closure-preserving sum theorem for the covering dimension.

The purpose of this paper is to study structural properties of the covers and, besides, by mean of that properties, to derive two general sum theorems for the covering dimension dim. Section 1 contains a characterization of order star-finite open covers. There are many results (cf. [2], [7], [8] and [9]) dealing with spaces endowed with two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ such that E_ξ is closed and has a topological property $\mathscr{P}$, while U_ξ is an open neighborhood of E_ξ for each $\xi<\alpha$. In Section 2 a structure of such spaces is described and, in particular, the order locally finite sum theorem for the covering dimension is established. Finally, Section 3 is concerned with closure-preserving closed covers consisting of countably compact sets, where the closure-preserving sum theorem for the covering dimension is proved. The last result turns out to be a special case of a statement established by a topological game.

The set of natural numbers $1, 2, 3, \ldots$ is denoted by N, while natural numbers by k, m and n. Ordinal numbers are denoted by α, ξ, η and ζ.

Let $\{A_i\colon i\in I\}$ be an indexed family of subsets of a space X. We shall denote by $\{A_i\colon i\in I\}^{\#}$ the set of all points $x\in X$ such that the set $\{i\in I\colon U\cap A_i\neq 0\}$ is infinite for each neighborhood U of x.

Let us note several properties of the operation $\#$:

(a) $\{A_i\colon i\in I\}^{\#}=0$ iff $\{A_i\colon i\in I\}$ is locally finite.

(b) $\{A_i\colon i\in I\}^{\#}$ is closed in X.

(c) If $U\supset\{A_i\colon i\in I\}^{\#}$, where U is open, then $\{A_i-U\colon i\in I\}^{\#}=0$.

(d) If $B_i\subset A_i$ for each $i\in I$, then $\{B_i\colon i\in I\}^{\#}\subset\{A_i\colon i\in I\}^{\#}$.

1. Order star-finite covers. A family $\{A_i\colon i\in I\}$ of subsets of a space X is said to be *order star-finite* [9], if one can introduce a well ordering $<$ in the index set I so that for each $i\in I$ the set A_i meets at most finitely many A_j with $j<i$. Since every well ordered set is order isomorphic to an initial segment of ordinal numbers, we may use the notation $\{A_\xi\colon \xi<\alpha\}$ instead of $\{A_i\colon i\in I\}$.