Algebraic topology for proper shape theory

by

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Abstract. Theorems are proved which provide analogs in the proper shape category for the classical Whitehead theorems and Hurewicz theorems. These provide an analog for the homology version of the classical Whitehead theorem. We define new algebraic functors which are obtained from known functors by a standard process and we provide an analog of standard obstruction theory which is valid in the proper category.

Let $f\colon X \to Y$ be a continuous map. In [12], Whitehead shows that if X and Y are CW complexes and if $f_*\colon \pi_*(X) \to \pi_*(Y)$ is an isomorphism then f is a homotopy equivalence. This theorem has been very useful in subsequent work and it has been modified by means of the Hurewicz theorem and Universal coefficient theorems so that it is more easily aoplied. In [4], Farrel, Taylor and Wagoner show that if f is in a certain nice category and f is properly o-connected and $f_*\colon \Delta(\underline{X}, \{x\}, \pi_*, \text{no cov}) \to \Delta(\underline{Y}, \{y\}, \pi_*, \text{no cov})$ is an isomorphism, then f is a proper homotopy equivalence. They have also obtained homology and cohomology versions of this theorem. In [5] and [6], Mardešić obtains shape versions of these theorems, using pro groups.

In this paper, corresponding theorems are obtained for the proper shape category, using a "Pro" version of the "proper functors" of [4]. In the course of developing this theory, we obtain a version of obstruction theory in the proper category, a Hurewicz theorem for proper shape theory and a method for associating an inverse system of finite dimensional locally finite simplicial complexes to a finite dimensional locally compact, paracompact Hausdorff space.

I. Notation and definitions.

I.1. Definition. A continuous function is *proper* if the inverse image of each compact set is compact. A topological pair (X, A) is *proper* if the inclusion of A into X is proper. $g \overset{p}{\simeq} h$ indicates that g and h are homotopic via a proper homotopy.

I.2. Notation. Let D represent the category whose objects are topological spaces and whose morphisms are continuous functions. Let G represent the category whose objects are groups and whose morphisms are homomorphisms. Suppose C is a subcategory of D. We indicate some other categories by the following notational system:

C^2 is the category of pairs of objects of C.

C' is the category whose objects are objects of C, but whose morphisms are proper morphisms of C.

C_0 is the category of objects of C with base point.

These modifications can be used in conjunction: C_0^2 is the category of pointed pairs, C_0' is the proper category of pointed spaces, $(C')^2$ is the proper category of pairs, etc. The following definition is used for C_0' and $(C_0')^2$: A set of base points for an object X of C', [4], is a set $\{x_\alpha\}_{\alpha\in\mathscr{A}}$ satisfying:

1) $\{x_\alpha\}_{\alpha\in\mathscr{A}}$ is a locally finite subset of X,

2) for any compact subset of X, any infinite component of the complement (i.e. one not contained in any compact subset of X) intersects $\{x_\alpha\}_{\alpha\in\mathscr{A}}$ and

3) any subset of $\{x_\alpha\}_{\alpha\in\mathscr{A}}$ satisfying 2 must have the same cardinality.

We will consider C_0 as a subcategory of C_0^2 by identifying $(\underline{X}, x_0)$ with $(\underline{X}, x_0, x_0)$. We will apply these constructions to each of the following three subcategories of D:

1) P is the full subcategory whose objects are simplicial complexes,

2) T is the full subcategory whose objects are locally compact metric spaces,

3) W is the full subcategory of T whose objects are ANR's for metric spaces.

II. Δ-groups and Pro-groups. The reader is referred to [5] for an excellent description of the Pro construction. Roughly, a pro-category is obtained from a category C as follows: An object of Pro-C is an inverse system of objects and morphisms of C, denoted $\underline{A} = (A_\lambda, p_{\lambda\lambda'}\colon A_{\lambda'} \to A_\lambda, \Lambda)$. A morphism in Pro-$C$ from $\underline{A}$ to $\underline{B} = (B_\mu, q_{\mu\mu'}, M)$ is a collection of morphisms of C: $\underline{f} = (f\colon M \to \Lambda, f_\mu\colon A_{f(\mu)} \to B_\mu)$. The bonding maps are required to commute with each other and the coordinate maps of each morphism are required to commute with the bonding maps, at least when composed with some bonding map of the domain. The category for which we obtain our main technical result is $\mathrm{Pro}(P_0^1)^2$.

We recall the "Δ" construction of [4]:

II.1. Definition. Let $F\colon P_0^2 \to G$ be a functor and let $(X, A, \{x_\alpha\}_{\alpha\in\mathscr{A}})$ be an object in $(P_0')^2$. $\Delta((X, A, \{x_\alpha\}), F)$ is the pullback in the following diagram:

$$\begin{tikzcd}
\Delta((X, A, \{x_\alpha\}), F) \arrow[r] \arrow[d] & \prod\limits_{\alpha\in\mathscr{A}} (F(X, A, x_\alpha)) \arrow[d] \\
\varprojlim\limits_{\substack{\mathrm{CPT} \\ K\subseteq X}} \dfrac{\prod_{\alpha\in\mathscr{A}} (F(X\setminus K, A\setminus K, x_\alpha))}{\sum_{\alpha\in\mathscr{A}} (F(X\setminus K, A\setminus K, x_\alpha))} \arrow[r] & \dfrac{\prod_{\alpha\in\mathscr{A}} (F(X, A, x_\alpha))}{\sum_{\alpha\in\mathscr{A}} (F(X, A, x_\alpha))}
\end{tikzcd}$$

where the inverse limit is taken over compact subsets K of X and $F(X\setminus K, A\setminus K, x_\alpha)$ is taken to be 0 if $x_\alpha \in K$.

If f is a morphism of $(P_0')^2$, the homomorphism induced by f will be denoted by $\Delta(f, F)$.

We will need the following lemma (c.f. Theorem I.2.4 of [11]).

II.2. Lemma. *Let $f\colon (X, A, \{x\}_{\alpha\in\mathscr{A}}) \to (Y, B, \{y\}_{\beta\in\mathscr{B}})$ be a morphism of $(P_0')^2$ and suppose $\Delta(f, \pi_k) = 0$. Then, for every compact subset C of Y, there is a compact subset D of X, $D \supseteq f^{-1}(C)$, such that $f_*\colon \pi_k(X\setminus D, A\setminus D, x_\alpha) \to \pi_k(Y\setminus C, B\setminus C, f(x_\alpha))$ is zero for all but finitely many x_α.*

Proof. Suppose not, then there is a C such that for all compact $D \supseteq f^{-1}(C)$, there are infinitely many x_α such that the map is non zero. For each such α and D, choose a $z_{\alpha D}$ in the domain which does not go to 0. We will construct an element γ of $\Delta((X, A, \{x\}), \pi_k)$ which goes nontrivially through the following composition:

$$\Delta((X, A, \{x\}), \pi_k) \to \Delta((Y, B, \{y\}), \pi_k) \to \frac{\prod_{\beta\in\mathscr{B}} \pi_k(Y\setminus C, B\setminus C, y_\beta)}{\sum_{\beta\in\mathscr{B}} \pi_k(Y\setminus C, B\setminus C, y_\beta)}.$$

First choose a compact exhaustion $\{D_i\}_{i=1}^\infty$ of X, with $D_1 = f^{-1}(C)$, and well-order $\mathscr{A}$. Now for each element $\alpha \in \mathscr{A}$, assign one of the D_i, say $D_{g(\alpha)}$ and an element $\gamma_\alpha \in \pi_k(X\setminus D_{g(\alpha)}, A\setminus D_{g(\alpha)}, x_\alpha)$ inductively as follows: assign D_1 to the first α and all those following it until you hit an α_1 for which a $z_{\alpha_1 D_1}$ is defined. Then to $\alpha < \alpha_1$, assign the element $0 \in \pi_k(X\setminus D_1, A\setminus D_1, x_\alpha)$ (recall that this is defined to be the zero group if $x_\alpha \in D_1$) and assign D_1 and $z_{\alpha_1 D_1}$ to α_1. Now, begin assigning D_2 and 0 until you hit an α_2 such that $z_{\alpha_2 D_2}$ is defined and $f(x_{\alpha_2}) \neq f(x_{\alpha_1})$. This can be done, as there are infinitely many α for which $z_{\alpha D_2}$ is defined and only finitely many α satisfy $\alpha < \alpha_1$ or $f(x_\alpha) = f(x_{\alpha_1})$. Of course, $\gamma_{\alpha_2} = z_{\alpha_2 D_2}$. Continuing in this way, obtain a sequence $\{\gamma_\alpha\}_{\alpha\in\mathscr{A}}$ such that $\gamma_\alpha \neq 0 \Rightarrow \gamma_\alpha = z_{\alpha D_{g(\alpha)}}$ and $\gamma_\alpha \neq 0 \neq \gamma_{\alpha'} \Rightarrow f(x_\alpha) \neq f(x_{\alpha'})$. This sequence determines an element, γ, of $\Delta(X, A, \{x\})$ in a fairly straightforward way: Its projection by the horizontal arrow has α-coordinate $i_*\gamma_\alpha$ where i is the inclusion. Its other projection has D_i-coordinate $[\{g_*\gamma_\alpha\}_{\alpha\in\mathscr{A}}]$ where

$$g\colon (X\setminus D_{g(\alpha)}, A\setminus D_{g(\alpha)}, x_\alpha) \to (X\setminus D_i, A\setminus D_i, x_\alpha)$$

is inclusion whenever $D_{g(\alpha)} \supseteq D_i$, and g_* is 0 otherwise, and the square brackets denote equivalence class in the quotient group. Observe that we have only used a cofinal sequence of the compact subsets of X, but this, of course, suffices to determine an element of the inverse limit. Also, we have used the standard construction of the inverse limit as a subspace of the cartesian product, so the D_i-coordinate makes sense. Note that the D_1-coordinate of γ goes nontrivially into the C-coordinate of $\Delta((Y, B, \{y\}), \pi_k)$, so γ is the element we wished to construct. An immediate corollary is that we may assume D is chosen so that f_* is zero for all base points simply by enlarging D to contain the finitely many exceptions.

II.3. Lemma. *Let $(X, A, \{x_\alpha\}_{\alpha\in\mathscr{A}})$ and $(Y, B, \{y_\beta\}_{\beta\in\mathscr{B}})$ be objects in $(P_0')^2$ with dimension of $X\setminus A \leqslant n$ and the $n-1$ skeleton of X, $X^{n-1} \subseteq A$. Then if $f\colon (X, A, \{x_\alpha\}_{\alpha\in\mathscr{A}})$*

$\to (Y, B, \{y_\beta\}_{\alpha\in\mathscr{B}})$ *is a proper map such that* $\Delta(f, \pi_n) = 0$, *then* f *deforms properly* rel$A$ *and* $\{x_\alpha\}_{\alpha\in\mathscr{A}}$ *to a map into* B.

Proof (cf. 3.2 of [4]). Write $X = A \cup P \cup Q$ where each of P, Q is the disjoint union of finite complexes $P = P_1 \cup P_2 \cup \ldots$, $Q = Q_1 \cup Q_2 \cup \ldots$ We show that $f|A \cup P$ can be properly deformed rel$A$ down into B. The proper homotopy extension theorem [4] and a repeat of the argument, then completes the theorem.

According to 3.3 of 4, it suffices to show: (i) For each $i \geqslant 1$ f: $(P_i, A \cap P_i) \to (Y, B)$ deforms rel$A \cap P_i$ down into B, and (ii) Given any compact set $C \subseteq Y$, there is a compact set $D \subseteq Y$, $D \supseteq C$ and a positive integer m such that f maps $(P_i, A \cap P_i)$ into $(Y \setminus D, A \setminus D)$ whenever $i \geqslant m$ and such that the map f: $(P_i, A \cap P_i) \to (Y \setminus C, B \setminus C)$ deforms rel$A \cap P_i$ down into $B \setminus C$ within $Y \setminus C$. Since $\Delta(f, \pi_n) = 0$, we get $\pi_n(f) = 0$ and so, we may push the n-skeleton of P_i down into B one simplex at a time. Extending to all of P_i via the homotopy extension theorem and stacking the homotopies provides a homotopy of P_i which carriers all of P_i^n into B. Thus, condition (i) is satisfied.

To get (ii), let $C \subseteq Y$ be compact. As $\Delta(f, \pi_n) = 0$, there is a compact set D containing C such that $\pi_n(f)$: $\pi_n(X \setminus f^{-1}(D), A \setminus f^{-1}(D)) \to \pi_n(Y \setminus C, B \setminus C)$ is the zero homomorphism. Choose m so large that whenever $j \geqslant m$, $f(P_j)$ is contained in an infinite component of $Y \setminus D$. Then $\pi_n(f)$: $\pi_n(P_j, P_j \cap A) \to \pi_n(X \setminus f^{-1}(D), A \setminus f^{-1}(D)) \to \pi_n(Y \setminus C, B \setminus C)$ is the zero homomorphism and as above, we obtain that f: $(P_i, A \cap P_i) \to (Y \setminus C, B \setminus C)$ deforms rel$A \cap P_i$ down into $B \setminus C$ within $Y \setminus C$. This choice of D and m satisfies hypothesis (ii) of [4] Lemma 3.3.

II.4. Definition. If $\underline{X} = ((X, A, \{x\}_\lambda), P_{\lambda\lambda'}, \Lambda)$ is an object of $\mathrm{Pro}(P_0')^2$ [5] then $\Delta(\underline{X}, \pi_n)$ is $((\Delta(X, A, \{x\}_\lambda), \pi_n), \Delta(P_{\lambda\lambda'}, \pi_n), \Lambda)$ and is called the n-*th* ProΔ-*homotopy group* of $\underline{X}$. This will play the role of the homotopy group in our theorem. We obtain from [4, Theorem 2.12] the following exact sequence of pro-groups and pro-pointed sets:

II.5. For $(\underline{X}, \underline{A}, \underline{\{x\}})$ an object of $(P_0')^2$: $\ldots \to \Delta(\underline{(A, \{x\})}, \pi_k) \to \Delta(\underline{X}, \underline{\{x\}}, \pi_k) \to \Delta(\underline{X}, \underline{A}, \underline{\{x\}}, \pi_k) \to \ldots \to \Delta(\underline{X}, \underline{\{x\}}, \pi_0)$ is exact.

III. The main technical results. For this section, $\simeq$ denotes proper homotopy. The principal result of this section is the following:

III.1. Theorem 1. *Let* $(\underline{X}, \underline{A}, \underline{\{x\}}) = ((X, A, \{x\})_\lambda, P_{\lambda\lambda'}, \Lambda)$ *and* $(\underline{Y}, \underline{B}, \underline{\{y\}}) = ((Y, B, \{y\})_\mu, q_{\mu\mu'}, M)$ *be objects of* $(P_0')^2$ *such that* $\dim(X_\lambda) \leqslant n$ *for all* λ. *If* $\Delta(\underline{Y}, \underline{B}, \underline{\{y\}}, \pi_k) = 0$ *for* $1 \leqslant k \leqslant n+1$ *and* M *is closure finite, then each morphism* $\underline{f}$: $(\underline{X}, \underline{A}, \underline{\{x\}}) \to (\underline{Y}, \underline{B}, \underline{\{y\}})$ *admits a morphism* $\underline{g}$: $(\underline{X}, \underline{\{x\}}) \to (\underline{B}, \underline{\{y\}})$ *such that* $\underline{j}\underline{g} = \underline{f}$: $(\underline{X}, \underline{\{x\}}) \to (\underline{Y}, \underline{\{y\}})$ *where* $\underline{j}$: $(\underline{B}, \underline{\{y\}}) \to (\underline{Y}, \underline{\{y\}})$ *is inclusion induced.*

The proof involves two lemmas which assume the hypotheses of the theorem.

III.2. Lemma 1. *For each* $\mu \in M$, *there is a* $\mu^* \geqslant \mu$ *such that for every relatively* $n+1$ *dimensional polyhedral pair* $(P, Q, \{x\}) \in (P_0')^2$ *and for every proper map* φ: $(P, Q, \{x\}) \to (Y, B, \{y\})_{\mu^*}$, *there is a proper map* ψ: $(P\ \{x\}) \to (B, \{y\})_\mu$ *such that*

1) $\psi \simeq H_{\mu\mu^*}\varphi$: $(P, \{x\}) \to (Y, \{x\})_\mu$ *and*

2) $\psi|(Q, \{x\}) \simeq H_{\mu\mu^*}\varphi|(Q, \{x\})$: $(Q, \{x\}) \to (B, \{y\}_\mu)$.

In fact, $\psi \simeq H_{\mu\mu^*}\varphi$: $(P, Q, \{x\}) \to (Y, B, \{y\})_\mu$ *where* $H_{\mu\mu^*}$ *is a map representing the proper homotopy class* $q_{\mu\mu^*}$. *Moreover, if* $\mu \leqslant \mu'$, *then we may take* $\mu^* \leqslant \mu'^*$.

Proof. Let $\mu = \mu_0 \leqslant \mu_1 \leqslant \ldots \leqslant \mu_{n+1} = \mu^*$ be a chain in M such that for each k: $1 \leqslant k \leqslant n+1$, the homomorphism $q_{\mu_j\mu_{j+1}}$: $\Delta((Y, B, \{y\})_{\mu'_{j+1}}\pi_k) \to \Delta((Y, B, \{y\})_{\mu_j}, \pi_k)$ is 0, for $j = n+1-k$.

We define, by induction, maps ψ_k: $P \times I$, $Q \times I \to (Y, B, \{y\})_{\mu_{n+1-k}}$ such that

1) $\psi_k(x, t) = H_{\mu_{n+1-k}, \mu_{n+1}}\varphi(x)$ if $t = 0$ or $x \in Q$ and

2) $\psi_k(P^k \times \{1\}) \subseteq B_{\mu_{n+1-k}}$.

By use of the lemma of Section II and the proper homotopy extension theorem [4], we obtain ψ_k by properly deforming the kth skeleton of P into $B_{\mu_{n+1-k}}$.

III.3. Lemma 2. *For each* μ *in* M, *let* $\mu^* \geqslant \mu$ *be as given by Lemma* 1. *Let* $(P, \{x_0\})$ *be an object of* P_0', $\dim(P) \leqslant n$. *Now, let* φ_0, φ_1: $(P, \{x\}) \to (B, \{y\})_{\mu^*}$ *be proper maps of pairs such that*

$$j_{\mu_*}\varphi_0 \simeq j_{\mu^*}\varphi_1\colon (P, \{x\}) \to (Y, \{y\})_{\mu^*}$$

then

$$H_{\mu\mu^*}\varphi_0 \simeq H_{\mu\mu^*}\varphi_1\colon (P, \{x\}) \to (B, \{y\})_\mu .$$

Proof. We apply Lemma 1 to the given homotopy.

Proof of Theorem 1. Let $\underline{f}$: $(\underline{X}, \underline{A}, \underline{\{x\}}) \to (\underline{Y}, \underline{B}, \underline{\{y\}})$ be given by f: $M \to \Lambda$ and relative proper homotopy classes of proper relative maps f_μ: $(X, A, \{x\})_{f(\mu)} \to (Y, B, \{y\})_\mu$ having representatives φ_μ. Since M is closure finite, we may assume f is increasing and that

$$\begin{array}{ccc} (X, A, \{x\})_{f(\mu)} & \xleftarrow{\ p\ } & (X, A, \{x\})_{f(\mu')} \\ \Big\downarrow{\scriptstyle f_\mu} & & \Big\downarrow{\scriptstyle f_{\mu'}} \\ (Y, B, \{y\}_\mu & \xleftarrow{\ q\ } & (Y, B, \{y)\}_{\mu'} \end{array} \qquad \text{commutes.}$$

For each μ in M, choose μ^* according to Lemma 1. Then for each μ in M, there is a unique ψ_{μ^*}: $(X, \{x\})_{f(\mu^{**})} \to (B, \{y\})_{\mu^*}$ such that

$$\begin{array}{ccc} (B, \{y\})_{\mu^*} & \xleftarrow{\ \psi\ } & (X, \{x\})_{f(\mu^{**})} \\ \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle \varphi} \\ (Y, \{y\})_{\mu^*} & \xleftarrow{\ H\ } & (Y, \{y\})_{\mu^{**}} \end{array}$$

and

$$\begin{tikzcd} & (A, \{x\})_{f(\mu)^{**}} \arrow[dl, "\psi"'] \arrow[d, "\varphi"] \\ (B, \{y\})_{\mu^*} & (B, \{y\})_{\mu^{**}} \arrow[l, "H"] \end{tikzcd} \qquad \simeq \text{commute.}$$

Now, define $g\colon M \to \Lambda$ by $g(\mu) = f(\mu^{**})$ and define $g_\mu\colon (X, \{x\})_{g(\mu)} \to (B, \{y\})_\mu$ by $g_\mu = q_{\mu\mu*}\psi_\mu{}^*$.

Claim. A) $\underline{g} = (g, \{g_\mu\})$ *is a map of systems and*
B) $(g, \{g_\mu\})$ *satisfies the conclusion of Theorem* 1.

Proof of A. Suppose $\mu \leqslant \mu'$, then

$$\begin{tikzcd}
(B, \{y\})_{\mu*} \arrow[d, "j"'] & (X, \{x\})_{f(\mu^{**})} \arrow[l, "\psi"'] \arrow[d, "\varphi"] \\
(Y, \{y\})_{\mu*} & (Y, \{y\})_{\mu**} \arrow[l, "H"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(B, \{y\})_{\mu'*} \arrow[d, "j"'] & (X, \{x\})_{f(\mu'^{**})} \arrow[l, "\psi"'] \arrow[d, "\varphi"'] \\
(Y, \{y\})_{\mu'*} & (Y, \{y\})_{\mu'**} \arrow[l, "H"']
\end{tikzcd} \qquad \text{commute}$$

Now, since

$$\begin{tikzcd}
(X, \{x\}) \arrow[r, "p"] \arrow[d, "f"'] & (X, \{x\})_{f(\mu'^{*})} \arrow[d, "f"'] \\
(Y, \{y\})_{\mu*} \arrow[r, "q"] & (Y, \{y\})_{\mu'*}
\end{tikzcd} \qquad \text{commutes}$$

we obtain

$$\begin{tikzcd}
& (X, \{x\})_{\mu**} \arrow[d, "\psi"'] & (X, \{x\})_{f(\mu'^{**})} \arrow[l, "p"'] \arrow[d, "\psi"'] \\
(Y, \{y\})_{\mu*} & (B, \{y\})_{\mu*} \arrow[l, "j"'] & (B, \{y\})_{\mu*'} \arrow[l, "H"']
\end{tikzcd} \qquad \text{commutes at } (Y, \{y\})_{\mu*}.$$

Thus, by Lemma 2, it commutes at $(B, \{y\})_{\mu*}$. i.e., $\underline{g}$ is a map of systems.

Proof of B. We have that

$$\begin{tikzcd}
(X, \{x\})_{f(\mu)} \arrow[dd, bend right, "f"'] & (X, \{x\})_{f(\mu^{*})} \arrow[l, "p"'] \arrow[dd, "f"] & (X, \{x\})_{f(\mu^{**})} \arrow[l, "p"'] \arrow[dll, "g"'] \\
(B, \{y\})_\mu \arrow[d, "j"] & & \\
(Y, \{y\})_\mu & (Y, \{y\})_{\mu*} \arrow[l, "q"'] &
\end{tikzcd} \qquad \text{commutes}$$

thus $\underline{j}\underline{g} = \underline{f}$.

We also have that:

$$\begin{tikzcd}
(A, \{x\})_{f(\mu)} \arrow[d, "f"'] & (A, \{x\})_{f(\mu^{**})} \arrow[l, "p"'] \arrow[dl, "g"] \arrow[d, "f"'] \\
(B, \{y\})_\mu & (A, \{x\})_{\mu**} \arrow[l, "q"']
\end{tikzcd} \qquad \text{commutes}$$

so, $\underline{g}|\underline{A} = \underline{f}|\underline{A}$.

This completes the proof of Theorem 1.

III.4. Theorem 2. *Let* $\underline{(X, A, \{x\})}$ *be an object of* $\mathrm{Pro}(P_0')^2$ *such that* $\dim(X_\lambda) \leqslant n$ *for all* λ, $\Delta(X, A, \{x\}, \pi_k) = 0$ *for* $1 \leqslant k \leqslant n+1$ *and* Λ *is closure finite. Then there is a morphism* $\underline{r}\colon \underline{(X, \{x\})} \to \underline{(A, \{x\})}$ *in* $\mathrm{Pro}P_0'$ *such that* $\underline{j}\underline{r} = \underline{l}$ *and* $\underline{r}\underline{j} = \underline{r}|\underline{A}, \{x\} = \underline{l}$. *That is, the morphism* $\underline{j}\colon \underline{(A, \{x\})} \to \underline{(X, \{x\})}$ *given by inclusions is an isomorphism in* $\mathrm{Pro}(P_0')$.

Proof. Apply Theorem 1 to $\underline{f} = \underline{l}\colon \underline{(X, A, \{x\})} \to \underline{(X, A, \{x\})}$. Note that Theorem 2 may be interpreted as a Whitehead theorem in $\mathrm{Pro}(P_0')$ for inclusions $\underline{(A, \{x\})} \hookrightarrow \underline{(X, \{x\})}$.

IV. Good covers and a Whitehead theorem for proper shape theory. In [1], Ball shows that to each locally compact metric space one may associate an inverse system of ANR's. However, the methods he uses provide systems in which each component space is infinite dimensional. We present here an alternate method, using the nerves of certain open covers. This approach has the advantage that it assigns to a finite dimensional space, X (finite dimensional in the sense of covering dimension) an inverse system in which each component space has dimension no greater than that of X. This will allow us to apply the results of the previous section to Proper Shape Theory.

IV.1. Definition 1. Let X be an object of T', a *good cover* of X is an open cover $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of X such that

1) $\overline{U}_\alpha$ is compact for all $\alpha \in \mathscr{A}$,

2) $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ is locally finite.

2. Let $\underline{X} = (X, \{x_\beta\}_{\beta \in \mathscr{A}})$ be an object of T_0' then a *good cover* of $\underline{X}$ is a good cover of $\underline{X}$ such that each x_β is contained in only one U_α.

3. Let $\underline{X} = (X, A, \{x_\beta\}_{\beta \in \mathscr{B}})$ be an object of $(T_0')^2$, then a *good cover* of $\underline{X}$ is a good cover of $(X, \{x_\beta\}_{\beta \in \mathscr{B}})$.

IV.2. It is easily seen that the set of good covers is cofinal in the set of all covers. Thus the inverse system of nerves of good covers of X is associated to X. (Theorem 1.3 of [9]). It is a matter of checking through the constructions in Lemmas 4.1 and 4.2 of [8] to see that it is in fact properly associated to X [1, Def. 3.1].

IV.3. Definition. For X an object of T', $\dim(X)$ is the covering dimension of X with respect to good covers.

IV.4. Note if X is finite dimensional with respect to either ordinary open covers or to good covers, then it is finite dimensional with respect to the other and the dimensions are the same. For if X has dimension n for ordinary open covers, let U be an arbitrary good cover and let U' be a dimension n open refinement, then U' is a good refinement by an easy argument. Thus, the dimension of x with respect to open covers is greater than or equal to $\dim(x)$. The reverse inequality is trivial.

Theorem 2 yields a Whitehead theorem for maps in proper shape theory:

IV.5. THEOREM. *If $f\colon (X, \{x\}) \to (Y, \{y\})$ is a properly o-connected morphism of T_0', $\max(\dim(X)+1, \dim(Y)) \leqslant n$ and f induces isomorphisms of $\Delta((X, \{x\}), \pi_k)$ to $\Delta((Y, \{y\}), \pi_k)$ for $0 \leqslant k \leqslant n$ and an epimorphism for $k = n+1$, then f is a proper shape equivalence.*

Proof. Choose an inverse system of nerves of good covers of $(M_f, X, \{x\})$ consisting of complexes of dimension less than or equal to $n+1$. Note that we may assume the system is closure finite by 2.3 of [5]. II.5 provides the algebraic hypotheses of Theorem 1 and so we obtain IV.5. Mf, of course, is the mapping cylinder of f.

V. The Hurewicz theorem and a homology version of the Whitehead theorem. We prove the following analog of the Hurewicz theorem (c.f. [7]).

V.1. *Let $((X, A, \{x\})_\lambda, P, \Lambda)$ be an object of $\operatorname{Pro}(P_0')^2$. If $n \geqslant 2$ and $\operatorname{Pro}(\Delta((X, A, \{x\}), \pi_k)) = 0$ for $k \leqslant n-1$ and $\operatorname{Pro}(\Delta((A, \{x\}), \pi_1)) = 0$ then*

(1) $\operatorname{Pro}(\Delta((X, A, \{x\}), H_k)) = 0$ for $1 \leqslant k \leqslant n-1$ and

(2) $\varphi_n\colon \operatorname{Pro}(\Delta((X, A, \{x\}), \pi_n)) \to \operatorname{Pro}(\Delta((X, A, \{x\}), H_n))$ is an isomorphism of pro groups.

φ_n is a Pro-map induced by the Hurewicz map of Δ-groups which is in turn induced by the ordinary Hurewicz map.

Proof. Let $L = [o, \infty)$ and for each λ in Λ, choose a proper map $u\colon X_\lambda^1 \to L$. This can be done as follows: choose a compact exhaustion $X_\lambda^1 = \bigcup_{i=1}^{\infty} K_i$, with $K_i \subseteq K_1^0$, each K_{i+1} compact, and $K_1 = \varphi$. Now choose Urysohn functions $u_i\colon K_i \setminus K_{i-1} \to [i, i-1]$ with u_i (Frontier (K_i)) $= \{i\}$. Let $u = \bigcup_{i=1}^{\infty} u_i$. Now, define $B_\lambda = A_\lambda \cup X_\lambda^n$ and $(Y, C, \{x\})_\lambda = (X_\lambda \cup M_u, B_\lambda \cup M_u, \{x\}_\lambda)$ for each $\lambda \in \Lambda$. Here M_u denotes the mapping cylinder of u. This construction plays the role of a "proper cone" on X^1 and the usual coning constructions go through with very little modification so that $((Y, C, \{y\})_\lambda, \bar{P}, \Lambda)$ becomes an inverse system. $\bar{P}$ is the "proper cone" on P. We will construct a Pro isomorphism of the system $(Y, C, \{x\})$ to the system $(X, A, \{x\})$. Since each component triple of $(Y, C, \{x\})$ satisfies the proper Hurewicz theorem [11], the system satisfies the pro version of the Hurewicz theorem. The Hurewicz homomorphism is easily seen to be a natural transformation of the functors $\operatorname{Pro}(\Delta(\cdot, \pi_k))$ and $\operatorname{Pro}(\Delta(\cdot\, H_k))$ and so we obtain the result for the original system, $(X, A, \{x\})$.

To obtain this pro-isomorphism, first note that the long exact sequence of pro-Δ homotopy groups of the triple $(X, A, \{x\})$ shows that $\operatorname{Pro}\Delta((X, \{x\}), \pi_1) = 0$. Thus, for each λ in Λ there is a $\lambda_0 > \lambda$ so that $\Delta((X, \{x\})_{\lambda_0}, \pi_1) \xrightarrow{P_*} \Delta((X, \{x\})_\lambda, \pi_1)$ is o. In addition, we choose a chain $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \ldots \leqslant \lambda_{n-1} = \lambda'$ so that for each k, $1 \leqslant k \leqslant n-1$ $\Delta((X, A, \{x\})_{\lambda_{j+1}}, \pi_k) \xrightarrow{P_*} \Delta((X, A, \{x\})_{\lambda_j}, \pi_k)$ is o, where $j = n-k-1$.

Now, using II.2 and the proper homotopy extension theorem as in the proof of III.1, we obtain, inductively, maps $h_{\lambda_j\lambda'}\colon (X, A, \{x\})_{\lambda'} \to (X, A, \{x\})_{\lambda_j}$ so that $h_{\lambda_j\lambda'}(X_{\lambda'}^k) \subseteq A_{\lambda_j}$, and $h_{\lambda_j\lambda'}$ is a member of the proper homotopy class $P_{\lambda_j\lambda'}$. Thus, we may consider $h_{\lambda_0\lambda'}$ as a map from $(X, B, \{x\})_{\lambda'}$ to $(X, A, \{x\})_{\lambda_0}$.

To complete the construction of the map, note that $P_{\lambda\lambda_0} h_{\lambda_0\lambda'}\colon (X, B, \{x\})_{\lambda'} \to (X, A, \{x\})_\lambda$ induces the zero map on $\Delta(\cdot, \pi_1)$ so that it extends to a proper map $h_{\lambda\lambda'}\colon (Y, C, \{x\})_{\lambda'} \to (X, A, \{x\})_\lambda$ which makes the following diagram proper homotopy commutative:

$$\begin{tikzcd}
 & (Y, C, \{x\})_{\lambda'} \arrow[dl, "h"'] \\
(X, A, \{x\})_\lambda & (X, A', \{x\})_{\lambda'} \arrow[l, "p"'] \arrow[u, "j"]
\end{tikzcd}$$

If one is careful to choose $\lambda' \leqslant \underline{\lambda}'$ whenever $\lambda \leqslant \underline{\lambda}$, then the resulting collection $(h_{\lambda\lambda'}, h(\lambda) = \lambda')$ is a map of systems and according to the above diagram, it is an isomorphism of systems. This completes the proof of V.1.

V.1 and IV.5 yield the following homology version of the Whitehead theorem:

V.2. THEOREM. *Let $f\colon (X, \{x\}) \to (Y, \{y\})$ be a properly o-connected map of properly o-connected finite dimensional locally compact metric spaces. Let $n = \max(\dim(X)+1, \dim(Y))$. If $\operatorname{Pro}\Delta(X, \{x\}, \pi_1) = 0$ and f induces an isomorphism of $\operatorname{Pro}(\Delta((X, \{x\}), H_k))$ to $\operatorname{Pro}(\Delta((Y, \{y\}), H_k))$ for $k \leqslant n$ and an epimorphism for $k = n+1$, then f is a proper shape equivalence.*

V.3. Remarks toward a cohomology version. It seems clear that there must be a corresponding version of the Whitehead theorem which would use a Pro-Δ version of cohomology as its obstruction group. However, the complexity in stating such a theorem may well outweigh the value in having it. The difficulty stems from the apparent necessity, in any proof, of using a "pro" form of the Universal coefficient theorems (c.f. [6], Theorem 7.2). This would require that the Δ groups at each stage be finitely generated and this last, unfortunately, is seldom the case.

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One-to-one Carathéodory representation theorem for multifunctions with uncountable values

by

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Abstract. It is shown that, given a measurable set-valued mapping M from a complete measure space T into a Polish space X such that all sets $M(t)$ are uncountable, there are another Polish space Z and a one-to-one Carathéodory mapping $f\colon T\times Z\to X$ such that $f(t, Z) = M(t)$ for all t.

§ 1. **Introduction.** In [6] Wesley proved that, given a set-valued mapping M from I into I (I being the unit real interval) with Borel graph and uncountable values, there exist a function $f\colon I\times I\to I$ which is Lebesgue measurable in the first variable and Borel isomorphism from I onto $M(t)$ in the second. Cenzer and Mauldin [1] strengthened this result having shown that, in the first argument, f can be chosen measurable with respect to the minimal σ-algebra containing Borel subsets of I and closed under A-operation of Souslin (which is a proper subalgebra of the algebra of Lebesgue measurable sets). On the other hand, we proved in [2] that, given a multifunction M from a measurable space $(T, \mathfrak{M})$ into a Polish space X with Souslin (in an appropriate sense) graph, there are another Polish space Z and a Carathéodory function $f\colon T\times Z\to X$ such that $f(t, Z) = M(t)$ for all t (such that $M(t)\neq\emptyset$).

The question arises if and how both results can be united. For instance, is it possible to replace in Wesley's theorem "Borel isomorphism" by "one-to-one and continuous"? An affirmative answer will be given here even in a more general setting that in [6] though not in so general as in [2]. The result to be proven here is stated as follows.

THEOREM. *Let $(T, \mathfrak{M}, \mu)$ be a measure space with σ-finite complete positive measure, and let X be an uncountable Polish space. Let M be a set-valued mapping from T into X such that*

(i) *every $M(t)$ is an uncountable subset of X;*

(ii) $\operatorname{Gr} M = \{(t, x)\in T\times X \mid x\in M(t)\}$, *the graph of M, belongs to $\mathfrak{M}\otimes\mathfrak{B}(X)$.*

Then M can be represented by a pair (Z, f), where Z is a Polish space, $f\colon T\times Z\to X$ is a Carathéodory function and for any $t\in T$, the mapping $z\to f(t, z)$ is one-to-one.