Finally, define the homeomorphism T from $f^{-1}(J^n) = \bigcup K_i$ onto $E^3 \times J^n$ by $T|K_i = \delta_i$. Clearly $\pi \circ T = f|f^{-1}(J^n)$.

Remark. Theorem 4.5 does not imply that f is completely regular. An example is given by Seidman [11, p. 465] of a metric for $E^1 \times E^1$ that yields the product topology, but that with respect to this metric, the projection mapping onto the first factor is not completely regular. However, if the usual metric on $E^3 \times E^n$ is imposed upon X under some homeomorphism which satisfies the conclusion of theorem, then f is completely regular.

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Accepté par la Rédaction le 8. 3. 1978

Neighborhoods of compacta in euclidean space

by

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Abstract. In this paper the question of when a compact subset of Euclidean n-space has arbitrarily small piecewise linear neighborhoods with k-dimensional spines is considered. A theory is developed which completely answers the question in terms of the fundamental dimension of the compactum and an embedding condition which is a weak form of the cellularity criterion. The theory is the shape theoretical analogue of the demension theory of M. A. Štańko.

1. Introduction. Suppose X is a compact subset of Euclidean n-space E^n. It is, of course, well-known that X has arbitrarily small piecewise linear (PL) neighborhoods with $(n-1)$-dimensional spines. We want to determine the smallest value of k such that X has arbitrarily small PL neighborhoods with k-dimensional spines. This problem leads naturally to shape theory, since if X has arbitrarily small PL neighborhoods with k-dimensional spines, then X has the shape of the inverse limit of an inverse sequence of k-dimensional polyhedra and thus has fundamental dimension $\leqslant k$. Hence we immediately see that the fundamental dimension of X is a lower bound for the possible values of k.

In this paper we present a theory which tells exactly when that lower bound is achieved and what the smallest value of k is otherwise. Our theory is the shape theory analogue of the demension theory of M. A. Štańko [13]. Štańko looks for neighborhoods which not only have k-dimensional spines, but also have small retractions onto the spines. (A precise statement of Štańko's results is given below). In our theory the fundamental dimension plays the role of the covering dimension in Štańko's theory and a weak form of McMillan's cellularity criterion (the inessential loops condition) plays the role of the 1-ULC property. Our theory unifies the various proofs of finite dimensional complement theorems which have appeared in [8], [3], [5], [6], [14] and [11] since the main step in each of those proofs involves finding small neighborhoods of compacta with k-dimensional spines where $2k+2 \leqslant n$. Before stating our main result (Theorem 1.4) we define the terms used.

DEFINITION 1.1 [1, p. 227]. The *fundamental dimension* of a compactum X is defined by $\mathrm{Fd}(X) = \min\{\dim Y | \mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)\}$.

* Research partially supported by grants from the National Science Foundation.

DEFINITION 1.2. Suppose X is a compact subset of the PL n-manifold M^n. The *fundamental dimension of embedding* of X (abbreviated $\mathrm{FDE}(X)$) is defined to be the smallest integer k such that for every neighborhood U of X there exists a PL neighborhood N of X such that $X \subset N \subset U$ and N has a k-dimensional spine. As remarked above, it is obvious that $\mathrm{FDE}(X) \geqslant \mathrm{Fd}(X)$.

DEFINITION 1.3 [14]. The compactum $X \subset M^n$ is said to satisfy the *inessential loops condition* (abbreviated ILC) if for every neighborhood U of X there exists a neighborhood V of X in U such that every loop in $V-X$ which is homotopically inessential in V is also inessential in $U-X$. In other words, if $i_\#$ and $j_\#$ are the inclusion induced homomorphisms in the diagram

$$\begin{tikzcd} & \pi_1(V) \\ \pi_1(V-X) \arrow[ur, "i_\#"] \arrow[dr, "j_\#"'] & \\ & \pi_1(U-X) \end{tikzcd}$$

then $\ker i_\# \subset \ker j_\#$.

The inessential loops condition is a condition on the embedding of X into M and is clearly weaker than the cellularity criterion of McMillan [7].

THEOREM 1.4. *Suppose X is a compact subset of E^n.*

Part I. *If $n \neq 3$ and $\mathrm{Fd}(X) \geqslant n-2$, then $\mathrm{FDE}(X) = \mathrm{Fd}(X)$.*

Part II. *If $n \neq 4$ and $\mathrm{Fd}(X) \leqslant n-3$, then either*
a) $\mathrm{FDE}(X) = \mathrm{Fd}(X)$ *and X satisfies* ILC, *or*
b) $\mathrm{FDE}(X) = n-2$ *and X does not satisfy* ILC.

So $\mathrm{FDE}(X)$ takes on one of only two values, $\mathrm{Fd}(X)$ or $n-2$, and the inessential loops condition can be used to detect which. Since the inessential loop condition and the fundamental dimension are invariant under ambient homeomorphism, we have the following corollary.

COROLLARY 1.5. *If $X \subset E^n$ $(n \neq 3, 4)$ and $h\colon E^n \to E^n$ is a topological homeomorphism, then* $\mathrm{FDE}(X) = \mathrm{FDE}(h(X))$.

Theorem 1.4 should be compared with the following theorem of Štańko which is the Main Theorem of [13]. (The terms dem and 1-ULC are defined in § 2.)

THEOREM 1.6 (M. A. Štańko). *Suppose X is a compact subset of E^n.*

Part I. *If $n \neq 3$ and $\dim X \geqslant n-2$, then* $\operatorname{dem} X = \dim X$.

Part II. *If $n \neq 4$ and $\dim X \leqslant n-3$, then either*
a) $\operatorname{dem} X = \dim X$ *and $E^n - X$ is* 1-ULC, *or*
b) $\operatorname{dem} X = n-2$ *and $E^n - X$ is not* 1-ULC.

It is necessary to exclude $n = 3$ from Part I of both Theorems 1.4 and 1.6 since, for example, the 1-dimensional compact subsets of E^3 constructed in [2] and [8] have fundamental dimension (dimension) 1 and fundamental dimension of embedding (demension) 2. Part II of Theorem 1.4 is true for $n = 4$ if the (apparently) stronger disk pushing property is substituted for ILC [11], but it is not known whether Part II of either Theorem 1.4 or 1.6 is true as stated for $n = 4$. Most parts of Theorem 1.4 extend straightforwardly to compacta embedded in arbitrary PL manifolds, so they will be stated in that generality whenever possible in later sections of this paper.

2. Definitions and a characterization of fundamental dimension. In this paper, all spaces are assumed to be subspaces of some finite dimensional Euclidean space. A *polyhedron* is the underlying set of some locally finite simplicial complex. A *piecewise linear* (or PL) *n-manifold* is a polyhedron in which the star of each vertex is a piecewise linear n-cell. Suppose M^n is a PL n-manifold, ϱ denotes the metric on M^n, $X \subset M^n$, and $\varepsilon > 0$. Then the *ε-neighborhood* of X in M is $N_\varepsilon(X) = \{p \in M \mid \varrho(X, p) < \varepsilon\}$. A polyhedron $K \subset \operatorname{int} M^n$ is called a *spine* of M^n (or equivalently, M^n is a *regular neighborhood* of K) if M^n simplicially collapses to K. If K is a spine of M, then (M, K) is homeomorphic with $(M(f), K)$ where $M(f)$ denotes the mapping cylinder of some PL map $f\colon \partial M \to K$. Call M an *ε-regular neighborhood* of K if f can be chosen so that each fiber of $M(f)$ has diameter $< \varepsilon$ in M.

The following definition is that given in [4] and is equivalent to the one in [13]. A compactum $X \subset E^n$ is said to have *dimension of embedding*, or *demension*, $\leqslant k$ (abbreviated $\operatorname{dem} X \leqslant k$) if for every $\varepsilon > 0$ there exists a compact PL manifold neighborhood M of X in E^n, $M \subset N_\varepsilon(X)$, such that M is an ε-regular neighborhood of some k-dimensional polyhedron $K \subset M$. An open set $U \subset E^n$ is said to be *uniformly locally simply connected* (or 1-ULC) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that each loop in U of diameter less than δ is homotopically inessential in a subset of U of diameter less than ε.

The notation $f \simeq g$ means that the maps f and g are homotopic and $f \overset{U}{\simeq} g$ means that f and g are homotopic via a homotopy whose track lies in U.

Whenever we write $\mathrm{Sh}(X)$, it is understood that X is compact. The following proposition characterizes those subsets of PL manifolds which have fundamental dimension $\leqslant k$. A similar characterization of fundamental dimension for subsets of the Hilbert cube is given in [9].

PROPOSITION 2.1. *Suppose M^n is a PL manifold and that X is a compact subset of $\operatorname{int} M$. Then $\mathrm{Fd}(X) \leqslant k$ if and only if for every neighborhood U of X there exists a neighborhood V of X in U and a compact polyhedron $K \subset U$, $\dim K \leqslant k$, such that the inclusion $V \hookrightarrow U$ is homotopic in U to a map of V into K.*

Proof. First suppose that the UV condition holds. Then there exists a sequence $\{U_i\}_{i=1}^{\infty}$ of neighborhoods of X and polyhedra $K_i \subset U_i$, $\dim K_i \leqslant k$, such that $X = \bigcap_{i=1}^{\infty} U_i$, $U_{i+1} \cup K_i \subset U_i$, and the inclusion map $\beta_i\colon U_{i+1} \hookrightarrow U_i$ is homotopic in U_i to a map $f_i\colon U_{i+1} \to K_i$. If $\gamma_i\colon K_i \hookrightarrow U_i$ denotes the inclusion map, then the following diagram commutes up to homotopy.

$$\begin{tikzcd}
U_1 & U_2 \arrow[l, "\beta_1"'] & U_3 \arrow[l, "\beta_2"'] & \ldots \arrow[l] & U_i \arrow[l] & U_{i+1} \arrow[l, "\beta_i"'] & \ldots \arrow[l] \\
K_1 \arrow[u, "\gamma_1"] & K_2 \arrow[u, "\gamma_2"] \arrow[l, "f_1\gamma_2"] & K_3 \arrow[u, "\gamma_3"] \arrow[l, "f_2\gamma_3"] & \ldots \arrow[l] & K_i \arrow[u, "\gamma_i"] \arrow[l] & K_{i+1} \arrow[u, "\gamma_{i+1}"'] \arrow[l, "f_i\gamma_{i+1}"] & \ldots \arrow[l]
\arrow[from=1-2, to=2-1, "f_1"']
\arrow[from=1-3, to=2-2, "f_2"']
\arrow[from=1-6, to=2-5, "f_i"']
\end{tikzcd}$$

So $\mathrm{Sh}(X) = \mathrm{Sh}(\underleftarrow{\lim}\{K_i, f_i\gamma_{i+1}\})$ and thus $\mathrm{Fd}(X) \leqslant k$.

Now suppose that $\mathrm{Fd}(X) \leqslant k$. We first consider the special case in which $M^n = E^n$. Let Y be a compact metric space such that $\dim Y = k$ and $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$. Embed Y in E^{2k+1} so that Y is in standard position [3]; i.e., so that $Y = \bigcap_{i=1}^{\infty} N_i$ where each N_i is a regular neighborhood of a polyhedron P_i in E^{2k+1}, $\dim P_i \leqslant k$, and $N_{i+1} \subset \operatorname{int} N_i$. Let $\underline{f} = \{f_i, X, Y\}_{E^n, E^{2k+1}}$ and $\underline{g} = \{g_i, Y, X\}_{E^{2k+1}, E^n}$ be fundamental sequences such that $\underline{g}\underline{f} \simeq \underline{1}_X$.

Let U be a neighborhood of X. Choose integers j and i_0 such that $g_i(N_j) \subset U$ for all $i \geqslant i_0$ and let $r\colon N_j \to P_j$ denote the retraction. Now choose a neighborhood V of X in U such that $f_i(V) \subset N_j$ and $g_i f_i|V \overset{U}{\simeq} \beta$ for all $i \geqslant i_1 \geqslant i_0$, where $\beta\colon V \hookrightarrow U$ is the inclusion map. It may be assumed that $g_{i_1}|P_j$ is PL. Then $\beta \overset{U}{\simeq} g_{i_1} f_{i_1}|V \overset{U}{\simeq} g_{i_1} r f_{i_1}|V$, so $K = g_{i_1}(P_j)$ is the polyhedron needed to complete the proof.

Now if $X \subset M^n \neq E^n$, we first embed M^n as a PL subset of E^{2n+1}. We can then use a PL retraction of a neighborhood of M^n onto M^n to push the homotopies constructed in the first part of the proof (the first part of the proof is now applied to $X \subset E^{2n+1}$) into M^n.

3. Codimension two. Suppose M^n is a PL n-manifold and that $X \subset M^n$ is compact. From results of Nowak [9, § 3], we see that it is only possible to have $\mathrm{Fd}(X) = n$ if $X = M^n$ and $\partial M^n = \emptyset$. Otherwise $\mathrm{Fd}(X) \leqslant n-1$ and (as remarked in the introduction) $\mathrm{FDE}(X) \leqslant n-1$. So Theorem 1.4 is easily proved for $\mathrm{Fd}(X) \geqslant n-1$. Also Theorem 1.4 follows easily from the well-known shape properties of plane compacta (see [1, Chapter VII, §§ 7 and 8]) in case $n \leqslant 2$. In this section we prove the only remaining case of Part I of Theorem 1.4, namely that in which $\mathrm{Fd}(X) = n-2$ and $n \geqslant 4$.

THEOREM 3.1. *Suppose M^n is a* PL *manifold and that X is a compact subset of* $\operatorname{int} M^n$ *with* $\mathrm{Fd}(X) \leqslant n-2$. *Then* $\pi_1(U, U-X) = 0$ *for every neighborhood U of X in M^n.*

COROLLARY 3.2. *If M^n and X are as in Theorem* 3.1 *and* $n \geqslant 4$, *then* $\mathrm{FDE}(X) \leqslant n-2$. *If* $\mathrm{Fd}(X) = n-2$, *then* $\mathrm{FDE}(X) = n-2$.

Proof of Corollary 3.2. The second part of Corollary 3.2 follows from the first since $\mathrm{FDE}(X) \geqslant \mathrm{Fd}(X)$ in every case. Given a neighborhood U of X, choose a PL manifold neighborhood N such that $X \subset \operatorname{int} N \subset N \subset U$. Let N^1 denote the 1-skeleton of N and let N_*^{n-2} denote the dual $(n-2)$-skeleton; i.e., N_*^{n-2} is the union of all simplices in the second barycentric subdivision of N which do not intersect N^1. It follows from Theorem 3.1 and Stallings' engulfing theorem [12] that there is a PL homeomorphism $h\colon M \to M$ such that $h|M^n - N = \mathrm{id}$ and $h(N^1) \cap X = \emptyset$. But then $X \subset R$ for some regular neighborhood R of $h(N_*^{n-2})$ in U and so the proof is complete.

Proof of Theorem 3.1. Let $U \supset X$ and $f\colon (\Delta^1, \partial\Delta^1) \to (U, U-X)$ be given, where Δ^1 denotes the unit interval $[0, 1]$. Since $\mathrm{Fd}(X) \leqslant n-2$, Proposition 2.1 implies that there exists a PL manifold neighborhood V of X and a polyhedron $K \subset U$ such that $\dim K \leqslant n-2$ and the inclusion $V \hookrightarrow U - F(\partial\Delta^1)$ is homotopic in $U - F(\partial\Delta^1)$ to a map of V into K. By general position we may assume that f is transverse to ∂V and that $f(\Delta^1) \cap K = \emptyset$. We also assume that U is connected by concentrating only on the component of U containing $f(\Delta^1)$ if necessary.

Let $\tilde{U}$ denote the universal covering space of U and let $p\colon \tilde{U} \to U$ denote the projection. Let $\tilde{V} = p^{-1}(V)$ and let $\tilde{f}\colon (\Delta^1, \partial\Delta^1) \to (\tilde{U}, \tilde{U} - p^{-1}(X))$ denote a lift of f. Since the diagram

$$\begin{tikzcd}
 & \tilde{U} \arrow[d, "p"] \\
\tilde{V} \arrow[ur] \arrow[r, "p|V"'] & U
\end{tikzcd}$$

commutes, the homotopy lifting property can be used to lift the homotopy of V downstairs to a homotopy $H\colon \tilde{V} \times I \to \tilde{U}$ such that $H_0 = $ inclusion and $H_1(\tilde{V}) \subset p^{-1}(K)$. Throw H and $\tilde{f}$ into general position.

We claim that $\partial\tilde{V}$ does not separate $\tilde{f}(0)$ from $\tilde{f}(1)$ in $\tilde{U}$. Let B denote a component of $\partial\tilde{V}$ and parametrize the path $\tilde{f}(\Delta^1)$ in the natural way. Suppose $\tilde{f}(\Delta^1) \cap B$ consists of more than one point, say q_1 and q_2 are two consecutive points of $\tilde{f}(\Delta^1) \cap B$. Note that $\tilde{f}(\Delta^1)$ must cross B in opposite directions at q_1 and q_2 since otherwise we could construct a simple closed curve in $\tilde{U}$ which only intersects B once. That is impossible since $\tilde{U}$ is simply connected. So $\tilde{f}(\Delta^1)$ crosses B in opposite directions at q_1 and q_2. We can then construct an arc from $\tilde{f}(0)$ to $\tilde{f}(1)$ with at least two fewer points of intersection with $\partial\tilde{V}$ by replacing the arc in $\tilde{f}(\Delta^1)$ from q_1 to q_2 with an arc in B and then pushing the latter arc a little to one side of B. Thus, if $\partial\tilde{V}$ separates $\tilde{f}(0)$ from $\tilde{f}(1)$, there must be a component B_0 of $\partial\tilde{V}$ such that $\tilde{f}(\Delta^1) \cap B_0$ consists of an odd number of points. But then $(H|B_0 \times I)^{-1}(\tilde{f}(\Delta^1))$ is a compact 1-manifold with an odd number of endpoints. Since no such 1-manifold exists, we can join $\tilde{f}(0)$ and $\tilde{f}(1)$ with an arc α which lies in $\tilde{U} - \partial\tilde{V}$.

Notice that since α does not intersect $\partial\tilde{V}$, $\alpha \cap \tilde{V} = \emptyset$ and so $\alpha \cap X = \emptyset$. Since $\tilde{U}$ is simply connected, $\tilde{f}(\Delta^1) \cup \alpha$ is null homotopic in $\tilde{U}$. Projecting that homotopy down, we see that f is homotopic $(\mathrm{rel}\,\partial\Delta^1)$ to $f'\colon \Delta^1 \to U - X$. Thus $\pi_1(U, U-X) = 0$.

4. Codimension three. In this section we prove Part II of Theorem 1.4 and then use a theorem of Nowak [10] to give a purely algebraic criterion for compacta to have fundamental dimension of embedding $\leqslant k$.

THEOREM 4.1. *Suppose M^n is a* PL *n-manifold, $n \geqslant 5$, and that X is a compact subset of* $\operatorname{int} M^n$ *with* $\mathrm{Fd}(X) \leqslant n-3$. *Then X satisfies* ILC *if and only if* $\mathrm{FDE}(X) = \mathrm{Fd}(X)$.

Proof. In the special case $M^n = E^n$, this is just Lemma 3 of [14]. However, the proof given in [14] will work in any manifold, providing Proposition 2.1 of the present paper is used to construct the homotopies needed in [14].

DEFINITION 4.2. Suppose M^n is a PL n-manifold. The compactum $X \subset M^n$ is said to have property 1-UV (or to be *approximatively* 1-*connected*) if for every neighborhood U of X there exists a neighborhood V of X such that each loop in V is null homotopic in U. Property 1-UV is a shape invariant [1, p. 145].

DEFINITION 4.3. The compactum $X \subset M^n$ is said to satisfy the *cellularity criterion* (abbreviated CC) if for every neighborhood U of X in M^n there exists a neighborhood V of X in U such that each loop in $V-X$ is null homotopic in $U-X$.

We omit the easy proof of the following lemma.

LEMMA 4.4. *Suppose M^n is a* PL *n-manifold and $X \subset \operatorname{int} M^n$ is compact. If X satisfies* ILC *and has property* 1-UV *then X satisfies CC.*

Proof of Theorem 1.4. Part II. For $n \geqslant 5$, Part II of Theorem 1.4 follows almost immediately from Theorem 4.1. Suppose $\mathrm{Fd}(X) \leqslant n-3$. By Corollary 3.2, $\mathrm{FDE}(X) \leqslant n-2$. So either $\mathrm{FDE}(X) = n-2$ or $\mathrm{FDE}(X) \leqslant n-3$. In the latter case X satisfies ILC (by general position) so Theorem 4.1 implies that $\mathrm{FDE}(X) = \mathrm{Fd}(X)$. If $\mathrm{FDE}(X) = n-2$, Theorem 4.1 implies that X does not satisfy ILC.

Suppose now that $n = 3$ and $\mathrm{Fd}(X) = 0$. Then X has the shape of a 0-dimensional set and consequently has property 1-UV. Given a neighborhood U of X, we can choose a PL manifold neighborhood M of X such that $M \subset U$. Since $\mathrm{Fd}(X) = 0$, X does not separate M and so we can assume that each component of M has a connected boundary. Applying Lemma 1 of [8] to each component of M, we see that $\mathrm{FDE}(X) \leqslant 1$.

It remains only to show that $\mathrm{FDE}(X) = 0$ if and only if X satisfies ILC. By Lemma 4.4, ILC is equivalent to CC in the case we are considering. If X satisfies ILC, then $\mathrm{FDE}(X) = 0$ by the proof of [8], Theorem 1'. If $\mathrm{FDE}(X) = 0$, then X satisfies ILC by general position. This completes the proof of Theorem 1.4.

To the author's knowledge, the following 3-dimensional Complement Theorem has not appeared before. The notation $\approx$ means "is homeomorphic with."

COROLLARY 4.5. *Suppose X and Y are compacta in E^3 such that $\mathrm{Fd}(X) = 0 = \mathrm{Fd}(Y)$ and both satisfy the cellularity criterion. Then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ if and only if $E^3 - X \approx E^3 - Y$.*

Proof. By Theorem 1.4, $X = \bigcap_{i=1}^{\infty} M_i$ where $M_{i+1} \subset \operatorname{int} M_i$ and each M_i is the disjoint union of a finite number of PL 3-cells; Y can be written as a similar intersection. Thus there exist 0-dimensional sets X' and Y' in E^3 such that $X' \cup Y' \subset E^1 \times 0 \times 0 \subset E^3$, $\mathrm{Sh}(X) = \mathrm{Sh}(X')$, $\mathrm{Sh}(Y) = \mathrm{Sh}(Y')$, $E^3 - X \approx E^3 - X'$ and $E^3 - Y \approx E^3 - Y'$. Theorem 1.2 of [5] implies that $\mathrm{Sh}(X') = \mathrm{Sh}(Y')$ if and only if $E^3 - X' \approx E^3 - Y'$. Hence $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ if and only if $E^3 - X \approx E^3 - Y$.

We wish to use the following theorem of Nowak to compute the fundamental dimension of embedding of compacta with property 1-UV. In the remainder of this paper, $\check{H}^*$ denotes Čech cohomology with integral coefficients.

THEOREM 4.6 (Nowak [10, Theorem 2.1]). *If X is a finite dimensional compact metric space with property* 1-UV *and if there exists a k such that $\check{H}^i(X) = 0$ for $i > k$, then* $\mathrm{Fd}(X) \leqslant k$.

Remark. Theorem 4.6 could be proved for $k > 1$ using the techniques of this paper as follows. First embed X in $E^{n-3} \subset E^n$ for some large n. Then X will satisfy ILC as a subset of E^n by general position. Thus Lemma 4.4 implies that X satisfies CC. Using the standard "disk trading" argument, we can construct arbitrarily small neighborhoods U of X such that $\pi_1(U) = \pi_1(U-X) = 0$. Then the relative Hurewicz theorem and Alexander duality (applied as in the proof of [14, Lemma 1]) imply that $\pi_i(U, U-X) = 0$ for $i \leqslant n-k-1$. Stallings' engulfing theorem (applied as in the proof of Corollary 3.2) implies that $\mathrm{FDE}(X) \leqslant k$ (providing $k \geqslant 2$) and thus $\mathrm{Fd}(X) \leqslant k$.

COROLLARY 4.7. *Suppose M^n is a* PL *n-manifold and X is a compact subset of* $\operatorname{int} M^n$ *satisfying* ILC *and having property* 1-UV. *If $\check{H}^i(X) = 0$ for $i > k$, then* $\mathrm{FDE}(X) \leqslant k$.

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Accepté par la Rédaction le 26. 6. 1978