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ERRATA

Page, line	For	Read
112^{12}	$\hat{\otimes}$	$\check{\otimes}$
116_{15}	$(G\hat{\otimes}G)$, $(E\hat{\otimes}E)$	$(G\check{\otimes}G)$, $(F\check{\otimes}F)$
120^{24}	Let $\bar{N} = \{co(N)$:	Let $\hat{N} = \{co(N)$:
120^{25}	Then $\bar{N}$	Then $\hat{N}$
120_2	$(x_{n_j})_{n=1}^{\infty}$	$(x_{n_j})_{j=1}^{\infty}$
122^2	$\sum_{n=1}^{\infty} t_j y_{n_j}$	$\sum_{j=1}^{\infty} t_j y_{n_j}$
123^3	unit-vectors	unit-vectors basis
125^5	$\{\|f_i(z_n)\| =$	$\{\|f_i(z_n)\|:$
128^{10}	$\sup\limits_{\|\|x\|\|<1}$	$\sup\limits_{\|\|x\|\|_1<1}$
128_3	$z_j = \sum_{i=n_{j-1}+1}^{n_j} = a_i e_i$	$z_j = \sum_{i=n_{j-1}+1}^{n_j} a_i e_i$
141_2	$C = c\big(c(E(S))\cup D\big)$	$\tilde{C} = c\big(\tilde{c}(E(S))\cup D\big)$
141_1	$\tilde{c}(ES))$	$\tilde{c}(E(S))$
204^{13}	$(x-ty, \xi/t)(y)\}\|$	$(x-ty, \xi/t)\varphi(y)\}\|$
211_4	$a>0.$	$a>0.\ \Gamma_a^h(x_0) = \{(x,t): \|x-x_0\| < at,\ t<h\}.$

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Differentiability of distributions at a single point

by

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Abstract. We develop tools needed to study the differentiability of distributions at a point of $\boldsymbol{R}^n$.

The purpose of this paper is to develop tools to study distributions at a point x_0. The integral expressions that we will be examining are closely related to the area integral of Lusin and the Littlewood–Paley g_λ^* functions. Rather than use the theory of harmonic functions we will consider analogous expressions that are independent of the mollifier that is used. The results of this paper are used to study tempered nontangential boundedness and convergence in [3].

Let φ be a Schwartz function with $\int \varphi(x)\,dx = 1$. Using this as a mollifier, form

$$u(x, t) = f*\varphi_t(x) \quad \text{where} \quad \varphi_t(x) = t^{-n}\varphi\left(\frac{x}{t}\right).$$

If f is a continuous function then $u(x, t)$ approaches $f(x)$ as $t\to 0$. It seems reasonable then that by examining $u(x, t)$ in the set $\Omega = \{(x, t)\colon x \in \boldsymbol{R}^n,\ 0<t<1\}$ we should be able to understand the behavior of f at a point, say x_0. We form a certain integral $\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0)$ of $u(x, t)$ over the set Ω. Of the various parameters involved the most important is $\gamma \in \boldsymbol{R}$. We will see that if $\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) < \infty$ then γ gives the order of differentiability of f at x_0.

If we add certain harmless terms to these $\mathscr{G}$ functions we can form norms $N_{\gamma,k}^{p,\lambda}$. Thus

$$N_{\gamma,k}^{p,\lambda}(f)(x_0) = \mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) + \text{“other terms”}.$$

In the second and third sections we show that both the $\mathscr{G}$ functions and the norm N are essentially independent of the mollifier φ. In addition, using a different k gives an equivalent norm.

With these norms we can define Banach spaces

$$\Lambda_\gamma^{p,\lambda}(x_0) = \{f \in \mathscr{S}'\colon N_{\gamma,k}^{p,\lambda}(f)(x_0) < \infty \text{ for some } k > \gamma + n/p\}.$$

Contrast these with the Sobolev spaces $L_k^p(\boldsymbol{R}^n)$ of functions that have derivatives up to order k in $L^p(\boldsymbol{R}^n)$. Rather than looking at global differentiability we are looking at a single point. These spaces $\Lambda_\gamma^{p,\lambda}(x_0)$ can be viewed as a pointwise version of the Besov spaces $\Lambda_\alpha^{p,\lambda}(\boldsymbol{R}^n)$ (see [5], [7]).

The fourth section is devoted to various elementary properties of these spaces. In § 5 we discuss the use of the Poisson kernel $\mathscr{P}$ as a mollifier. Since $\mathscr{P}$ is not rapidly decreasing at infinity, certain modifications have to be made. § 6 concerns inclusion relations. In § 7 it is shown that certain pseudo-differential operators are bounded on the $\Lambda_\gamma^{p,\lambda}(x_0)$. Section eight deals with differentiation and powers of $(I-\Delta)$. For example we show that $(I-\Delta)^{\alpha/2}$ is a Banach space isomorphism from $\Lambda_\gamma^{p,\lambda}(x_0)$ to $\Lambda_{\gamma-\alpha}^{p,\lambda}(x_0)$. Finally in § 9 we discuss an alternate difinition of $\Lambda_\gamma^{\infty}(x_0) = \bigcup_{\lambda>0} \Lambda_\gamma^{\infty,\lambda}(x_0)$, as a Lipschitz space.

In the sense that we are trying to free certain expressions in harmonic analysis from their dependence on the Poisson kernel, this paper can be considered a local version of the work of Fefferman and Stein on the real variable theory of H^p spaces. In their paper [2], they demonstrated that the H^p spaces could be defined using any mollifier rather than just the Poisson kernel.

An earlier theory of differentiability at a single point can be found in Calderón and Zygmund [1]. There they consider spaces of locally integrable functions that are differentiable in a certain sense at a point x_0.

I am happy to have this opportunity to thank my adviser, Charles Fefferman, for the help and encouragement that he has given me. This paper and [3] comprised my doctoral dissertation written under his direction.

§ 1. Introduction. The goal of Sections 2 to 4 will be to define the spaces $\Lambda_\gamma^p(x_0)$. In this section we will introduce some notation and give a summary of the results leading up to the definition of $\Lambda_\gamma^p(x_0)$. We postpone defining these spaces in order to study the norms $N_\gamma^{p,\lambda}(f)(x_0)$ and to show that they are essentially independent of k and the mollifier $\varphi\in\mathscr{S}$, $\int\varphi(x)\,dx\neq 0$.

The dilates of φ will be written as $\varphi_t(x) = t^{-n}\varphi(x/t)$. We adopt the convention that whenever s or t appears as a subscript then that subscript denotes dilation.

Let $\Omega = \{(x,t)\colon x\in\boldsymbol{R}^n,\ 0<t<1\}$. We will write Z_0 for the non-negative integers and Z_0^n for the corresponding n-fold Cartesian product. If $\beta = (\beta_1,\ldots,\beta_n)\in Z_0^n$ then $|\beta| = \sum_{i=1}^n \beta_i$, $x^\beta = x_1^{\beta_1}\ldots x_n^{\beta_n}$ and $\left(\frac{\partial}{\partial x}\right)^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1}\ldots\left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$.

We now give the integral expressions that will be our measure of integrability. Define

$$(1)\qquad G_{\gamma,\beta}^{p,\lambda}(f)(x_0) = \left[\iint_\Omega \left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}\left\{t^{|\beta|-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|\right\}^p \frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$

where $u(x,t) = f*\varphi_t(x)$, $\varphi\in\mathscr{S}$, $\int\varphi(x)\,dx\neq 0$ and $1\leqslant p\leqslant\infty$, $\lambda>0$, $\beta\in Z_0^n$. For $p=\infty$ the expression in (1) is to be interpreted as

$$G_{\gamma,\beta}^{\infty,\lambda}(f)(x_0) = \sup_{(x,t)\in\Omega}\left(\frac{t}{t+|x-x_0|}\right)^\lambda t^{|\beta|-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|.$$

As we have mentioned, γ will give the order of smoothness of f at x_0.

At certain times we will be interested in the derivatives of u with respect to t. Thus if we replace $\left(\frac{\partial}{\partial x}\right)^\beta$ in (1) by $\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial t}\right)^k$ then we will denote the resulting expression by $G_{\gamma,\beta,k}^{p,\lambda}(f)(x_0)$. We will need to have control of these expressions for example when we prove that $R(f)(x_0)$ and $N(f)(x_0)$ are essentially independent of the approximate identity φ_t.

If $G_{\gamma,\beta}^{p,\lambda}(f)(x_0)<\infty$, then we have control over $\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)$. Usually we will want to be able to control all the derivatives of a specific order. Therefore it is natural to define

$$\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) = \left[\sum_{|\beta|=k}\{G_{\gamma,\beta}^{p,\lambda}(f)(x_0)\}^p\right]^{1/p} = \left[\sum_{|\beta|=k}\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}\left\{t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|\right\}^p\frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$

and

$$\mathscr{G}_{\gamma,k}^{\infty,\lambda}(f)(x_0) = \sup_{|\beta|=k} G_{\gamma,\beta}^{\infty,\lambda}(f)(x_0) = \sup_{|\beta|=k}\sup_{(x,t)\in\Omega}\left(\frac{t}{t+|x-x_0|}\right)^\lambda t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|.$$

The most important special case will be $p=2$. With smoothness $\gamma=0$ the expression $\mathscr{G}(f)(x_0)$ becomes the Littlewood–Paley function $g_{2\lambda/n}^*(f)(x_0)$:

$$\mathscr{G}_{0,1}^{2,\lambda}(f)(x_0) = \left[\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{2\lambda}|\nabla u(x_0-y,t)|^2\,t^{1-n}\,dy\,dt\right]^{1/2} = g_{2\lambda/n}^*(f)(x_0).$$

This case will be important in the next chapter when we study the connection between the $\mathscr{G}$ functions and tempered nontangential boundedness.

Our first goal will be to show that if we had used a different function φ in the definition of $G(f)(x_0)$ we would have gotten a similar quantity. Thus we will prove a change of approximate identity theorem.

Since we are concerned with the effect of a change of mollifier φ, we will consider f and x_0 to be fixed and we will emphasize the dependence of G on φ. Consequently we will denote $G^{p,\lambda}_{\gamma,\beta}(f)(x_0)$ by $G^{p,\lambda}_{\gamma,\beta}(\varphi)$.

We begin Section 2 by studying the effect of a change of approximate identity on G. We will prove that

$$G^{p,\lambda}_{\gamma,\beta}(\Phi) \leqslant A\,\|\Phi\| G^{p,\lambda}_{\gamma,\beta}(\varphi)$$

where A is a constant independent of f and x_0, and $\|\Phi\|$ is a norm of Φ. Thus changing the mollifier gives an equivalent G function.

Before proving the same theorem for G functions involving derivatives in t, it is necessary for us to examine certain differential operators $\mathscr{D}$. After doing this we prove that

$$G^{p,\lambda}_{\gamma,\beta,j}(\Phi) \leqslant A\,\|\Phi\| \mathscr{G}^{p,\lambda}_{\gamma,|\beta|+j}(\varphi).$$

The ultimate objective of the third section is to define the spaces $A^{p,\lambda}_{\gamma}(x_0)$. Since the G functions contain only derivatives of high order, certain harmless lower order terms must be added to give a norm. These lower order terms are defined as follows:

$$R^{p,\lambda}_{\beta}(f)(x_0) = \left[\int_{\boldsymbol{R}^n} \frac{1}{(1+|x-x_0|)^{p\lambda}} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x,1)\right|^p dx\right]^{1/p}$$

and

$$R^{\infty,\lambda}_{\beta}(f)(x_0) = \sup_{x\in\boldsymbol{R}^n} \frac{1}{(1+|x-x_0|)^{\lambda}} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x,1)\right|$$

where as usual $u(x,t) = f*\varphi_t(x)$ and $1\leqslant p\leqslant\infty$, $\lambda>0$, $\beta\in Z^n_0$.

These terms are harmless because if f is in $\mathscr{S}'$, then $u(x,1) = f*\varphi(x)$ is a tempered C^∞ function. Therefore $R^{p,\lambda}_{\beta}(f)(x_0)$ is finite for all λ sufficiently large.

Just as before while studying the effect of a change of mollifier we will write these terms as $R^{p,\lambda}_{\beta}(\varphi)$.

After several preparatory lemmas we prove the change of approximate identity theorem for these lower order terms:

$$R^{p,\lambda}_{\beta}(\Phi) \leqslant A\,\|\Phi\| \{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi) + R^{p,\lambda}_{\beta}(\varphi)\}.$$

Having done this we then define the norms

$$N^{p,\lambda}_{\gamma,k}(f)(x_0) = \mathscr{G}^{p,\lambda}_{\gamma,k}(f)(x_0) + \sum_{|\beta|<k} R^{p,\lambda}_{\beta}(f)(x_0).$$

That changing the mollifier gives an equivalent norm follows from the change of approximate identity theorems for the G and R functions.

In the norm $N^{p,\lambda}_{\gamma,k}(f)(x_0)$ the most critical information comes from the derivatives of $u(x,t)$ of order k (from the $\mathscr{G}$ function). It turns out though that for any $k>\gamma+n/p$ we get an equivalent norm.

$$A\,N^{p,\lambda}_{\gamma,k+1}(f)(x_0) \leqslant N^{p,\lambda}_{\gamma,k}(f)(x_0) \leqslant A\,N^{p,\lambda}_{\gamma,k+1}(f)(x_0).$$

Now that we know that the norms are independent of $\varphi\in\mathscr{S}$ and $k>\gamma+n/p$ we are in a position to define the spaces $A^{p,\lambda}_{\gamma}(x_0)$.

$$A^{p,\lambda}_{\gamma}(x_0) = \{f\in\mathscr{S}'\colon\ N^{p,\lambda}_{\gamma,k}(f)(x_0) < \infty \text{ for some } k>\gamma+n/p,\ \hat{\varphi}(0)\neq 0\}.$$

The λ gives the order of growth as we examine $u(x,t) = f*\varphi_t(x)$ over cones of larger and larger aperture. Often, particularly in [3], this parameter will be unimportant. In these cases we consider instead the spaces

$$A^{p}_{\gamma}(x_0) = \bigcup_{\lambda>0} A^{p,\lambda}_{\gamma}(x_0).$$

We close section three by proving that $A^{p,\lambda}_{\gamma}(x_0)$ is a Banach space and by showing that $\bigcup_{\gamma\in\boldsymbol{R}} A^{p}_{\gamma}(x_0) = \mathscr{S}'$.

§ 2. The G functions. In this section we will prove change of approximate identity theorems for the G functions defined by (1).

Note in the definition of the G functions that integrating over Ω is equivalent to considering cones that have been truncated at height one. This is done strictly for convenience and all results are valid for any height of truncation. In fact it can be shown that changing this height gives an equivalent expression.

In several proofs of this section and the next we will change from one mollifier φ to another Φ. The key to these proofs is a certain decomposition of Φ into pieces closely related to φ.

Let $N_0 = \{x\in\boldsymbol{R}^n\colon |x|<1\}$, $N_j = \{x\in\boldsymbol{R}^n\colon 2^{j-2}<|x|<2^j\}$ for $j=1,2,\ldots$ Let η_j be a C^∞ partition of unity subordinate to this open covering of $\boldsymbol{R}^n$ such that $\eta_j\geqslant 0$ for all j and

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\eta_j(x)\right| \leqslant A_\beta 2^{-j|\beta|} \quad \text{for all } \beta\in Z^n_0,\ j\geqslant 1.$$

We assume that $\int\varphi(x)\,dx\neq 0$, and that $\hat{\varphi}$ is C^∞ in a small neighborhood of the origin. Thus for $\varepsilon<1$ small enough $\hat{\varphi}$ is of class C^N and bounded away from 0 in $\{|x|<\varepsilon\}$.

Define ψ_j by the condition that

$$\hat{\psi}_j(\xi) = \frac{\hat{\Phi}(\xi)\eta_j(\xi)}{\hat{\varphi}(\varepsilon\xi 2^{-j})} \quad \text{for } j=0,1,\ldots \tag{2}$$

By our choice of ε we see that the denominator is smooth and bounded away from zero in the support of the numerator. With ψ_j defined as above we get the desired decomposition of Φ:

$$\Phi(x) = \sum_{j=0}^{\infty} \varphi_{\varepsilon 2^{-j}} * \psi_j(x).$$

The subscript $\varepsilon 2^{-j}$ here denotes dilation. By considering φ_ε, instead of φ, as our original mollifier we may assume that $\varepsilon = 1$. (The change of mollifier from φ to φ_ε is controlled by a change of variable in t.)

If we write $u(x,t) = f*\varphi_t(x)$ and $U(x,t) = f*\Phi_t(x)$ then we have

$$U(x,t) = \sum_{j=0}^{\infty} \int u(x-ty, t2^{-j})\psi_j(y)\,dy. \tag{3}$$

This equation will be the starting point of many of the change of approximate identity results.

In the following we will need estimates of the functions ψ_j in terms of Φ. The next lemma gives the necessary information.

LEMMA 1. *Let N be an integer such that $N > n+\lambda$. Then*

$$\sum_{j=0}^{\infty} 2^{ja} \int (1+|y|)^{\lambda} |\psi_j(y)|\,dy \leq A\,|||\Phi|||$$

where

$$|||\Phi||| = \sum_{|\beta|\leq N} \int (1+|\xi|)^{a} \left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{\Phi}(\xi)\right| d\xi$$

and A is a constant depending on φ but independent of Φ.

Proof. By applying Leibnitz rule to (2) it is not difficult to show that

$$|(I-\Delta)^{N/2}\hat{\psi}_j(\xi)| \leq A \sum_{|\beta|\leq N} \left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{\Phi}(\xi)\right|.$$

Since $\hat{\psi}_j$ is supported in N_j then we have

$$\int |(I-\Delta)^{N/2}\hat{\psi}_j(\xi)|\,d\xi \leq A \sum_{|\beta|\leq N} \int_{N_j} \left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{\Phi}(\xi)\right| d\xi.$$

Thus

$$\int (1+|y|)^{\lambda}|\psi_j(y)|\,dy \leq \sup_{x\in \mathbf{R}^n} \{(1+|x|^2)^{N/2}|\psi_j(x)|\} \int (1+|y|)^{\lambda-N}\,dy$$
$$\leq A \sum_{|\beta|\leq N} \int_{N_j} \left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{\Phi}(\xi)\right| d\xi.$$

Now by recalling the definition of the sets N_j we see that

$$\sum_{j=0}^{\infty} 2^{ja} \int (1+|y|)^{\lambda}|\psi_j(y)|\,dy \leq A \sum_{|\beta|\leq N} \sum_{j=0}^{\infty} 2^{ja} \int_{N_j} \left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{\Phi}(\xi)\right| d\xi$$
$$\leq A \sum_{|\beta|\leq N} \sum_{j=0}^{\infty} \int_{N_j} (1+|\xi|)^{a} \left|\left(\frac{\partial}{\partial \xi}\right)^{\beta} \hat{\Phi}(\xi)\right| d\xi = A\,|||\Phi|||.$$

The first application of this decomposition of Φ is the following theorem stating that the G functions are essentially independent of the approximate identity φ_t.

THEOREM 1. *Suppose that N is an integer $> n+\lambda$, $\lambda > 0$, and $1 \leq p \leq \infty$. Let φ be a C^∞ function such that $\hat{\varphi}(0) \neq 0$ and $\hat{\varphi}$ is of class C^N in a neighborhood of the origin. Then for every C^∞ function Φ,*

$$G^{p,\lambda}_{\gamma,\beta}(\Phi) \leq A\,|||\Phi|||\,G^{p,\lambda}_{\gamma,\beta}(\varphi)$$

where

$$|||\Phi||| \leq \sum_{|\delta|\leq N} \int (1+|z|)^{\lambda+|\beta|-\delta} \left|\left(\frac{\partial}{\partial t}\right)^{\delta} \hat{\Phi}(z)\right| dz \tag{4}$$

and A is a constant independent of Φ, f, and x_0.

Proof. Let $u(x,t) = f*\varphi_t(x)$ and $U(x,t) = f*\Phi_t(x)$. It follows from equation (3) that

$$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda} t^{|\beta|-\gamma} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} U(x,t)\right| \tag{5}$$
$$\leq \sum_j \int \left(\frac{t}{t+|x-x_0|}\right)^{\lambda} t^{|\beta|-\gamma} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x-ty, t2^{-j})\right| |\psi_j(y)|\,dy.$$

For $p = \infty$, we have by the definition of $G^{\infty,\lambda}_{\gamma,\beta}$

$$|u(x-ty, t2^{-j})| \leq \left(\frac{2^j}{t}\right)^{|\beta|-\gamma} \left(1+\frac{|x-ty-x_0|}{t2^{-j}}\right)^{\lambda} G^{\infty,\lambda}_{\gamma,\beta}(\varphi).$$

But

$$\left(1+\frac{|x-ty-x_0|}{t2^{-j}}\right) \leq A2^j \left(1+\frac{|x-x_0|}{t}\right)(1+|y|). \tag{6}$$

Putting this into (5) and simplifying gives

$$G^{\infty,\lambda}_{\gamma,\beta}(\Phi) = \sup_{(x,t)\in\Omega} \left(\frac{t}{t+|x-x_0|}\right)^{\lambda} t^{|\beta|-\gamma} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} U(x,t)\right| \tag{7}$$
$$\leq A\left\{\sum_j 2^{j(\lambda+|\beta|-\gamma)}(1+|y|)^{\lambda}|\psi_j(y)|\,dy\right\} G^{\infty,\lambda}_{\gamma,\beta}(\varphi).$$

We will now try to prove that this inequality is also true for $p<\infty$. First we insert (5) into the definition for $G^{p,\lambda}_{\gamma,\beta}$ and apply Minkowski's integral inequality. This results in

$$G^{p,\lambda}_{\gamma,\beta}(\Phi)\leqslant A\sum_j\int\Big[\iint_\Omega\Big(\frac{t}{t+|x-x_0|}\Big)^{p\lambda}\times \times\Big\{t^{|\beta|-\gamma}\Big|\Big(\frac{\partial}{\partial x}\Big)^\beta u(x-ty,\,t2^{-j})\Big|\Big\}^p\frac{dx\,dt}{t^{n+1}}\Big]^{1/p}|\psi_j(y)|\,dy.$$

We use the following estimate similar to (6)

$$\Big(\frac{t}{t+|x-x_0|}\Big)\leqslant A2^j\Big(\frac{t2^{-j}}{t2^{-j}+|x-ty-x_0|}\Big)(1+|y|). \tag{8}$$

Using this and applying the change of variables $z=x-ty$ and $s=t2^{-j}$ produces

$$G^{p,\lambda}_{\gamma,\beta}(\Phi)\leqslant A\sum_j\int\Big[\iint_\Omega 2^{jp(\lambda+|\beta|-\gamma)}\Big(\frac{s}{s+|z-x_0|}\Big)^{p\lambda}\times \times\Big\{s^{|\beta|-\gamma}\Big|\Big(\frac{\partial}{\partial z}\Big)^\beta u(z,\,s)\Big|\Big\}^p 2^{-jn}\frac{dz\,ds}{s^{n+1}}\Big]^{1/p}(1+|y|)^\lambda|\psi_j(y)|\,dy \leqslant AG^{p,\lambda}_{\gamma,\beta}(\varphi)\Big\{\sum_j 2^{j(\lambda+|\beta|-\gamma)}(1+|y|)^\gamma|\psi_j(y)|\,dy\Big\}.$$

So we have succeeded in showing that (7) is true for all p, $1\leqslant p\leqslant\infty$. Lemma 1 now tells us that

$$\sum_j 2^{j(\lambda+|\beta|-\gamma)}\int(1+|y|)^\lambda|\psi_j(y)|\,dy\leqslant A\,|||\Phi|||,$$

where $|||\Phi|||$ is defined by (4). Therefore

$$G^{p,\lambda}_{\gamma,\beta}(\Phi)\leqslant A\,|||\Phi|||\,G^{p,\lambda}_{\gamma,\beta}(\varphi).$$

This completes the proof of Theorem 1.

We have now shown that if G contains only derivatives of the form $\Big(\dfrac{\partial}{\partial x}\Big)^\beta$ then it is well behaved under a change of mollifier. Our next objective is to prove that if we have this control over all the derivatives $\Big(\dfrac{\partial}{\partial x}\Big)^\beta$, $|\beta|=k$, then we can dominate $G^{p,\lambda}_{\gamma,\beta,k_0}$ whenever $|\beta|+k_0=k$. Before doing this however it will be useful to know more about the derivatives with respect to t.

Define $\mathscr{D}$ by

$$(\mathscr{D}\varphi)(x)=-n\varphi(x)-\sum_{i=1}^n x_i\frac{\partial\varphi}{\partial x_i}(x)=-\sum_{i=1}^n\Big(\frac{\partial}{\partial x_i}\Big)(x_i\varphi)$$

and define $\mathscr{D}_k$ recursively by $\mathscr{D}_1=\mathscr{D}$, $\mathscr{D}_{k+1}=(\mathscr{D}-k)\mathscr{D}_k$.

LEMMA 2. (i) $\dfrac{\partial}{\partial t}(\varphi_t)(x)=\dfrac{1}{t}(\mathscr{D}\varphi)_t(x)$,

(ii) $\dfrac{\partial^k}{\partial t^k}(\varphi_t)(x)=\dfrac{1}{t^k}(\mathscr{D}_k\varphi)_t(x)$,

(iii) $\int x^\beta(\mathscr{D}_k\varphi)(x)\,dx=0$ *for all* $|\beta|\leqslant k-1$,

(iv) $\mathscr{D}_k\varphi=(-1)^k\sum_{|\beta|=k}c_\beta\Big(\dfrac{\partial}{\partial x}\Big)^\beta(x^\beta\varphi)$.

Proof. The proof of (i) is simply a matter of differentiating $\varphi_t(x)=\dfrac{1}{t^n}\varphi\Big(\dfrac{x}{t}\Big)$ with respect to t. Assertion (ii) is proven by induction. For (iii), if $k=1$ then

$$\int(\mathscr{D}\varphi)(x)\,dx=-\sum_{i=1}^n\int\Big(\frac{\partial}{\partial x_i}\Big)(x_i\varphi)(x)\,dx=0.$$

Assume that (iii) is true for $k=j$. Then

$$\int x^\beta(\mathscr{D}_{j+1}\varphi)(x)\,dx=\int x^\beta(\mathscr{D}-j)(\mathscr{D}_j\varphi)(x)\,dx =-(n+j)\int x^\beta(\mathscr{D}_j\varphi)(x)\,dx-\sum_{i=1}^n\int x^\beta x_i\frac{\partial}{\partial x_i}(\mathscr{D}_j\varphi)(x)\,dx.$$

Integration by parts gives

$$\int x^\beta(\mathscr{D}_{j+1}\varphi)(x)\,dx=\Big[-n-j+\sum_{i=1}^n(\beta_i+1)\Big]\int x^\beta(\mathscr{D}_j\varphi)(x)\,dx =(|\beta|-j)\int x^\beta(\mathscr{D}_j\varphi)(x)\,dx.$$

This expression is clearly zero if $|\beta|=j$ and if $|\beta|\leqslant j-1$ we use the induction hypothesis.

Assertion (iv) is obviously true for $k=1$. Assume that it holds for $k=j$. Then

$$\mathscr{D}_{j+1}\varphi=\mathscr{D}_j(\mathscr{D}-j)\varphi=(-1)^j\sum_{|\beta|=j}c_\beta\Big(\frac{\partial}{\partial x}\Big)^\beta\Big[x^\beta\Big(-\sum_{i=1}^n\Big(\frac{\partial}{\partial x_i}\Big)(x_i\varphi)-j\varphi\Big)\Big] =(-1)^{j+1}\sum_{|\beta|=j}c_\beta\Big(\frac{\partial}{\partial x}\Big)^\beta\Big[\sum_{i=1}^n\Big(\frac{\partial}{\partial x_i}(x_ix^\beta\varphi)-\beta_ix^\beta\varphi\Big)+jx^\beta\varphi\Big].$$

But since $\sum_{i=1}^n\beta_i=j$, it follows that

$$\mathscr{D}_{j+1}\varphi=(-1)^{j+1}\sum_{|\beta|=j}\sum_{i=1}^n c_\beta\Big(\frac{\partial}{\partial x}\Big)^\beta\Big(\frac{\partial}{\partial x_i}\Big)(x_ix^\beta\varphi)=(-1)^{j+1}\sum_{|\delta|=j+1}c_\delta\Big(\frac{\partial}{\partial x}\Big)^\delta(x^\delta\varphi).$$

Thus the proof is completed by induction.

The next theorem states that if we have control over all derivatives of the form $\left(\frac{\partial}{\partial x}\right)^\beta$, $|\beta| = k$, then we have control over all derivatives $\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial t}\right)^j$ where $|\beta|+j = k$.

THEOREM 2. *Let φ and Φ be as in Theorem 1. Suppose that $|\beta|+j = k$. Then*

$$G^{p,\lambda}_{\gamma,\beta,j}(\Phi) \leqslant A\,\|\Phi\|\mathscr{G}^{p,\beta}_{\gamma,k}(\varphi)$$

where

(9) $$\|\Phi\| = \sum_{|\delta|\leqslant N+j}\int (1+|z|)^{\lambda+|\beta|-\gamma}\left|\left(\frac{\partial}{\partial z}\right)^\delta \hat{\Phi}(z)\right| dz$$

and A is independent of Φ, f, and x_0. Note in particular that

$$G^{p,\lambda}_{\gamma,0,k}(\Phi) \leqslant A\,\|\Phi\|\mathscr{G}^{p,\lambda}_{\gamma,k}(\varphi).$$

Proof. By Lemma 2 we have

$$\mathscr{D}_j\Phi = (-1)^j\sum_{|\delta|=j} c_\delta\left(\frac{\partial}{\partial x}\right)^\delta (x^\delta\Phi).$$

Consequently

$$\left(\frac{\partial}{\partial x}\right)^\beta \mathscr{D}_j\Phi = (-1)^j\sum_{|\delta|=j} c_\delta\left(\frac{\partial}{\partial x}\right)^{\delta+\beta} (x^\delta\Phi).$$

Therefore remembering Lemma 2(ii),

$$\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial t}\right)^j U(x,t) = (-1)^j\sum_{|\delta|=j} c_\delta\left(\frac{\partial}{\partial x}\right)^{\delta+\beta} f*(x^\delta\Phi)_t(x).$$

Hence inserting this into the definition of $G^{p,\lambda}_{\gamma,\beta,j}(\Phi)$ and using Minkowski's inequality gives

(10) $$G^{p,\lambda}_{\gamma,\beta,j}(\Phi) \leqslant A\sum_{|\delta|=j} G^{p,\lambda}_{\gamma,\beta+\delta}(x^\delta\Phi).$$

Now we can apply Theorem 1 to the right-hand side. If $\|\Phi\|_*$ is the norm given in Theorem 1 then notice that $\|x^\delta\Phi\|_* \leqslant \|\Phi\|$ where the latter norm is the one in (9). Thus (10) becomes

$$G^{p,\lambda}_{\gamma,\beta,j}(\Phi) \leqslant A\,\|\Phi\|\sum_{|\delta|=j} G^{p,\lambda}_{\gamma,\beta+\delta}(\varphi) \leqslant A\,\|\Phi\|\sum_{|\delta|=k} G^{p,\lambda}_{\gamma,\delta}(\varphi) = A\,\|\Phi\|\mathscr{G}^{p,\lambda}_{\gamma,k}(\varphi).$$

This completes the proof of the theorem.

§ 3. The norms $N(f)(x_0)$.

The G functions introduced in the last section can be used to define norms for spaces of distributions. Since the G functions contain only derivatives of high order, certain harmless lower order terms must be added to give a norm. We will examine these $R(f)(x_0)$ terms now and will study how they are affected by a change of mollifier.

After showing that these terms are well-behaved under a change of approximate identity, we will be able to combine them with the G functions to form norms. We then will show that these norms are essentially independent of k, the order of the derivatives in the G function (for $k > \gamma+n/p$).

Recall that

$$R^{p,\lambda}_\beta(f)(x_0) = \left[\int_{R^n}\frac{1}{(1+|x-x_0|)^{p\lambda}}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right|^p dx\right]^{1/p}$$

and that these terms are always finite for λ sufficiently large.

In this section we write $R^{p,\lambda}_\beta(\varphi)$ to mean $R^{p,\lambda}_\beta(f)(x_0)$ with φ.

LEMMA 3. *Suppose that $k > \gamma$. Then*

(11) $$\mathscr{G}^{\infty,\lambda}_{\gamma,k}(\varphi) \leqslant A\Big\{\mathscr{G}^{\infty,\lambda}_{\gamma,k+1}(\varphi) + \sum_{|\beta|=k} R^{\infty,\lambda}_\beta(\varphi)\Big\}.$$

Proof. It will suffice to prove

$$G^{\infty,\lambda}_{\gamma,\beta}(\varphi) \leqslant A\{\mathscr{G}^{\infty,\lambda}_{\gamma,k+1}(\varphi) + R^{\infty,\lambda}_\beta(\varphi)\} \quad \text{for every } |\beta| = k.$$

Write

$$u(x,t) = -\int_t^1\left(\frac{\partial}{\partial s}\right)u(x,s)\,ds + u(x,1).$$

Then

(12) $$t^{k-\gamma}\left(\frac{\partial}{\partial x}\right)^\beta u(x,t) = -t^{k-\gamma}\int_t^1\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\,ds + t^{k-\gamma}\left(\frac{\partial}{\partial x}\right)^\beta u(x,1).$$

But by Theorem 2 and the definition of $\mathscr{G}^{\infty,\lambda}_{\gamma,k+1}(\varphi)$

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right| \leqslant As^{-k-1+\gamma}\left(1+\frac{|x-x_0|}{s}\right)^\lambda \|\varphi\|\,\mathscr{G}^{\infty,\lambda}_{\gamma,k+1}(\varphi).$$

Since

$$\left(1+\frac{|x-x_0|}{s}\right)^\lambda \leqslant \left(1+\frac{|x-x_0|}{t}\right)^\lambda \quad\text{and}\quad t^{k-\gamma}\int_t^1 s^{-k-1+\gamma}\,ds \leqslant A,$$

it follows that the first term on the right-hand side of (12) is dominated by

(13) $$At^{k-\gamma}\left(1+\frac{|x-x_0|}{t}\right)^\lambda \|\varphi\|\,\mathscr{G}^{\infty,\gamma}_{\gamma,k+1}(\varphi)\int_t^1 s^{-k-1+\gamma}\,ds$$
$$\leqslant A\mathscr{G}^{\infty,k}_{\gamma,k+1}(\varphi)\left(1+\frac{|x-x_0|}{t}\right)^\lambda.$$

Now we want to majorize the second term in (12). By the definition of $R_\beta^{\infty,\lambda}$ we have

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right| \leqslant R_\beta^{\infty,\lambda}(\varphi)(1+|x-x_0|)^\lambda.$$

Since $k>\gamma$, $t^{k-\gamma}\leqslant 1$. Also

$$(1+|x-x_0|)^\lambda \leqslant \left(1+\frac{|x-x_0|}{t}\right)^\lambda.$$

Therefore

$$(14)\qquad t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right| \leqslant R_\beta^{\infty,\lambda}(\varphi)\left(1+\frac{|x-x_0|}{t}\right)^\lambda.$$

Inserting (14) and (13) into (12) yields

$$t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right| \leqslant A\{\mathcal{G}_{\gamma,k+1}^{\infty,\lambda}(\varphi)+R_\beta^{\infty,\lambda}(\varphi)\}\left(1+\frac{|x-x_0|}{t}\right)^\lambda.$$

But the fact that this is true for all (x,t) in Ω is exactly the statement of (11).

LEMMA 4. *If $k>\gamma+n/p$ then*

$$\mathcal{G}_{\gamma,k}^{p,\lambda}(\varphi)\leqslant A\left\{\mathcal{G}_{\gamma,k+1}^{p,\lambda}(\varphi)+\sum_{|\beta|=k}R_\beta^{p,\lambda}(\varphi)\right\}.$$

Proof. Again it suffices to be able to dominate $G_{\gamma,\beta}^{p,\lambda}(\varphi)$ for every $|\beta|=k$. As before we start with

$$(15)\qquad t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right| \leqslant t^{k-\gamma}\int_t^1\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|ds+t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right|.$$

Thus

$$(16)\qquad G_{\gamma,\beta}^{p,\lambda}(\varphi) = \left[\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda}\left\{t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|\right\}^p\frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$
$$\leqslant\left[\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda}\left\{t^{k-\gamma}\int_t^1\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|ds\right\}^p\frac{dx\,dt}{t^{n+1}}\right]^{1/p}+$$
$$+\left[\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda}\left\{t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right|\right\}^p\frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$
$$=J_1+J_2.$$

We will first examine J_1. Let $1/p+1/q=1$ and let $\varepsilon>0$ be so small that

$$(17)\qquad k>\gamma+n/p+\varepsilon.$$

By Hölder's inequality

$$(18)\qquad \int_t^1\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|ds$$
$$\leqslant\left[\int_t^1 s^{p(1/q+\varepsilon)}\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|^p ds\right]^{1/p}\left[\int_t^1 s^{-q(1/q+\varepsilon)}ds\right]^{1/q}.$$

But notice that

$$\int_t^1 s^{-q(1/q+\varepsilon)}ds=\int_t^1 s^{-\varepsilon q-1}ds=\left(1+\frac{t^{-\varepsilon q}}{\varepsilon q}\right)\leqslant At^{-\varepsilon q}.$$

Therefore from (18)

$$\left[\int_t^1\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|ds\right]^p\leqslant At^{-\varepsilon p}\int_t^1 s^{p(1/q+\varepsilon)}\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|^p ds$$

and

$$J_1\leqslant A\left[\iint_\Omega\int_t^1\left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}t^{p(k-\gamma)-\varepsilon p}s^{p(1/q+\varepsilon)}\times\right.$$
$$\left.\times\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|^p\frac{ds\,dx\,dt}{t^{n+1}}\right]^{1/p}.$$

Observe that $\dfrac{t}{t+|x-x_0|}\leqslant\dfrac{s}{s+|x-x_0|}$. Consequently by changing the order of integration in s and t it follows that

$$J_1\leqslant A\left[\int_{R^n}\int_0^1\left(\int_0^s t^{p(k-\gamma-\varepsilon)-n-1}dt\right)\left(\frac{s}{s+|x-x_0|}\right)^{\lambda p}s^{p(1/q+\varepsilon)}\times\right.$$
$$\left.\times\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|^p dx\,ds\right]^{1/p}.$$

Since we know that $k>\gamma+n/p+\varepsilon$ then the integral in t equals $As^{p(k-\gamma-\varepsilon)-n}$. Hence

$$J_1\leqslant A\left[\int_{R^n}\int_0^1\left(\frac{s}{s+|x-x_0|}\right)^{\lambda p}s^{p(k-\gamma)+p/q-n}\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial s}\right)u(x,s)\right|^p dx\,ds\right]^{1/p}.$$

Because $p(k-\gamma)+p/q-n=p(k+1-\gamma)-n-1$ what we have shown is that $J_1 \leqslant A G^{p,\lambda}_{\gamma,\beta,1}(\varphi)$. In view of Theorem 2 this means that

$$J_1 \leqslant A \mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi). \tag{19}$$

Now we estimate J_2. Since

$$\left(\frac{t}{t+|x-x_0|}\right) \leqslant \left(\frac{1}{1+|x-x_0|}\right)$$

then

$$J_2 \leqslant \Big[\int_{R^n} \frac{1}{(1+|x-x_0|)^{\lambda p}} \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x,1)\right|^p dx\Big]^{1/p} \Big[\int_0^1 t^{p(k-\gamma)-n-1} dt\Big]^{1/p}$$
$$\leqslant A R^{p,\lambda}_{\beta}(\varphi).$$

As a result, this and (19) combine with (16) to complete the proof of the lemma:

$$G^{p,\lambda}_{\gamma,\beta}(\varphi) \leqslant J_1 + J_2 \leqslant A\{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi) + R^{p,\lambda}_{\beta}(\varphi)\}.$$

In the applications we will want to be able to use any $k > \gamma$. The proof of the last lemma can be adjusted so that the result is true for all $k > \gamma$. There is the minor annoyance however that the λ must be increased.

Corollary 4. *If $k > \gamma$ then*

$$\mathscr{G}^{p,\lambda+2}_{\gamma,k}(\varphi) \leqslant A\Big\{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi) + \sum_{|\beta|=k} R^{p,\lambda+2}_{\gamma,\beta}(\varphi)\Big\}.$$

Proof. This time we will integrate over line segments directed toward x_0 rather than vertical line segments. Consider the line segment $L = \{(x_0 + tz, t)\colon 0 < t < 1\}$. Let $\partial/\partial r$ denote differentiation along this line. Then

$$\left|\left(\frac{\partial^\beta u}{\partial x^\beta}\right)(x_0+tz, t)\right| \leqslant \int_t^1 \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial r}\right) u(x_0+sz, s)\right| ds \sqrt{1+|z|^2} + \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x_0+z, 1)\right|. \tag{20}$$

Let $\varepsilon > 0$ be so small that $k > \gamma + \varepsilon$. By using Hölder's inequality as in (18) it follows that

$$\Big[\int_t^1 \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial r}\right) u(x_0+sz, s)\right| ds\Big]^p$$
$$\leqslant A t^{-\varepsilon p} \int_t^1 s^{p(1/q+\varepsilon)} \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial r}\right) u(x_0+sz, s)\right|^p ds.$$

Since $\dfrac{\partial}{\partial r} = z_1 \dfrac{\partial}{\partial x_1} + \ldots + z_n \dfrac{\partial}{\partial x_n} + \dfrac{\partial}{\partial t}$, if we write $x_{n+1} = t$, then

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial r}\right) u(x_0+sz, s)\right|^p \leqslant (1+|z|) A \sum_{i=1}^{n+1} \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial x_i}\right) u(x_0+sz, s)\right|^p.$$

Hence putting this into (20) produces

$$\left|\left(\frac{\partial^\beta u}{\partial x^\beta}\right)(x_0+tz, t)\right|$$
$$\leqslant A(1+|z|)^2 t^{-\varepsilon}\Big[\sum_{i=1}^{n+1}\int_t^1 s^{p(1/q+\varepsilon)} \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial x_i}\right) u(x_0+sz, s)\right|^p ds\Big]^{1/p} +$$
$$+ \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x_0+z, 1)\right|.$$

Now multiply this by $t^{(k-\gamma)p-1}$ and integrate

$$\Big[\int_0^1 t^{(k-\gamma)p-1} \left|\frac{\partial^\beta u}{\partial x^\beta}(x_0+tz, t)\right|^p dt\Big]^{1/p} \leqslant A(1+|z|^2) \times$$
$$\times \Big[\int_0^1 t^{p(k-\gamma)-1} t^{-\varepsilon p} \sum_{i=1}^{n+1}\int_t^1 s^{p(1/q+\varepsilon)} \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial x_i}\right) u(x_0+sz, s)\right|^p ds dt\Big]^{1/p} +$$
$$+ \left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x_0+z, 1)\right| \Big[\int_0^1 t^{p(k-\gamma)-1} dt\Big]^{1/p} \leqslant A(1+|z|^2) \times$$
$$\times \Big[\sum_{i=1}^{n+1}\int_0^1 \Big(\int_0^s t^{(k-\gamma)p-1} t^{-\varepsilon p} dt\Big) s^{(1/q+\varepsilon)p} \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial x_i}\right) u(x_0+sz, s)\right|^p ds\Big]^{1/p} +$$
$$+ A\left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x_0+z, 1)\right|.$$

Note first that

$$\int_0^s t^{p(k-\gamma-\varepsilon)-1} dt = A s^{p(k-\gamma-\varepsilon)} \quad \text{and} \quad A s^{p(k-\gamma-\varepsilon)} s^{p(1/q+\varepsilon)} = A s^{p(k+1-\gamma)-1}.$$

Hence

$$\Big[\int_0^1 t^{p(k-\gamma)-1} \left|\frac{\partial^\beta u}{\partial x^\beta}(x_0+tz, t)\right|^p dt\Big]$$
$$\leqslant A(1+|z|^2)\Big[\sum_{i=1}^{n+1}\int_0^1 s^{p(k+1-\gamma)-1} \left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial x_i}\right) u(x_0+sz, s)\right|^p ds\Big]^{1/p} +$$
$$+ A\left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x_0+z, 1)\right|.$$

Now multiply by $\frac{1}{(1+|z|)^{\lambda+2}}$ and integrate over $\boldsymbol{R}^n$ in the z variable. Then change variables to $x = x_0+tz$. So $dz = t^{-n}dx$ and

$$\frac{1}{1+|z|} = \frac{t}{t+|x-x_0|}.$$

When we do this the result will be

$$G^{p,\lambda+2}_{\gamma,\beta}(\varphi) \leqslant A\{\mathcal{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda+2}_{\beta}(\varphi)\}.$$

LEMMA 5. *Suppose that* $\varphi\in\mathcal{S}$, $\int\varphi(x)\,dx\neq 0$. *If* $\gamma>\gamma_0$, $k>\gamma_0$, $|\beta|=k$, *then*

$$G^{\infty,\lambda}_{\gamma_0,\beta}(\varphi)\leqslant A\{\mathcal{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}.$$

Proof. Let χ be a C^∞ function supported in $\{|x|\leqslant 1\}$. Define $\phi=\varphi*\chi$ and $v(x,t)=f*\phi_t(x)$. As in the last lemma we have

(21) $$v(x,t) = -\int_t^1 \left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)v(x,s)\,ds+\left(\frac{\partial}{\partial x}\right)^{\beta} v(x,1).$$

Since $v(x,s)=f*\varphi_s*\chi_s(x)$ then

$$\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)v(x,s)=\left[\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)f*\varphi_s\right]*\chi_s(x)+\left[\left(\frac{\partial}{\partial x}\right)^{\beta}f*\varphi_s\right]*\left[\left(\frac{\partial}{\partial s}\right)\chi_s\right](x).$$

This allows us to dominate the first term of (21):

(22)

$$J_1=\int_t^1\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)v(x,s)\right|ds\leqslant\|\chi\|_\infty\int_t^1\int_{|y-x|\leqslant s}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)f*\varphi_s(y)\right|s^{-n}\,dy\,ds$$
$$+\|\mathcal{D}\chi\|_\infty\int_t^1\int_{|x-y|\leqslant s}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}f*\varphi_s(y)\right|s^{-n-1}\,dy\,ds$$
$$\leqslant A\sum_{j=0}^{1}\int_t^1\int_{|y-x|\leqslant s}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)^{j}f*\varphi_s(y)\right|s^{-n-1+j}\,dy\,ds.$$

For $t\leqslant s$ and $k\geqslant\gamma_0$, (22) can be used to give

(23) $$\left(\frac{t}{t+|x-x_0|}\right)t^{k-\gamma_0}J_1$$
$$\leqslant A\sum_{j=0}^{1}\int_t^1\int_{|y-x|\leqslant s}\left(\frac{s}{s+|x-x_0|}\right)^{\lambda}s^{k-\gamma_0}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)^{j}f*\varphi_s(y)\right|s^{-n-1+j}\,dy\,ds.$$

Use the fact that

$$\left(\frac{s}{s+|x-x_0|}\right)\leqslant\left(\frac{s}{s+|y-x_0|}\right)\left(1+\frac{|x-y|}{s}\right)\leqslant 2\left(\frac{s}{s+|y-x_0|}\right)$$

and apply Hölder's inequality to the right-hand side of (23). This produces

(24)

$$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}t^{k-\gamma_0}J_1\leqslant A\sum_{j=0}^{1}G^{p,\lambda}_{\gamma,\beta,j}(\varphi)\left[\int_t^1\int_{|y-x|\leqslant s}s^{q(\gamma-\gamma_0)-n-1}\,dy\,ds\right]^{1/q}$$
$$=A\sum_{j=0}^{1}G^{p,\lambda}_{\gamma,\beta,j}(\varphi)$$

where $1/p+1/q=1$. But by Theorem 2 and Lemma 4,

(25) $$G^{p,\lambda}_{\gamma,\beta}(\varphi)+G^{p,\lambda}_{\gamma,\beta,1}(\varphi)\leqslant A\{\mathcal{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}.$$

Therefore we have shown that

(26) $$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}t^{k-\gamma_0}J_1\leqslant A\{\mathcal{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}.$$

Now we will take care of the second term in (21)

$$J_2=\left|\left(\frac{\partial}{\partial x}\right)^{\beta}v(x,1)\right|=\left|\left(\frac{\partial}{\partial x}\right)^{\beta}f*\varphi*\chi(x)\right|.$$

Therefore

(27) $$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}t^{k-\gamma_0}J_2\leqslant\frac{1}{(1+|x-x_0|)^{\lambda}}J_2$$
$$\leqslant\int_{\boldsymbol{R}^n}\frac{1}{(1+|y-x_0|)^{\lambda}}\left|\left(\frac{\partial}{\partial y}\right)^{\beta}f*\varphi(y)\right|(1+|x-y|)^{\lambda}|\chi(x-y)|\,dy.$$

By Hölder's inequality the last integral is dominated by

$$R^{p,\lambda}_{\beta}(\varphi)\left[\int_{\boldsymbol{R}^n}(1+|y|)^{\lambda q}|\chi(y)|^{q}\,dy\right]^{1/q}\quad\text{where}\quad 1/p+1/q=1.$$

Hence we now have

(28) $$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}t^{k-\gamma_0}J_2\leqslant AR^{p,\lambda}_{\beta}(\varphi)\quad\text{for all }(x,t)\in\Omega.$$

Combining (26) and (28) then gives

$$G^{\infty,\lambda}_{\gamma_0,\beta}(\chi*\varphi) \leqslant \sup_{(x,t)\in R_1^{n+1}} \left(\frac{t}{t+|x-x_0|}\right)^{\lambda} t^{k-\gamma_0}(J_1+J_2)$$
$$\leqslant A\{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}.$$

Hence an application of Theorem 1 completes the proof.

In the proof of Theorem 5 below we will need a slight variant of this lemma.

COROLLARY 5. *Let φ and χ be as in Lemma 5 and let*

$$v(x,t) = f*\Phi_t(x) = f*\varphi_t*\chi_t(x).$$

If $\gamma_0<\gamma$, $|\beta| = k$, $k\geqslant\gamma_0$, then

$$\text{(29)}\qquad \sup_{0<t<1} t^{k-\gamma_0}\left[\int_{R^n}\left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} v(x,t)\right|^p dx\right]^{1/p}$$
$$\leqslant A\{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}.$$

Proof. Let J_1 and J_2 be defined as in Lemma 5. So $\left|\left(\frac{\partial}{\partial x}\right)^{\beta} v(x,t)\right| \leqslant J_1+J_2$. Using (22) we get, instead of (23),

$$\text{(30)}\qquad t^{k-\gamma_0}\left[\int_{R^n}\left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}(J_1)^p dx\right]^{1/p}$$
$$\leqslant A\sum_{j=0}^{1}\left[\int_{R^n}\left\{\int_t^1\int_{|y-x|\leqslant s}\left(\frac{s}{s+|x-x_0|}\right)^{\lambda} s^{k-\gamma_0}\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial s}\right)^{j} f*\varphi_s(y)\right|\frac{dy\,ds}{s^{n+1-j}}\right\}^p dx\right]^{1/p}.$$

Then just as in (24) an application of Hölder's inequality simplifies the integral on the right of (30). Imitating steps (25) and (26) then completes the analysis of this term.

The second term J_2 is very easy to handle. This is done in the same way as in (27).

This completes the proof of Corollary 5.

Now we can deal with the problem of the effect of a change of mollifier on the terms $R^{p,\lambda}_{\beta}(\varphi)$. The following theorem then is a change of approximate identity theorem very much similar to Theorem 1 of the second section.

THEOREM 4. *Suppose that N is an integer greater than $n+\lambda$ and $1\leqslant p \leqslant\infty$. Let φ be a C^∞ function such that $\hat{\varphi}(0)\neq 0$ and $\hat{\varphi}$ is of class C^N in a neighborhood of the origin. Assume that $|\beta| = k$, $k>\gamma$. Then for every C^∞ function Φ*

$$R^{p,\lambda}_{\beta}(\Phi)\leqslant A\|\Phi\|\{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}$$

where

$$\text{(31)}\qquad \|\Phi\| = \sum_{|\delta|\leqslant N}\int(1+|z|)^{k-\gamma_0+\lambda}\left|\left(\frac{\partial}{\partial z}\right)^{\delta}\hat{\Phi}(z)\right| dz$$

for any fixed $\gamma_0<\gamma$. Furthermore

$$\Big\{\mathscr{G}^{p,\lambda}_{\gamma,k}(\Phi)+\sum_{|\beta|<k}R^{p,\lambda}_{\beta}(\Phi)\Big\}\leqslant A\|\Phi\|\Big\{\mathscr{G}^{p,\lambda}_{\gamma,k}(\varphi)+\sum_{|\beta|<k}R^{p,\lambda}_{\beta}(\varphi)\Big\}$$

where

$$\|\Phi\| = \sum_{|\delta|\leqslant N}\int(1+|z|)^{k+\lambda+\max(0,-\gamma_0)}\left|\left(\frac{\partial}{\partial z}\right)^{\delta}\hat{\Phi}(z)\right| dz.$$

Proof. Let χ be a C^∞ function supported in $\{|x|\leqslant 1\}$. Consider $\phi = \chi*\varphi$. We will now construct Φ as in the first chapter but this time we start from the mollifier ϕ and not from φ. We use the notation $U(x,t) = f*\Phi_t(x)$ and $v(x,t) = f*\phi_t(x)$. Therefore, as in (2) and (3), we have

$$\hat{\psi}_j(z) = \frac{\eta_j(z)\hat{\Phi}(z)}{\hat{\phi}(z2^{-j})}$$

and

$$U(x,t) = \sum_j v(x-ty, t2^{-j})\psi_j(y)\,dy.$$

Hence differentiating and setting $t = 1$ yields

$$\text{(32)}\qquad \left|\left(\frac{\partial}{\partial x}\right)^{\beta}U(x,1)\right|\leqslant\sum_j\int\left|\left(\frac{\partial}{\partial x}\right)^{\beta}v(x-y,2^{-j})\right||\psi_j(y)|\,dy.$$

Therefore

$$\text{(33)}\qquad R^{p,\lambda}_{\beta}(\Phi)$$
$$=\left[\int_{R^n}\frac{1}{(1+|x-x_0|)^{\lambda p}}\left|\left(\frac{\partial}{\partial x}\right)^{\beta}U(x,1)\right|^p dx\right]^{1/p}$$
$$\leqslant\sum_j\int\left[\int_{R^n}\frac{1}{(1+|x-x_0|)^{\lambda p}}\left|\left(\frac{\partial}{\partial x}\right)^{\beta}v(x-y,2^{-j})\right|^p dx\right]^{1/p}|\psi_j(y)|\,dy.$$

To simplify the estimate (33) we will apply Corollary 5. If we write $B = \{\mathscr{G}^{p,\lambda}_{\gamma,k+1}(\varphi)+R^{p,\lambda}_{\beta}(\varphi)\}$ and notice that

$$\frac{1}{(1+|x-x_0|)}\leqslant\left(\frac{2^{-j}}{2^{-j}+|x-y-x_0|}\right)2^j(1+|y|)$$

then by Corollary 5

$$\left[\int_{R^n}\frac{1}{(1+|x-x_0|)^{\lambda p}}\left|\left(\frac{\partial}{\partial x}\right)^{\beta}v(x-y,2^{-j})\right|^p dx\right]^{1/p}\leqslant A2^{j(\lambda+k-\gamma_0)}(1+|y|)^{\lambda}B.$$

Thus (33) becomes

$$R_\beta^{p,\lambda}(\Phi) \leqslant AB \sum_j 2^{j(\lambda+k-\gamma_0)} \int (1+|y|)^\lambda |\psi_j(y)|\,dy. \tag{34}$$

Lemma 1 allows us to gain control of the sum over j by using the norm defined in (31). Therefore (34) simplifies to

$$R_\beta^{p,\lambda}(\Phi) \leqslant A\,\|\Phi\| B.$$

The rest of the theorem is a direct consequence of this and Theorem 1.

Now that we know how the terms $R(f)(x_0)$ change with a new approximate identity we are able to examine the norms

$$N_{\gamma,k}^{p,\lambda}(f)(x_0) = \mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) + \sum_{|\beta|<k} R_\beta^{p,\lambda}(f)(x_0) \quad \text{for} \quad k > \gamma+n/p. \tag{35}$$

In Theorem 4, we have just proven that

$$N_{\gamma,k}^{p,\lambda}(\Phi) \leqslant A\,\|\Phi\| N_{\gamma,k}^{p,\lambda}(\varphi).$$

Therefore these expressions do not depend in any essential way on the mollifier φ.

The $N_{\gamma,k}^{p,\lambda}$ are in fact norms since

(i) $N_{\gamma,k}^{p,\lambda}(f_1+f_2)(x_0) \leqslant N_{\gamma,k}^{p,\lambda}(f_1)(x_0) + N_{\gamma,k}^{p,\lambda}(f_2)(x_0)$,

(ii) $N_{\gamma,k}^{p,\lambda}(cf)(x_0) = |c|\,N_{\gamma,k}^{p,\lambda}(f)(x_0)$ for all $c \in \boldsymbol{C}$,

(iii) $N_{\gamma,k}^{p,\lambda}(f)(x_0) \geqslant 0$ and if $N_{\gamma,k}^{p,\lambda}(f)(x_0) = 0$ then $f = 0$.

All of these properties are evident except possibly for the second property in (iii).

Suppose that $N_{\gamma,k}^{p,\lambda}(f)(x_0) = 0$. We emphasize the mollifier by writing $N_{\gamma,k}^{p,\lambda}(\varphi)$. Let $\gamma_0 < 0$. Using (35) and Lemma 5 we can show that

$$G_{\gamma_0,0}^{\infty,\lambda}(\varphi) \leqslant A N_{\gamma,k}^{p,\lambda}(\varphi).$$

But since $N_{\gamma,k}^{p,\lambda}(\varphi) = 0$ then

$$|f(\tilde\varphi)| = |f*\varphi(0)| \leqslant G_{\gamma_0,0}^{\infty,\lambda}(\varphi) \leqslant A N_{\gamma,k}^{p,\lambda}(\varphi) = 0 \tag{36}$$

where $\tilde\varphi(x) = \varphi(-x)$. Since by Theorem 4 this is true for all $\varphi \in \mathscr{S}'$ we have proven that $f = 0$. Therefore N is a norm.

Our next objective will be to show that the dependence of N on k is unimportant. That is, if we use a different k we get an equivalent norm.

THEOREM 5. *Let $1 \leqslant p \leqslant \infty$. If k is a positive integer $> \gamma+n/p$, then*

$$A N_{\gamma,k+1}^{p,\lambda}(f)(x_0) \leqslant N_{\gamma,k}^{p,\lambda}(f)(x_0) \leqslant A N_{\gamma,k+1}^{p,\lambda}(f)(x_0) \tag{37}$$

for some positive constants A independent of k, f, and x_0.

Proof. Let $\delta \in Z_0^n$, $|\delta| = k+1$; then there exist $\beta \in Z_0^n$ and $i \in \{1, \ldots, n\}$ such that $\left(\frac{\partial}{\partial x}\right)^\delta = \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial x_i}\right)$. Therefore

$$t^{k+1-\gamma}\left(\frac{\partial}{\partial x}\right)^\delta (f*\varphi_t) = t^{k-\gamma}\left(\frac{\partial}{\partial x}\right)^\beta \left(f*\left(\frac{\partial\varphi}{\partial x_i}\right)_t\right)$$

and

$$G_{\gamma,\delta}^{p,\lambda}(\varphi) = G_{\gamma,\beta}^{p,\lambda}\left(\frac{\partial\varphi}{\partial x_i}\right) \leqslant A G_{\gamma,\beta}^{p,\lambda}(\varphi).$$

Combining this estimate for all $|\delta| = k+1$ gives

$$\mathscr{G}_{\gamma,k+1}^{p,\lambda}(\varphi) \leqslant A \mathscr{G}_{\gamma,k}^{p,\lambda}(\varphi). \tag{38}$$

Similarly since $k > 0$, for every $|\delta| = k$ there exist β and i such that $\left(\frac{\partial}{\partial x}\right)^\delta = \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial x_i}\right)$, and $|\beta| = k-1$. Hence

$$\left(\frac{\partial}{\partial x}\right)^\delta (f*\varphi) = \left(\frac{\partial}{\partial x}\right)^\beta \left(f*\left(\frac{\partial\varphi}{\partial x_i}\right)\right)$$

and so

$$R_\delta^{p,\lambda}(\varphi) = R_\beta^{p,\lambda}\left(\frac{\partial\varphi}{\partial x_i}\right) \leqslant A\,\{\mathscr{G}_{\gamma,k}^{p,\lambda}(\varphi) + R_\beta^{p,\lambda}(\varphi)\}. \tag{39}$$

Therefore using (38) and (39) we get

$$\begin{aligned} N_{\gamma,k+1}^{p,\lambda}(f)(x_0) &= \mathscr{G}_{\gamma,k+1}^{p,\lambda}(f)(x_0) + \sum_{|\beta|=k} R_\beta^{p,\lambda}(f)(x_0) + \sum_{|\beta|<k} R_\beta^{p,\lambda}(f)(x_0) \\ &\leqslant A\left\{\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) + \sum_{|\beta|<k} R_\beta^{p,\lambda}(f)(x_0)\right\} = A N_{\gamma,k}^{p,\lambda}(f)(x_0). \end{aligned}$$

The second inequality in (37) is a direct consequence of the definition and Lemma 4.

We have shown that any $k > \gamma+n/p$ gives an equivalent norm. Therefore when we write $N_\gamma^{p,\lambda}$ we will assume that k is the smallest integer greater than $\gamma+n/p$.

§ 4. The spaces $\Lambda_\gamma^{p,\lambda}(x_0)$. The norms of the preceding section determine certain spaces $\Lambda_\gamma^{p,\lambda}(x_0)$. For $1 \leqslant p \leqslant \infty$, $\lambda > 0$, $\gamma \in \boldsymbol{R}$, define

$$\begin{aligned} \Lambda_\gamma^{p,\lambda}(x_0) &= \Big\{f \in \mathscr{S}': N_{\gamma,k}^{p,\lambda}(f)(x_0) < \infty \text{ for some } k > \gamma + \\ &\qquad + n/p,\ \varphi \in \mathscr{S},\ \int \varphi(x)\,dx \neq 0\Big\} \\ &= \Big\{f \in \mathscr{S}': N_{\gamma,k}^{p,\lambda}(f)(x_0) < \infty \text{ for all } k > \gamma + \\ &\qquad + n/p,\ \varphi \in \mathscr{S},\ \int \varphi(x)\,dx \neq 0\Big\}. \end{aligned}$$

We have shown already that we have the freedom to choose k and φ as we like. Making such a change gives us an equivalent norm for $\Lambda_\gamma^{p,\lambda}(x_0)$.

As mentioned in the introduction to this article the parameter γ gives the order of smoothness of f, the p tells the type of integral being used. The λ gives the order of growth as we examine $u(x,t) = f*\varphi_t(x)$ over cones of larger and larger aperture. On certain occasions when we have little control over the parameter λ it will be natural for us to examine the spaces

$$\Lambda_\gamma^p(x_0) = \bigcup_{\lambda>0} \Lambda_\gamma^{p,\lambda}(x_0).$$

For these spaces it is possible to use any $k>\gamma$ in the norm (rather than $k>\gamma+n/p$). By the first part of the proof of Theorem 5 we can show that

$$\mathscr{G}_{\gamma,k+1}^{p,\lambda}(\varphi) \leqslant A\mathscr{G}_{\gamma,k}^{p,\lambda}(\varphi).$$

(This is (38).) Also by Corollary 5,

$$\mathscr{G}_{\gamma,k}^{p,\lambda+2}(\varphi) \leqslant A\Big\{\mathscr{G}_{\gamma,k+1}^{p,\lambda}(\varphi) + \sum_{|\beta|=k} R_\beta^{p,\lambda+2}(\varphi)\Big\}.$$

Remember that the R terms will be finite for λ large enough. Therefore these two estimates combine to prove that

$$\Lambda_\gamma^p(x_0) = \Big\{f\in\mathscr{S}'\colon N_{\gamma,k}^{p,\lambda}<\infty \text{ for some } k>\gamma,\ \varphi\in\mathscr{S},\ \int\varphi(x)\,dx\neq 0\Big\}.$$

These spaces $\Lambda_\gamma^p(x_0)$ are of interest in particular when we consider the connection between the G functions and tempered nontangential boundedness (see [3]). In this context it will be important to be able to use the lower values of k.

We now mention some elementary facts about these spaces.

THEOREM 6. *$\Lambda_\gamma^{p,\lambda}(x_0)$ is a Banach space.*

Proof. Let $\{f_m\}$ be a convergent sequence on $\Lambda_\gamma^{p,\lambda}(x_0)$. Thus

$$N_{\gamma,k}^{p,\lambda}(f_{m_1}-f_{m_2})(x_0)\to 0 \quad \text{as} \quad m_1, m_2\to\infty.$$

As in (36) we have

$$|f_{m_1}(\Phi)-f_{m_2}(\Phi)| \leqslant A\,\|\Phi\|\,N_{\gamma,k}^{p,\lambda}(f_{m_1}-f_{m_2})(x_0)\to 0 \tag{40}$$

where $N_{\gamma,k}^{p,\lambda}$ depends on one fixed mollifier φ. Define $f(\Phi)=\lim_{m\to\infty} f_m(\Phi)$. From (40) it follows that $|f(\Phi)|\leqslant A\,\|\Phi\|$. So $f\in\mathscr{S}'$. Also

$$(f_m*\varphi_t)(x) = f_m(\phi)\to f(\phi) = f*\varphi_t(x) \tag{41}$$

for all $(x,t)\in\Omega$ where $\phi(y)=\varphi_t(x-y)$. Since

$$\left(\frac{\partial}{\partial x}\right)^\beta (f*\varphi_t)(x) = t^{-|\beta|} f*\left[\left(\frac{\partial}{\partial x}\right)^\beta\varphi\right]_t(x),$$

this also shows that all the derivatives converge pointwise.

Let

$$V_m(x,t) = \left(\frac{t}{t+|x-x_0|}\right)^\lambda t^{|\beta|-\gamma}\left(\frac{\partial}{\partial x}\right)^\beta (f_m*\varphi_t)(x)\,t^{-(n+1)/p}.$$

Since $G_{\gamma,\beta}^{p,\lambda}(f_{m_1}-f_{m_2})(x_0)\to 0$ and $G_{\gamma,\beta}^{p,\lambda}(f_m)(x_0) = \|V_m\|_{L^p(\Omega)}$ then V_m converges in the $L^p(\Omega)$ norm to some function $V\in L^p(\Omega)$. But by (41) we know that $V_m(x,t)$ converges pointwise to

$$V(x,t) = \left(\frac{t}{t+|x-x_0|}\right)^\lambda t^{k-\gamma}(f*\varphi_t)(x)\,t^{-(n+1)/p}. \tag{42}$$

Therefore $G_{\gamma,\beta}^{p,\lambda}(f)(x_0)<\infty$ and $G_{\gamma,\beta}^{p,\lambda}(f-f_m)(x_0)\to 0$ as $m\to\infty$.

A similar argument proves the same thing for the $R_\beta^{p,\lambda}$ terms. Combining these results gives

$$N_{\gamma,k}^{p,\lambda}(f)(x_0)<\infty \quad\text{and}\quad N_{\gamma,k}^{p,\lambda}(f-f_m)(x_0)\to 0 \quad \text{as} \quad m\to\infty.$$

Hence f is the limit of $\{f_m\}$ and $f\in\Lambda_\gamma^{p,\lambda}(x_0)$. So $\Lambda_\gamma^{p,\lambda}(x_0)$ is a Banach space.

LEMMA 6. $\bigcup_{\gamma\in\mathbf{R}} \Lambda_\gamma^p(x_0) = \mathscr{S}'$.

Proof. If $f\in\mathscr{S}'$, then $f=\sum_{|\delta|\leqslant M} c_\delta\left(\frac{\partial}{\partial x}\right)^\delta g_\delta$ where c_δ are constants and g_δ are slowly increasing continuous functions. Therefore it is easy to show that $|f*\varphi(x)|\leqslant A(1+|x-x_0|)^{M_1}$ and

$$\left(\frac{\partial}{\partial x_i}\right) f*\varphi_t(x) \leqslant \frac{A}{t^N}\left(1+\frac{|x-x_0|}{t}\right)^{M_1} \quad \text{for} \quad i=1,\dots,n$$

for some M_1 and N sufficiently large.

Therefore consider $\gamma<0$.

$$G_{\gamma,1}^{p,\lambda}(f)(x_0) \leqslant A\left[\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{p(\lambda-M_1)} t^{p(1-\gamma-N)}\frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$

$$= A\left[\int_{\mathbf{R}^n}\left(\frac{1}{1+|x|}\right)^{p(\lambda-M_1)}dx\right]^{1/p}\left[\int_0^1 t^{p(1-\gamma-N)}\frac{dt}{t}\right]^{1/p}.$$

This last expression is finite if $\lambda>M_1+n/p$ and $\gamma<1-N$. Since the R terms are always finite for large enough values of λ, we have shown that $N_{\gamma,1}^{p,\lambda}(f)(x_0)<\infty$ for λ large enough and $\gamma<1-N$. Therefore $f\in\Lambda_\gamma^p(x_0)$ for γ small enough.

§ 5. The Poisson kernel. Because the Poisson kernel does not decrease rapidly at infinity the problem of changing to and from the Poisson kernel must be dealt with separately.

$G^{p,\lambda*}_{\gamma,\beta}(f)(x_0)$ will denote the expression corresponding to $G^{p,\lambda}_{\gamma,\beta}$ but obtained by integrating over $\Omega^* = \{(x,t)\colon x \in \boldsymbol{R}^n, t > 0\}$ rather than over Ω. This corresponds to looking at untruncated cones rather than truncated ones.

THEOREM 7. *Let $\varphi \in \mathscr{S}$ and let $\mathscr{P}$ be the Poisson kernel. Also $1 \leqslant p \leqslant \infty$.*

(i) If $\int \varphi(x)\,dx \neq 0$ then $G^{p,\lambda*}_{\gamma,\beta}(\mathscr{P}) \leqslant A G^{p,\lambda*}_{\gamma,\beta}(\varphi)$ for $|\beta| > \gamma + n/p - 1$. In addition, if $\mathscr{G}^{p,\lambda*}_{\gamma,k}(\varphi)$ is finite ($k > \gamma$), then $\mathscr{G}^{p,\lambda_1 *}_{\gamma,k}(\mathscr{P}) < \infty$ for λ_1 large enough.

(ii) For all $\varphi \in \mathscr{S}$, $G^{p,\lambda}_{\gamma,\beta}(\varphi) \leqslant A G^{p,\lambda}_{\gamma,\beta}(\mathscr{P})$.

The constants A here depend on φ and $\mathscr{P}$ but not on f or x_0.

Proof. Part (i). We mention first how the second assertion follows from the first. Let $k > \gamma + n/p$. By Theorem 5, $\mathscr{G}^{p,\lambda*}_{\gamma,k}(\varphi) < \infty$. Part (i) of this theorem implies that $\mathscr{G}^{p,\lambda*}_{\gamma,k}(\mathscr{P}) < \infty$. Then the proof of Corollary 4 shows that $\mathscr{G}^{p,\lambda_1 *}_{\gamma,k}(\mathscr{P}) < \infty$ for λ_1 large enough.

Now for the first assertion in (i). We write

$$\mathscr{P}(x) \equiv A(1+|x|^2)^{-(n+1)/2} = \sum_{j=0}^{\infty} \Phi_j(x 2^{-j})$$

where Φ_j is of the form $\Phi_j(x) = \eta_1(x)\mathscr{P}(2^j x)$. Hence if $U_j(x,t) = f*(\Phi_j)_t(x)$ then

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}(f*\mathscr{P}_t)(x)\right| \leqslant \sum_{j=0}^{\infty} 2^{jn}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} U_j(x, 2^j t)\right|.$$

Put this into the definition of $G^{p,\lambda*}_{\gamma,\beta}$ and use Minkowski's inequality. This gives

$$G^{p,\lambda*}_{\gamma,\beta}(\mathscr{P}) \leqslant \sum_{j=0}^{\infty} 2^{jn}\left[\iint_{\Omega^*}\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda} t^{p(|\beta|-\gamma)}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} U_j(x,2^j t)\right|^p \frac{dx\,dt}{t^{n+1}}\right]^{1/p}.$$

By using a change of variable $s = 2^j t$ and $\left(\dfrac{t}{t+|x-x_0|}\right) \leqslant A\left(\dfrac{2^j t}{t2^j + |x-x_0|}\right)$ it follows that

$$G^{p,\lambda*}_{\gamma,\beta}(\mathscr{P}) \leqslant A \sum_{j=0}^{\infty} 2^{jn} 2^{-j(|\beta|-\gamma)} 2^{j\frac{n}{p}} G^{p,\lambda*}_{\gamma,\beta}(\Phi_j)$$

$$\leqslant A G^{p,\lambda*}_{\gamma,\beta}(\varphi) \sum_{j=0}^{\infty} \|\Phi_j\| 2^{j\left(n+\frac{n}{p}-|\beta|+\gamma\right)}.$$

By the proof of Theorem 3 of Chapter I of [3] we already know that $\|\Phi_j\| \leqslant A 2^{-j(n+1)}$. Hence

$$G^{p,\lambda*}_{\gamma,\beta}(\mathscr{P}) \leqslant A G^{p,\lambda*}_{\gamma,\beta}(\varphi) \sum_{j=0}^{\infty} 2^{j\left(\frac{n}{p}-|\beta|+\gamma-1\right)}.$$

Since this series is convergent for $|\beta| > \gamma + n/p - 1$, the proof of part (i) is complete.

Part (ii). We write $\varphi(x) = \int \mathscr{P}_s(x) g(s)\,ds$ where $g(s)$ is supported in $[1, 2]$ and

$$\int s^k g(s)\,ds = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, \dots, M. \end{cases}$$

If M is sufficiently large then the norm $\|\varphi\|$ in (4) will be finite. Thus the problem again reduces to changing from $\mathscr{P}$ to this φ. But we have $\varphi_t(x) = \int \mathscr{P}_{st}(x) g(s)\,ds$. So if we write $u(x,t) = f*\varphi_t(x)$ and $U(x,t) = f*\mathscr{P}_t(x)$ then

$$\left(\frac{\partial}{\partial x}\right)^{\beta} u(x,t) = \int \left(\frac{\partial}{\partial x}\right)^{\beta} U(x, st) g(s)\,ds.$$

Therefore

$$G^{p,\lambda}_{\gamma,\beta}(\varphi)$$

$$\leqslant \left[\iint_{\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda}\left\{t^{|\beta|-\gamma}\int\left|\left(\frac{\partial}{\partial x}\right)^{\beta} U(x,st)\right| |g(s)|\,ds\right\}^p \frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$

$$\leqslant \left[\iint_{\Omega}\left\{\int\left(\frac{st}{st+|x-x_0|}\right)^{p\lambda}(st)^{p(|\beta|-\gamma)}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} U(x,st)\right| |g(s)|\,ds\right\}^p \frac{dx\,dt}{t^{n+1}}\right]^{1/p}.$$

We have used here that $s \geqslant 1$. So by Minkowski's inequality,

$$G^{p,\lambda}_{\gamma,\beta}(\varphi) \leqslant \int\left[\iint_{\Omega}\left(\frac{st}{st+|x-x_0|}\right)^{p\lambda}(st)^{p(|\beta|-\gamma)}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} U(x,st)\right|^p \frac{dx\,dt}{t^{n+1}}\right] |g(s)|\,ds$$

$$= \left(\int s^n |g(s)|\,ds\right) G^{p,\lambda}_{\gamma,\beta}(\mathscr{P}) = A G^{p,\lambda}_{\gamma,\beta}(\mathscr{P}).$$

This completes the proof of Theorem 7.

In the applications to tempered nontangential boundedness we will change the mollifier to the Poisson kernel in order to take advantage of certain results about harmonic functions. To do this however we must be able to control $f*\varphi_t(x)$ over all of Ω^* rather than just Ω. We will now discuss how that will be done.

LEMMA 7. *Let $\psi(x)$ be a tempered C^∞ function such that $\psi(x) \equiv 0$ for all x in some open set W. Let $f \in \mathscr{S}'$. Then for all $x_0 \in W$ and for all $\gamma \in \boldsymbol{R}$, $\psi f \in A^p_\gamma(x_0)$.*

Proof. Fix x_0 and γ. Also take a particular $k > \gamma + n/p$. Suppose that $\varphi \in \mathscr{S}$ with support in $\{|x| \leqslant 1\}$. We will use this function as our mollifier. Since φ has compact support and $x_0 \in W$, there exists an $\varepsilon > 0$

such that

$$\left(\frac{\partial}{\partial x}\right)^{\beta}(\psi f)*\varphi_t(x) = 0$$

for all $\beta\in Z_0^n$ and all (x,t) in Ω^* such that $|x-x_0|<\varepsilon-t$. Also as we have noted before there exist M and N such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}(\psi f*\varphi_t)(x)\right| \leqslant \frac{A}{t^N}\left(1+\frac{|x-x_0|}{t}\right)^M$$

for all $|\beta|\leqslant k$ and $(x,t)\in\Omega^*$. Therefore

$$(42)\qquad G^{p,\lambda}_{\gamma,\beta}(\psi f)(x_0) \leqslant \Big[\iint\limits_{\substack{\Omega\\ |x-x_0|\geqslant\varepsilon-t}} \left(\frac{t}{t+|x-x_0|}\right)^{p(\lambda-M)} \left\{t^{k-\gamma}\frac{A}{t^N}\right\}^p \frac{dx\,dt}{t^{n+1}}\Big]^{1/p}$$
$$\leqslant A\Big[\int_0^1 \int\limits_{|x|>\varepsilon-t} \left(\frac{t}{t+|x|}\right)^{p(\lambda-M)}\frac{dx}{t^n}\, t^{(k-\gamma-N)p-1}\,dt\Big]^{1/p}.$$

But

$$(43)\qquad \int\limits_{|x|>\varepsilon-t}\left(\frac{t}{t+|x|}\right)^{p(\lambda-M)}\frac{dx}{t^n} = \int\limits_{|y|>\varepsilon/t-1}\frac{dy}{(1+|y|)^{p(\lambda-M)}}.$$

For all t the integral in (43) is bounded by

$$\int\limits_{R^n}\frac{dy}{(1+|y|)^{p(\lambda-M)}} = A \quad\text{ if }\quad \lambda > M+\frac{n}{p}.$$

But if $t\leqslant\varepsilon/2$ then (43) is bounded by

$$\int\limits_{|y|>\varepsilon/2t}\frac{dy}{|y|^{p(\lambda-M)}} = At^{p(\lambda-M)-n}.$$

Therefore there exists an A such that (43) is bounded by $At^{p(\lambda-M)-n}$ for all $t\,(\lambda>M+n/p)$. Putting this into (42) gives

$$G^{p,\lambda}_{\gamma,\beta}(\psi f)(x_0) \leqslant A\Big[\int_0^1 At^{p(\lambda-M)-n}t^{(|\beta|-\gamma-N)p-1}\,dt\Big]^{1/p}.$$

Since this last integral is finite for λ large enough, we have shown that $G^{p,\lambda}_{\gamma,\beta}(\psi f)(x_0)<\infty$ for λ sufficiently large. Since this is true for all $|\beta|=k$ then $\psi f\in A^{p,\lambda}_{\gamma}(x_0)$ for λ large enough. Thus we have proven the assertion of the theorem.

THEOREM 8. *Consider a fixed x_0 and a tempered C^∞ function χ such that $\chi\equiv 1$ in a neighborhood of x_0. Then*

$$f\in A^p_\gamma(x_0) \quad\textit{iff}\quad \chi f\in A^p_\gamma(x_0).$$

Proof. Since $f=\chi f+(1-\chi)f$ this is a direct consequence of Lemma 7 with $\psi=1-\chi$.

LEMMA 8. *Let E be a bounded set and let W be a bounded open set containing E. If $f\in A^p_\gamma(x_0)$ for every x_0 in E, then there exists a distribution g with compact support such that $f=g$ in W and $g\in A^{p*}_\gamma(x_0)$ where the asterisk means that integration is over Ω^* rather than Ω.*

Proof. Every tempered distribution can be written as $f=\left(\frac{\partial}{\partial x}\right)^{\delta}h$ where h is a slowly increasing continuous function. h can be integrated to produce a slowly increasing continuous function h_1 such that $h=\left(\frac{\partial}{\partial x_1}\right)^{M}h_1$, where M is some large number. Let χ be a C^∞ function of compact support such that $\chi=1$ in W. Define

$$(44)\qquad g=\left(\frac{\partial}{\partial x}\right)^{\delta}\left(\frac{\partial}{\partial x_1}\right)^{M}(\chi h_1).$$

We will now make M so large that $g\in A^{p*}_\gamma(x_0)$. By expanding (44) using the product rule it follows from Lemma 7 and Theorem 8 that if $f\in A^{p,\lambda}_\gamma(x_0)$ then so is g. Therefore we need only worry about the integral $G^{p,\lambda}_{\gamma,k}$ of g over the set $\{(x,t)\colon x\in R^n,\ t\geqslant 1\}$. So it suffices to prove that

$$(45)\qquad \int\limits_{R^n}\int_1^\infty\left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}\left\{t^{k-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^{\beta}g*\varphi_t\right|\right\}^p\frac{dx\,dt}{t^{n+1}}<\infty$$

for all $|\beta|=k$. From its definition we see that

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\beta}g*\varphi_t\right| = \frac{1}{t^{k+|\delta|+M}}\left|\chi h_1*\left[\left(\frac{\partial}{\partial x_1}\right)^{M}\left(\frac{\partial}{\partial x}\right)^{\delta}\varphi\right]_t\right|$$
$$\leqslant\frac{\|\chi h_1\|_\infty\left\|\left(\frac{\partial}{\partial x_1}\right)^{M}\left(\frac{\partial}{\partial x}\right)^{\delta}\varphi\right\|_1}{t^{k+|\delta|+M}}\leqslant\frac{A}{t^{k+M}}.$$

Putting this into (45) gives

$$(46)\qquad \int\limits_{R^n}\int_1^\infty\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda}\left\{t^{k-\gamma}\frac{A}{t^{k+M}}\right\}^p\frac{dx\,dt}{t^{n+1}}.$$

Notice that

$$\int\limits_{R^n}\left(\frac{t}{t+|x-x_0|}\right)^{p\lambda}\frac{dx}{t^n} = \int\limits_{R^n}\frac{1}{(1+|y|)^{\lambda p}}\,dy$$

is finite if $\lambda > n/p$. So if necessary we make λ large enough that this true. Now (46) is dominated by $A\int\limits_1^\infty t^{-p(\gamma+M)-1}\,dt$. Choose M so large that this integral is finite. Then the integral in (45) will be finite and so $g \in \Lambda_\gamma^{p,\lambda^*}(x_0)$.

§ 6. Inclusion relations. The space $\Lambda_\gamma^{p,\lambda}(x_0)$ should be thought of as the set of tempered distributions with derivatives of order up to and including γ. We will show now that as the order of differentiability γ increases the spaces $\Lambda_\gamma^{p,\lambda}(x_0)$ become smaller.

THEOREM 9. *If $\gamma_2 < \gamma_1$ are any real numbers and $1 \leqslant p \leqslant \infty$, $\lambda > 0$ then*

$$\Lambda_{\gamma_1}^{p,\lambda}(x_0) \subset \Lambda_{\gamma_2}^{p,\lambda}(x_0)$$

and the inclusion map is bounded. That is,

$$N_{\gamma_2}^{p,\lambda}(f)(x_0) \leqslant A N_{\gamma_1}^{p,\lambda}(f)(x_0) \quad \text{for all } f \in \Lambda_{\gamma_1}^{p,\lambda}(x_0)$$

where A is a constant independent of f and x_0.

Proof. In view of Theorem 5 we may take any $k > \gamma_1 + n/p$. Since $\gamma_2 < \gamma_1$ then $t^{k-\gamma_2} \leqslant t^{k-\gamma_1}$ for $0 < t < 1$. Therefore

$$G_{\gamma_2,k}^{p,\lambda}(f)(x_0) \leqslant G_{\gamma_1,k}^{p,\lambda}(f)(x_0).$$

Since the remainder terms are the same this implies that

$$N_{\gamma_2,k}^{p,\lambda}(f)(x_0) \leqslant N_{\gamma_1,k}^{p,\lambda}(f)(x_0).$$

Hence an application of Theorem 5 completes the proof.

THEOREM 10. *If $1 \leqslant q \leqslant p \leqslant \infty$ and $\gamma_1 > \gamma_2$ then $\Lambda_{\gamma_1}^{p}(x_0) \subset \Lambda_{\gamma_2}^{q}(x_0)$. If $\lambda_2 > \lambda_1 + n\left(\frac{1}{q} - \frac{1}{p}\right)$ and $k > \gamma_1 + n/p$ then for all $|\beta| = k$*

$$G_{\gamma_2,\beta}^{q,\lambda_2}(f)(x_0) \leqslant A G_{\gamma_1,\beta}^{p,\lambda_1}(f)(x_0)$$

where A is a constant independent of f and x_0.

Proof. By using the last theorem we may assume that $p \neq q$. Also let $p < \infty$. Write $a = \lambda_2 - \lambda_1$ and suppose that p' is the conjugate exponent of p/q; that is, $q/p + 1/p' = 1$. Observe that

$$G_{\gamma_2,\beta}^{q,\lambda_2}(f)(x_0) = \Bigg[\iint\limits_{\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{q\lambda_1} \times$$
$$\times \left\{t^{k-\gamma_1}\left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x,t)\right|\right\}^{q}\left(\frac{t}{t+|x-x_0|}\right)^{qa} t^{q(\gamma_1-\gamma_2)}\,\frac{dx\,dt}{t^{n+1}}\Bigg]^{1/q}.$$

We will apply Hölder's inequality to this integral using the exponents p/q and p'. This will give

$$(47)\qquad G_{\gamma_2,\beta}^{q,\lambda_2}(f)(x_0) \leqslant G_{\gamma_1,\beta}^{p,\lambda_1}(f)(x_0)\left[\iint\limits_{\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{qap'} t^{qp'(\gamma_1-\gamma_2)}\,\frac{dx\,dt}{t^{n+1}}\right]^{1/q}.$$

Note that

$$\int\limits_{R^n}\left(\frac{t}{t+|x-x_0|}\right)^{qap'}\frac{dx}{t^n} = \int\limits_{R^n}\frac{dx}{(1+|x|)^{qap'}}.$$

Also

$$qap' = \frac{pq}{(p-q)}(\lambda_2-\lambda_1) > n.$$

Therefore

$$\left[\iint\limits_{\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{qap'} t^{qp'(\gamma_1-\gamma_2)}\,\frac{dx\,dt}{t^{n+1}}\right]^{1/p'} \leqslant A\left[\int\limits_0^1 t^{qp'(\gamma_1-\gamma_2)-1}\,dt\right]^{1/p'} < \infty$$

since $qp'(\gamma_1-\gamma_2) > 0$. Hence (47) becomes

$$G_{\gamma_2,\beta}^{q,\lambda}(f)(x_0) \leqslant A G_{\gamma_1,\beta}^{p,\lambda}(f)(x_0).$$

If $p = \infty$ then the same argument works but with certain obvious modifications.

We finish this section with a result stating the relationship between tempered nontangential boundedness and the spaces $\Lambda_\gamma^{\infty,\lambda}(x_0)$. We will write

$$\mathscr{M}^\lambda(f)(x_0) = \sup_{(x,t)\in\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}|f*\varphi_t(x)|$$

and we define

$$\mathscr{B}^\lambda(x_0) = \{f \in \mathscr{S}' \colon \mathscr{M}^\lambda(f)(x_0) < \infty\}.$$

THEOREM 11. *For all $\gamma > 0$*

$$\Lambda_\gamma^{\infty,\lambda}(x_0) \subset \mathscr{B}^\lambda(x_0) \subset \Lambda_0^{\infty,\lambda}(x_0)$$

and

$$(48)\qquad A N_{0,k}^{\infty,\lambda}(f)(x_0) \leqslant \mathscr{M}^\lambda(f)(x_0) \leqslant N_{\gamma,k}^{\infty,\lambda}(f)(x_0)$$

where A is a positive constant not depending on f or x_0.

Proof. We will write $\mathscr{M}^\lambda(\varphi)$ to emphasize the mollifier. If we do this the first inequality in (48) becomes trivial because we can use the change of approximate identity theorem for tempered nontangential boundedness. Thus because

$$t^k\left(\frac{\partial}{\partial t}\right)^k (f*\varphi_t)(x) = f*(\mathscr{D}_k\varphi)_t(x)$$

we have

$$N^{\infty,\lambda}_{0,k}(\varphi) = \sup_{(x,t)\in\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}|f*(\mathcal{D}_k\varphi)_t(x)| + $$

$$+\sup_{(x,t)\in\Omega}\frac{1}{(1+|x-x_0|)^{\lambda}}|f*\varphi(x)|$$

$$\leqslant \mathcal{M}^{\lambda}(\mathcal{D}_k\varphi)+\mathcal{M}^{\lambda}(\varphi)\leqslant A\|\varphi\|\mathcal{M}^{\lambda}(\varphi).$$

Hence

$$N^{\infty,\lambda}_{0,k}(f)(x_0)\leqslant A\mathcal{M}^{\lambda}(f)(x_0).$$

To prove the second inequality it suffices by Theorem 9 to consider $0<\gamma<1$. But this means that we are free to take $k=1$. Since

$$f*\varphi_t(x) = f*\varphi(x) - \int_t^1\left(\frac{\partial}{\partial s}\right)(f*\varphi_s)(x)\,ds$$

it follows that

$$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}|f*\varphi_t(x)|$$

$$\leqslant\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}|f*\varphi(x)|+\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}\int_t^1\left|\left(\frac{\partial}{\partial s}\right)f*\varphi_s(x)\right|ds$$

$$\leqslant\frac{1}{(1+|x-x_0|)^{\lambda}}|f*\varphi(x)|+\int_t^1\left(\frac{s}{s+|x-x_0|}\right)^{\lambda}\left|\left(\frac{\partial}{\partial s}\right)f*\varphi_s(x)\right|ds$$

$$\leqslant R^{\infty,\lambda}_{0}(f)(x_0)+\int_t^1 G^{\infty,\lambda}_{\gamma,1}(f)(x_0)s^{-1+\gamma}\,ds\leqslant AN^{\infty,\lambda}_{\gamma,1}(f)(x_0).$$

Therefore

$$\mathcal{M}^{\lambda}(f)(x_0)\leqslant AN^{\infty,\lambda}_{\gamma,1}(f)(x_0).$$

§ 7. Pseudo-differential operators. In this section we examine the effect of certain pseudo-differential operators on the spaces $\Lambda^{p,\lambda}_{\gamma}(x_0)$.

Let M and m be real numbers. $a\in C^{\infty}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ will be called *a symbol of order m*, $a\in S^m_M$, if for all $\alpha,\beta\in\boldsymbol{Z}^n_0$ there exists a constant A such that

(49) $$|D^{\alpha}_{x}D^{\beta}_{\xi}a(x,\xi)|\leqslant A(1+|x|)^{M}(1+|\xi|)^{m}|\xi|^{-|\beta|}$$

for all $x\in\boldsymbol{R}^n$, $\xi\in\boldsymbol{R}^n$.

Let a be a symbol of order m in S^m_M and let φ be a C^{∞} function supported in $\{|x|\leqslant 1\}$ with $\int\varphi(x)\,dx\neq 0$. If $f\in\mathcal{S}'$ is sufficiently smooth that

$$\int(1+|\xi|)^{m}|\hat{f}(\xi)|\,d\xi<\infty,$$

then the pseudo-differential operator T corresponding to a is defined by

$$(Tf)(x) = (2\pi)^{-n}\int e^{ix\xi}a(x,\xi)\hat{f}(\xi)\,d\xi.$$

The main result of this section is the following:

THEOREM 12. *Let $1\leqslant p<\infty$. Suppose that a is a symbol in S^m_M. If a is independent of x and $f\in\mathcal{S}'$, then there exists a constant A such that*

$$\mathcal{G}^{p,\lambda}_{\gamma,4h}(Tf)(x_0)\leqslant A\mathcal{G}^{p,\lambda}_{\gamma+m,2h}(f)(x_0).$$

For general $a\in S^m_M$, if $f\in\mathcal{S}'$ then

$$N^{p,\lambda}_{\gamma}\big(T(-\Delta)^{h}f\big)(x_0)\leqslant AN^{p,\lambda+M}_{\gamma+m+2h}(f)(x_0)$$

where h is the smallest nonnegative integer greater than $\lambda+n$. The constant A is independent of f and x_0.

There are a number of special cases that are of interest.

EXAMPLE 1. If $a(x,\xi)=\psi(x)$ and $|D^{\alpha}\psi(x)|\leqslant A(1+|x|)^{M}$ for all $\alpha\in\boldsymbol{Z}^n_0$, then $Tf(x)=\psi(x)f(x)$.

EXAMPLE 2. If $a(x,\xi)=m(\xi)$ and $|D^{\beta}m(\xi)|\leqslant A|\xi|^{-|\beta|}$ for all $\beta\in\boldsymbol{Z}^n_0$, then the multiplier transformation determined by $\widehat{Tf}(\xi)=m(\xi)\hat{f}(\xi)$ is bounded from $\Lambda^{p,\lambda}_{\gamma}(x_0)$ to itself.

EXAMPLE 3. If $a(x,\xi)=\xi^{\beta}$, $\beta\in\boldsymbol{Z}^n_0$, then $(Tf)(x)=i^{-|\beta|}D^{\beta}f(x)$. Thus this differentiation is bounded from $\Lambda^{p,\lambda}_{\gamma}(x_0)$ to $\Lambda^{p,\lambda}_{\gamma-|\beta|}(x_0)$.

EXAMPLE 4. If $m\in\boldsymbol{R}$ and $a(x,\xi)=(1+|\xi|^2)^{m/2}$, then we see that $(I-\Delta)^{m/2}$ is a Banach space isomorphism from $\Lambda^{p,\lambda}_{\gamma}(x_0)$ to $\Lambda^{p,\lambda}_{\gamma-m/2}(x_0)$, for $1\leqslant p<\infty$. This will be proven in Theorem 14.

In this section we will prove Theorem 12 only for the case $m\geqslant 0$. In § 8 (Theorem 14) we will show that $(I-\Delta)^{m/2}$ is bounded from $\Lambda^{p,\lambda}_{\gamma}(x_0)$ to $\Lambda^{p,\lambda}_{\gamma-m/2}(x_0)$ for all $m\in\boldsymbol{R}$. Suppose that a is a symbol in S^m_M. Define

$$b(x,\xi)=a(x,\xi)(1+|\xi|^2)^{(\mu-m)/2}$$

and let S be the pseudo-differential operator corresponding to b. Then b is a symbol in S^{μ}_{M} and $S(I-\Delta)^{(m-\mu)/2}=T$. Therefore if S is bounded then T is bounded. This reduces Theorem 12 to the case $m\geqslant 0$.

If we write $U(x,t)=\big(T(-\Delta)^{h}f\big)*\varphi_t(x)$, then

(50) $$U(x,t)=\iint e^{i(x-y)\xi}a(x-y,\xi)|\xi|^{2h}\hat{f}(\xi)\,d\xi\,\varphi_t(y)\,dy$$

$$=\int e^{ix\xi}\hat{f}(\xi)|\xi|^{2h}\left\{\int e^{-ity\xi}a(x-ty,\xi)\varphi(y)\,dy\right\}d\xi$$

$$=\int e^{ix\xi}\hat{f}(\xi)|\xi|^{2h}J(x,t,t\xi)\,d\xi$$

where J is defined by

(51) $$J(x,t,\xi)=e^{-iy\xi}a(x-ty,\xi t^{-1})\varphi(y)\,dy.$$

Our first step is to get a favorable estimate of $J(x,t,\xi)$.

LEMMA 9. *Let φ be a C^∞ function of compact support and let a be a symbol in S^m_M with $m \geqslant 0$. Then for all $N \in \boldsymbol{R}$, α, $\delta \in \boldsymbol{Z}^n_0$, there exists a constant A such that*

$$|\xi|^{\delta} |D^\delta_\xi D^\alpha_x J(x, t, \xi)| \leqslant A t^{-m} (1+|x|)^M (1+|\xi|)^{-N}$$

for all x, $\xi \in \boldsymbol{R}^n$, $0 < t \leqslant 1$. A is independent of x, t, ξ.

Proof. We may assume that $|\delta| < N$. To prove the lemma we estimate the expression $\xi^\beta D^\delta_\xi D^\alpha_x J(x, t, \xi)$. Consider $|\delta| \leqslant |\beta|$. Then

$$\xi^\beta D^\delta_\xi D^\alpha_x J(x, t, \xi)$$
$$= \xi^\beta \sum A \int (-iy)^{\delta_1} e^{-iy\xi} t^{-|\delta_2|} [D^{\delta_2}_\xi D^\alpha_x a](x - ty, \xi/t) \varphi(y)\, dy$$

where the sum is over $\delta_1 + \delta_2 = \delta$. Let ν be such that $|\nu| = |\delta_2|$ and $\nu \leqslant \beta$. Integrate by parts using $D^{\beta-\nu}_y$. Notice first that by differentiating and using (49) it is easy to show that

$$|(\xi/t)^\nu D^{\beta-\nu}_y \{y^{\delta_1} [D^{\delta_2}_\xi D^\alpha_x a](x - ty, \xi/t)(y)\}|$$
$$\leqslant A \sum_{0 \leqslant \gamma \leqslant \beta - \nu} |D^\gamma_y (y^{\delta_1} \varphi)| (1 + |x - ty|)^M (1 + |\xi|/t)^m$$
$$\leqslant A t^{-m} (1+|x|)^M (1+|\xi|)^m \psi(y)$$

where ψ is a C^∞ function of compact support. Therefore the integration by parts gives

$$|\xi^\beta D^\delta_\xi D^\alpha_x J(x, t, \xi)| \leqslant \sum_{\delta_1+\delta_2=\delta} \int |e^{-iy\xi}| A t^{-m} (1+|x|)^M (1+|\xi|)^m \psi(y)\, dy$$
$$\leqslant A t^{-m} (1+|x|)^M (1+|\xi|)^m.$$

These estimates for β such that $|\beta| = |\delta|$ or $|\beta| = |\delta| + N$ combine to complete the proof of the lemma.

Proof of Theorem 12. Since $f \in \mathscr{S}'$, Lemma 9 shows that for h sufficiently large (50) can be used to define $U(x, t)$. We will estimate $N^{p, \lambda+M}_\gamma (T(-\Delta)^h f)(x_0)$ by using (50) and by breaking $J(x, t, \xi)$ into pieces.

Let $\{\eta_j\}$ be a C^∞_0 partition of unity on $\boldsymbol{R}^n$ such that $\eta_j \geqslant 0$ for all j, η_0 is supported in $N_0 = \{|x| < 1\}$, η_j is supported in $N_j = \{x \in \boldsymbol{R}^n: 2^{j-2} < |x| < 2^j\}$, and

$$|D^\beta \eta_j(x)| \leqslant A_\beta 2^{-j|\beta|} \quad \text{for all } \beta \in \boldsymbol{Z}^n_0,\ j \geqslant 1.$$

Define ψ_j for $j = 0, 1, 2, \ldots$ so that

$$\hat\psi_j(\xi) = \eta_j(\xi) J(x, t, \xi)/\hat\varphi(\xi 2^{-j})$$

where $\hat\varphi$ is chosen so that $\{\hat\varphi(\xi): |\xi| \leqslant 1\}$ is bounded away from zero. Note that ψ_j depends on x and t and will be written $\hat\psi_j(\xi)$ for $\hat\psi_j(x, t, \xi)$.

From equation (50) it follows that

$$D^\beta_x U(x, t) = \sum\sum A \int e^{ix\xi} (i\xi)^{\beta_1} |\xi|^{2h} \hat f(\xi) D^{\beta_2}_x J(x, t, t\xi) \eta_j(t\xi)\, d\xi$$
$$= \sum\sum A \int e^{ix\xi} \xi^{\beta_1} |\xi|^{2h} \hat f(\xi) [\widehat{D^{\beta_2}_x \psi_j}](t\xi) \hat\varphi(t\xi 2^{-j})\, d\xi$$

where the sums are over $\beta_1 + \beta_2 = \beta$ and $0 \leqslant j < \infty$. If $u(x, t) = f * \varphi_t(x)$, then by Plancherel's theorem $D^\beta_x U(x, t)$ is a linear combination of integrals of the form

$$(52) \qquad t^{-2h-n} \int D^{\beta_1}_x u(x - y, t2^{-j}) (D_* \psi_j)(x, t, y/t)\, dy$$

where $D_* = (-\Delta_y)^h D^{\beta_2}_x$.

Note that $\psi_j(y) = \psi_j(x, t, y)$ and if $a(x, \xi)$ is independent of x then so is ψ_j. Thus the only nonzero term of the form (52) is the one with $\beta_1 = \beta$ and $\beta_2 = 0$. Instead of (52), $D^\beta_x U(x, t)$ is equal to

$$(52') \qquad A t^{-2h-n} \int D^{\beta_1}_x u(x - y, t2^{-j}) (D^{\beta_2}_y \psi_j)(y/t)\, dy$$

where $|\beta_1| = |\beta_2| = \frac{1}{2}|\beta| = 2h$. In this case let $D_* = D^\beta_y \psi_j$. In fact, equation (52′) is how $D^\beta_x(Tf * \varphi_t)(x)$ is to be defined for $|\beta| = k$ sufficiently large. If $f \in \mathscr{S}'$ then $D^{\beta_1}_x u$ is a tempered C^∞ function and we will see shortly that if $|\beta_2| = \frac{1}{2}k$ is large enough then the integral (52′) is absolutely convergent.

The inequalities given in Theorem 12 will be proven simultaneously. If a is independent of x define $S = T$. This will be referred to as case (i). Otherwise define $S = T(-\Delta)^h$. This is case (ii). In both cases $U(x, t) = Sf * \varphi_t(x)$.

Observe that $\hat\psi_j$ is supported in N_j. From Lemma 9, it follows that

$$\left| D^\delta_\xi \{|\xi|^{2h} D^{\beta_2}_x \psi_j(\xi)\} \right| \leqslant A t^{-m} (1+|x|)^M (1+|\xi|)^{-N} 2^{-j(N-n)}$$

for all $|\delta| \leqslant h$. Therefore since $\|\hat g\|_\infty \leqslant \|g\|_1$ for all $g \in L^1$,

$$|y^\delta (-\Delta_y)^h D^{\beta_2}_x \psi_j(y)| \leqslant A t^{-m} (1+|x|)^M 2^{-j(N-n)}$$

and

$$|D_* \psi_j(y)| \leqslant A t^{-m} (1+|x|)^M 2^{-j(N-n)} (1+|y|)^{-h}.$$

The same estimate holds in case (i) if k is sufficiently large. In particular, if $|\beta| = k > 4h$ then the fact that $\mathscr{G}^{p,\lambda}_{\gamma+m,k}(f)(x_0)$ is finite will guarantee that the integral in (52′) is absolutely convergent.

This simplifies (52) or (52′) to

$$(53) \qquad |D^\beta_x U(x, t)| \leqslant A^{-2h} \sum_{\beta_1} \sum_j \int |D^{\beta_1}_x u(x - ty, t2^{-j})| |D_* \psi_j(y)|\, dy$$
$$\leqslant A_{x,t} \sum_{\beta_1} \sum_j 2^{-j(N-n)} \int |D^{\beta_1}_x u(x - ty, t2^{-j})| (1+|y|)^{-h}\, dy$$

where $A_{x,t} = At^{-2h-m}(1+|x|)^M$. In case (i) there is only one β_1. In case (ii) the sum is over all $\beta_1 \leqslant \beta$. Henceforth this sum will be denoted by $\sum^1$.

Put this last estimate into the definition for $G_{\gamma,\beta}^{p,\lambda+M}(Tf)(x_0)$. Using Minkowski's inequality and the fact that

$$\left(\frac{t}{t+|x-x_0|}\right)^{\lambda+M}(1+|x|)^M \leqslant (1+|x_0|)^M\left(\frac{t}{t+|x-x_0|}\right)^{\lambda} = A\left(\frac{t}{t+|x-x_0|}\right)^{\lambda}$$

gives the result that

$$(54)\qquad G_{\gamma,\beta}^{p,\lambda+M}(Sf)(x_0) \leqslant A\sum^1\sum_j 2^{-j(N-n)}\int\Big[\iint_{\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{\lambda p}\times$$
$$\times\{t^{k-2h-\gamma-m}|D_z^{\beta_1}u(z,s)|\}^p t^{-n-1}\,dx\,dt\Big]^{1/p}(1+|y|)^{-h}\,dy$$

where $z = x-ty$ and $s = t2^{-j}$. Notice that

$$\left(1+\frac{|z-x_0|}{s}\right) = \left(1+\frac{|x-x_0-ty|}{s}\right)A \leqslant 2^{-j}\left(1+\frac{|x-x_0|}{t}\right)(1+|y|).$$

Thus the change of variables $z = x-ty$, $s = t2^{-j}$ transforms (54) into

$$G_{\gamma,\beta}^{p,\lambda+M}(Sf)(x_0) \leqslant A\sum^1\sum_j 2^{-j\nu}\int\Big[\iint_{\Omega_j}\left(\frac{s}{s+|z-x_0|}\right)^{\lambda p}s^{\gamma_1}\times$$
$$\times|D_z^{\beta_1}u(z,s)|^p\,dz\,ds\Big]^{1/p}(1+|y|)^{\lambda-h}\,dy$$

where $\nu = N-\lambda+2h-k+\gamma+m$, $\gamma_1 = (k-2h-\gamma-m)p-n-1$ and $\Omega_j = \{(x,t)\colon x\in \boldsymbol{R}^n,\ 0<t<2^{-j}\}$. Choose N so large that $\nu \geqslant 1$. If $\gamma_2 = \gamma+m+|\beta_1|+2h-k$, then since $h > n+\lambda$, it follows that

$$(55)\qquad G_{\gamma,\beta}^{p,\lambda+M}(Sf)(x_0) \leqslant A\sum^1\sum_j 2^{-j}G_{\gamma_2,\beta_1}^{p,\lambda}(f)(x_0)$$
$$= A\sum^1 G_{\gamma_2,\beta_1}^{p,\lambda}(f)(x_0).$$

In case (i), $k = 4h$ and $\gamma_2 = \gamma+m$ and (55) for all $|\beta| = 4h$ shows that

$$\mathscr{G}_{\gamma,4h}^{p,\lambda+M}(Tf)(x_0) \leqslant A\mathscr{G}_{\gamma+m,2h}^{p,\lambda}(f)(x_0).$$

In case (ii), if $k > \gamma+m+2h+n/p$, then by Lemma 4, (55) proves that

$$(56)\qquad \mathscr{G}_{\gamma,k}^{p,\lambda+M}\big(T(-\Delta)^h f\big)(x_0) \leqslant A\sum^1 N_{\gamma_2}^{p,\lambda}(f)(x_0) \leqslant AN_{\gamma+m+2h}^{p,\lambda}(f)(x_0).$$

Consider next the R terms for case (ii). Estimate (53) and the definition of $R(Sf)(x_0)$ give

$$(57)\qquad R_\beta^{p,\lambda+M}(Sf)(x_0) \leqslant A\sum^1\sum_j 2^{-j(N-n)}\int\Big\{\int(1+|x-x_0|)^{-p(\lambda+M)}\times$$
$$\times|D_x^{\beta_1}u(x-y,2^{-j})|^p(1+|x|)^{Mp}\,dx\Big\}^{1/p}(1+|y|)^{-h}\,dy$$
$$\leqslant A\sum^1\sum_j 2^{-j(N-n)}\Big\{\int(1+|x-x_0|)^{-\lambda p}|D_x^{\beta_1}u(x,2^{-j})|^p\,dx\Big\}^{1/p}.$$

But by Corollary 5 of Section 3,

$$\Big\{\int(1+|x-x_0|)^{-\lambda p}|D_x^{\beta_1}u(x,2^{-j})|^p\,dx\Big\}^{1/p}$$
$$\leqslant\Big\{\int\left(\frac{2^{-j}}{2^{-j}+|x-x_0|}\right)^{\lambda p}|D_x^{\beta_1}u(x,2^{-j})|^p\,dx\Big\}^{1/p}$$
$$\leqslant A2^{-j(\gamma_0-b)}\{\mathscr{G}_{\gamma_1,b+1}^{p,\lambda}(f)(x_0)+R_{\beta_1}^{p,\lambda}(f)(x_0)\}$$
$$\leqslant A2^{-j(\gamma_0-b)}N_{\gamma_1}^{p,\lambda}(f)(x_0)$$

where $\gamma_0 < \gamma_1$, $\gamma_0 \leqslant b = |\beta_1|$, $\gamma_1 = \gamma+m+2h$. Therefore for N sufficiently large

$$(58)\qquad R_\beta^{p,\lambda+M}(Sf)(x_0) \leqslant A\sum^1\Big\{\sum_j 2^{-j}\Big\}N_{\gamma+m+2h}^{p,\lambda}(f)(x_0) = AN_{\gamma+m+2h}^{p,\lambda}(f)(x_0).$$

This completes the proof of Theorem 12.

If a is independent of x, we have shown that it is possible to dominate the higher order terms of $N(Tf)$. More precisely,

$$\mathscr{G}_{\gamma,4h}^{p,\lambda}(Tf)(x_0)+\sum R_\delta^{p,\lambda}(Tf)(x_0) \leqslant AN_{\gamma+m}^{p,\lambda}(f)(x_0)$$

where the sum is over $2h \leqslant |\delta| < 4h$.

Lemma 10. *The tempered C^∞ functions are dense in $\Lambda_\gamma^{p,\lambda}(x_0)$ for $1 \leqslant p < \infty$.*

Proof. A C^∞ function f is tempered if for all $\alpha\in \boldsymbol{Z}_0^n$ there exist A and N such that

$$|D^\alpha f(x)| \leqslant A(1+|x|)^N.$$

Let $f\in\Lambda_\gamma^{p,\lambda}(x_0)$ and let $\varphi\in\mathscr{S}$ such that $\hat\varphi(\xi) = 1$ for $|\xi|\leqslant 1$. Consider $a^\delta(x,\xi) = 1-\hat\varphi(\delta\xi)$. a^δ is a symbol in S_0^0. If we denote the corresponding pseudo-differential operator by T^δ, then $T^\delta f = f-f*\varphi_\delta$.

We will write

$$Q(\xi) = \begin{cases}0 & \text{if } |\xi|\leqslant 1,\\ 1 & \text{if } |\xi|>1.\end{cases}$$

If $J(x,t,\xi)$ is defined as in (51), then

$$J(x,t,\xi) = a^\delta(0,\xi/t)\hat\varphi(\xi) = \hat\varphi(\xi)-\hat\varphi(\delta\xi/t)\hat\varphi(\xi).$$

Instead of Lemma 9 we can show that for every $N \in \boldsymbol{R}$, $\alpha \in Z_0^n$ there exists a constant A such that

$$|\xi|_{\Delta}^{\alpha}|D_\xi^\alpha J| \leqslant AQ(\delta\xi/t)(1+|\xi|)^{-N}.$$

If $|\nu| = 2h$, then

$$|D_\xi^\alpha\{\xi^\nu\hat{\psi}_j(\xi)\}| \leqslant AQ(\delta\xi/t)(1+|\xi|)^{-N}2^{-j(N-n)}$$

and

$$|y^\alpha D_y^\nu \psi_j(y)| \leqslant A2^{-j(N-n)}\int\limits_{|\xi|>t/\delta}(1+|\xi|)^{-N}d\xi.$$

If this integral is called $E(t/\delta)$, then the estimate of $D_x^\beta U(x,t) = D_x^\beta T^\delta f*\varphi_t(x)$ is the same as (53) except that $A_{x,t}$ now equals $At^{-2h}E(t/\delta)$. The proof then is the same as that of Theorem 12, leading to

$$G_{\gamma,\beta}^{p,\lambda}(T^\delta f)(x_0) \leqslant A\sum_j 2^{-j}\Big[\iint\limits_{\Omega_j}\Big(\frac{s}{s+|z-x_0|}\Big)^{p\lambda}s^{\gamma_1}E(2^js/\delta)|D_z^{\beta_1}u(z,s)|^p dz ds\Big]^{1/p}.$$

Since this series converges absolutely and each term approaches zero as $\delta\to 0$, we have

$$G_{\gamma,\beta}^{p,\lambda}(T^\delta f)(x_0)\to 0 \quad \text{as} \quad \delta\to 0.$$

Let $F \in \mathscr{S}'$ be such that $D^{\beta_1}F = f$ and $|\beta_1| = 2h$. Then

$$R_\beta^{p,\lambda}(T^\delta f) = R_{\beta+\beta_1}^{p,\lambda}(T^\delta D^\beta f).$$

As in the case (ii) of Theorem 12

$$R_{\beta+\beta_1}^{p,\lambda}(T^\delta D^\beta F)(x_0) \leqslant AN_{\gamma+2h}^{p,\lambda}(F)(x_0)$$

and as in the case of the G function $R_{\beta+\beta_1}^{p,\lambda}(T^\delta D^\beta F)(x_0)\to 0$ as $\delta\to 0$. Therefore

$$N_\gamma^{p,\lambda}(T^\delta f)(x_0)\to 0 \quad \text{as} \quad \delta\to 0.$$

Since $T^\delta f = f - f*\varphi_\delta$, we have shown that any $f \in \Lambda_\gamma^{p,\lambda}(x_0)$ can be approximated arbitrarily closely by a tempered C^∞ function $f*\varphi_\delta$.

§ 8. Differentiation and powers of $(I-\Delta)$. In this section we study the effect of differentiation on the spaces $\Lambda_\gamma^p(x_0)$. It is here that we see that the parameter γ is indeed a measure of differentiability.

THEOREM 13. *If $f \in \Lambda_\gamma^{p,\lambda}(x_0)$, then $\frac{\partial f}{\partial x_i} \in \Lambda_{\gamma-1}^{p,\lambda}(x_0)$ and*

$$N_{\gamma-1}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) \leqslant AN_\gamma^{p,\lambda}(f)(x_0). \tag{60}$$

Conversely, if $\frac{\partial f}{\partial x_i} \in \Lambda_{\gamma-1}^p(x_0)$ for all $i = 1,\ldots,n$ then $f \in \Lambda_\gamma^p(x_0)$. In fact we have

$$\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) \leqslant A\sum_{i=1}^n \mathscr{G}_{\gamma-1,k}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0). \tag{61}$$

If λ is big enough that $R_0^{p,\lambda}(f)(x_0) < \infty$, then

$$N_\gamma^{p,\lambda}(f)(x_0) \leqslant A\sum_{i=1}^n N_{\gamma-1}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) + R_0^{p,\lambda}(f)(x_0)$$

where A is independent of f and x_0.

Proof. Consider $\beta \in Z_0^n$ such that $|\beta| = k-1$. Let $\beta^i = (\beta_1,\ldots,\beta_i^{i+1},\ldots\ldots,\beta_n)$. Then

$$|\beta^i| = k \quad \text{and} \quad \Big(\frac{\partial}{\partial x}\Big)^\beta\Big(\frac{\partial}{\partial x_i}\Big) = \Big(\frac{\partial}{\partial x}\Big)^{\beta^i}.$$

Hence

$$t^{(k-1)-(\gamma-1)}\Big(\frac{\partial}{\partial x}\Big)^\beta\Big[\Big(\frac{\partial f}{\partial x_i}\Big)*\varphi_t\Big] = t^{k-\gamma}\Big(\frac{\partial}{\partial x}\Big)^{\beta^i}[f*\varphi_t].$$

Putting this into the definition for $G_{\gamma,\beta}^{p,\lambda}$ results in

$$G_{\gamma-1,\beta}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) = G_{\gamma,\beta^i}^{p,\lambda}(f)(x_0). \tag{62}$$

Notice that if $(\beta_1)^i \neq (\beta_2)^i$ then $\beta_1 \neq \beta_2$. Thus by combining the estimates for all $|\beta| = k-1$ we get

$$\mathscr{G}_{\gamma-1,k-1}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) = \Big[\sum_{|\beta|=k-1}\Big\{G_{\gamma-1,\beta}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0)\Big\}^p\Big]^{1/p} \tag{63}$$

$$\leqslant\Big[\sum_{|\beta|=k}\Big\{G_{\gamma,\beta}^{p,\lambda}(f)(x_0)\Big\}^p\Big]^{1/p} = \mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0).$$

The terms $R_\beta^{p,\lambda}$ satisfy a similar equation

$$R_\beta^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) \leqslant R_{\beta^i}^{p,\lambda}(f)(x_0).$$

Therefore

$$\sum_{|\beta|<k-1}R_\beta^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) = \sum_{|\beta|<k-1}R_{\beta^i}^{p,\lambda}(f)(x_0) \leqslant \sum_{|\beta|<k}R_\beta^{p,\lambda}(f)(x_0). \tag{64}$$

Hence combining (63) and (64) completes the proof of (64):

$$N_{\gamma-1}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) = \mathscr{G}_{\gamma-1,k-1}^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) + \sum_{|\beta|<k-1}R_\beta^{p,\lambda}\Big(\frac{\partial f}{\partial x_i}\Big)(x_0) \leqslant N_\gamma^{p,\lambda}(f)(x_0).$$

The converse is proven using the same ideas. If $|\beta| = k$ then there exist δ and i such that $\delta^i = \beta$ and $|\delta| = k-1$. So as in (62) we get

$$G_{\gamma,\beta}^{p,\lambda}(f)(x_0) = G_{\gamma-1,\delta}^{p,\lambda}\left(\frac{\partial f}{\partial x_i}\right)(x_0) \leqslant \sum_{i=1}^{n} \mathcal{G}_{\gamma-1,k-1}^{p,\lambda}\left(\frac{\partial f}{\partial x_i}\right)(x_0). \tag{65}$$

Summing over $|\beta| = k$ completes the proof of (61). Similarly if $\beta > k$, $\beta \neq 0$, there exist δ and i such that $\delta^i = \beta$. Consequently,

$$R_\beta^{p,\lambda}(f)(x_0) = R_\delta^{p,\lambda}\left(\frac{\partial f}{\partial x_i}\right)(x_0) \leqslant \sum_{i=1}^{n} \sum_{|\delta|<k-1} R_\delta^{p,\lambda}\left(\frac{\partial f}{\partial x_i}\right)(x_0). \tag{66}$$

Combining (65) and (66) gives

$$N_\gamma^{p,\lambda}(f)(x_0) \leqslant A \sum_{i=1}^{n} N_\gamma^{p,\lambda}\left(\frac{\partial f}{\partial x_i}\right)(x_0) + R_0^{p,\lambda}(f)(x_0).$$

This completes the proof of Theorem 13.

LEMMA 11. *Let $1 \leqslant p < \infty$ and let h be the smallest nonnegative integer greater than $n+\lambda$. Then*

$$N_\gamma^{p,\lambda}(f)(x_0) \leqslant A N_{\gamma-2}^{p,\lambda}(\Delta f)(x_0) + \sum_{|\delta|<2+2h} R_\delta^{p,\lambda}(f)(x_0). \tag{67}$$

Proof. It follows from Theorem 13 that

$$N_\gamma^{p,\lambda}(f)(x_0) \leqslant A\left\{\sum_{|\alpha|=2} N^{p,\lambda}\left(\left(\frac{\partial}{\partial x}\right)^\alpha f\right)(x_0) + \sum_{|\delta|<2} R_\delta^{p,\lambda}(f)(x_0)\right\}.$$

Let $\mathcal{R}_i$ denote the ith Riesz transform and let $\mathcal{R}^\alpha = \mathcal{R}_1^{\alpha_1}\ldots\mathcal{R}_n^{\alpha_n}$ for $\alpha \in \mathbf{Z}_0^n$. Then $\mathcal{R}^\alpha(-\Delta) = (\partial/\partial x)^\alpha$. By Theorem 12, the higher order terms in $N_{\gamma-2}^{p,\lambda}\big((\partial/\partial x)^\alpha f\big)(x_0)$ are dominated by $AN_{\gamma-\alpha}^{p,\lambda}(\Delta f)(x_0)$. Therefore

$$N_{\gamma-\alpha}^{p,\lambda}\left(\left(\frac{\partial}{\partial x}\right)^\alpha f\right)(x_0) \leqslant AN_{\gamma-\alpha}^{p,\lambda}(\Delta f)(x_0) + \sum_{|\beta|\leqslant 2h} R_\beta^{p,\lambda}\left(\left(\frac{\partial}{\partial x}\right)^\alpha f\right)(x_0).$$

These two estimates yield (67).

THEOREM 14. *Let $1 \leqslant p < \infty$ and $\alpha \in \mathbf{R}$. Then $(I-\Delta)^{\alpha/2}$ is a Banach space isomorphism from $\Lambda_\gamma^{p,\lambda}(x_0)$ to $\Lambda_{\gamma-\alpha}^{p,\lambda}(x_0)$. There exist constants A_1 and A_2 such that*

$$A_1 N_\gamma^{p,\lambda}(f)(x_0) \leqslant N_{\gamma-\alpha}^{p,\lambda}\big((I-\Delta)^{\alpha/2} f\big)(x_0) \leqslant A_2 N_\gamma^{p,\lambda}(f)(x_0)$$

for all $f \in \Lambda_\gamma^{p,\lambda}(x_0)$.

Proof. Consider $(I-\Delta)^{\alpha/2}$ where $\alpha \geqslant 0$. Theorem 12 shows that we can dominate all the high order terms in $N_{\gamma-\alpha}^{p,\lambda}\big((I-\Delta)^{\alpha/2} f\big)(x_0)$. In particular,

$$N_{\gamma-\alpha}^{p,\lambda}\big((I-\Delta)^{\alpha/2} f\big)(x_0) \leqslant AN_\gamma^{p,\lambda}(f)(x_0) + \sum_{|\beta|\leqslant 2h} R_\beta^{p,\lambda}\big((I-\Delta)^{\alpha/2} f\big)(x_0).$$

But $(I-\Delta)^{\alpha/2}\varphi \in \mathcal{S}$ if $\varphi \in \mathcal{S}$. Then by a change of mollifier φ we see that these lower order R terms are also bounded by $AN_\gamma^{p,\lambda}(f)(x_0)$. Hence $(I-\Delta)^{\alpha/2}$ is a bounded operator for $\alpha \geqslant 0$. To show that $(I-\Delta)^{-\alpha/2}$ is bounded for $\alpha > 0$ it suffices to consider the operator $(I-\Delta)^{-1}$. Let $F = (I-\Delta)^{-1} f$. Then, by Lemma 11,

$$N_\gamma^{p,\lambda}(F)(x_0) \leqslant AN_{\gamma-2}^{p,\lambda}(\Delta F)(x_0) + \sum_{|\delta|<2+2h} R_\delta^{p,\lambda}(F)(x_0).$$

But

$$\begin{aligned} N_{\gamma-2}^{p,\lambda}(\Delta F)(x_0) &\leqslant N_{\gamma-2}^{p,\lambda}\big((I-\Delta)F\big)(x_0) + N_{\gamma-2}^{p,\lambda}(F)(x_0) \\ &= N_{\gamma-2}^{p,\lambda}(f)(x_0) + N_{\gamma-2}^{p,\lambda}(F)(x_0) \end{aligned}$$

and

$$\sum_{|\delta|<2+2h} R_\delta^{p,\lambda}(F)(x_0) \leqslant AN_{\gamma-2}^{p,\lambda}(F)(x_0).$$

Therefore,

$$N_\gamma^{p,\lambda}(F)(x_0) \leqslant AN_{\gamma-2}^{p,\lambda}(f)(x_0) + N_{\gamma-2}^{p,\lambda}(F)(x_0).$$

Thus in order to show that $(I-\Delta)^{-1}$ is bounded we need only show that

$$N_{\gamma-2}^{p,\lambda}(F)(x_0) \leqslant AN_{\gamma-2}^{p,\lambda}(f)(x_0).$$

Since $(I-\Delta)^{-1}$ is a pseudo-differential operator with symbol in S_0^0, Theorem 12 takes care of the higher order terms. The lower order R terms are covered by a change of approximate identity from φ to $(I-\Delta)_{\mathcal{S}}^{-1}\varphi$. This completes the proof of Theorem 14.

§ 9. The Lipschitz spaces $\Lambda_\gamma^\infty(x_0)$. We finish with the observation that the spaces $\Lambda_\gamma^\infty(x_0)$, $0 < \gamma < 1$, can be defined by a modulus of continuity. We will see that this modulus of continuity is closely related to the notion of tempered nontangential convergence.

THEOREM 15. *Let $0 < \gamma < 1$. $f \in \Lambda_\gamma^\infty(x_0)$ if and only if for some $\lambda > 0$, $A > 0$*

$$\sup_{(x_1,t_1),(x_2,t_2)\in\Gamma_a^h(x_0)} |u(x_1,t_1) - u(x_2,t_2)| \leqslant Ah^\gamma(1+a)^\lambda \tag{68}$$

for all $0 < h < 1$, $a > 0$.

Proof. First suppose that $f \in \Lambda_\gamma^\infty(x_0)$. Consider (x_1, t) and (x_2, t) in $\Gamma_a^h(x_0)$. For some c between x_1 and x_2

$$|u(x_1, t) - u(x_2, t)| \leqslant |x_1 - x_2|\,|\nabla u(c, t)|.$$

Since $f \in \Lambda_\gamma^\infty(x_0)$ then

$$\left(\frac{t}{t+|x_0-c|}\right)^\lambda t^{1-\gamma}|\nabla u(c,t)| \leqslant A.$$

So

$$|\nabla u(c,t)| \leqslant At^{\gamma-1}\left(1+\frac{|x_0-c|}{t}\right) \leqslant A(1+\alpha)^\lambda t^{\gamma-1}.$$

Since $|x_1-x_2| \leqslant 2\alpha t$ we know that

(69) $$|u(x_1,t)-u(x_2,t)| \leqslant A(1+\alpha)^{\lambda+1}t^\gamma \leqslant A(1+\alpha)^{\lambda+1}h^\gamma.$$

Now consider two points (x_0, t_1) and (x_0, t_2) in $\Gamma_\alpha^h(x_0)$. Then recall that Theorem 2 says that we can control the derivatives with respect to t. Therefore

$$|u(x_0,t_1)-u(x_0\,t_2)| = \left|\int_{t_2}^{t_1}\frac{\partial u}{\partial s}(x_0,s)\,ds\right|$$
$$\leqslant A\left|\int_{t_2}^{t_1}\left(1+\frac{|x_0-x_0|}{s}\right)^\lambda s^{\gamma-1}\,ds\right| \leqslant A\int_0^h s^{\gamma-1}\,ds = Ah^\gamma.$$

Now combining this with (69) proves that (68) is true.

Assume that (68) is true. Let $e_i = (0, \ldots, 1, \ldots, 0)$. Suppose that $\varphi \in \mathcal{S}$ and $\int\varphi(x)\,dx \neq 0$. Then there exists $\Phi \in \mathcal{S}$ such that

$$\left(\frac{\partial}{\partial x_i}\right)^s\Phi(x) = \varphi(x+e_i)-\varphi(x).$$

If $(x,t) \in \Gamma_\alpha^h(x_0) - \Gamma_{\alpha/2}^h(x_0)$,

$$t\left|\left(\frac{\partial}{\partial x_i}\right)(f*\Phi_t)(x)\right| = |f*\varphi_t(x+te_i)-f*\varphi_t(x)| \leqslant At^\gamma(1+\alpha)^\lambda.$$

Since $\left(\frac{t}{t+|x-x_0|}\right) \leqslant \frac{2}{(1+\alpha)}$ this shows that $G_{\gamma,e_i}^{\infty,\lambda}(\Phi) < \infty$. Because $\int\Phi(x)\,dx \neq 0$ then we can use the change of mollifier Theorem 1 to prove that $G_{\gamma,e_i}^{\infty,\lambda}(\varphi) < \infty$. Since this is true for all $1 \leqslant i \leqslant n$ then $\mathcal{G}_{\gamma,1}^{\infty,\lambda}(\varphi) < \infty$. Therefore $f \in \Lambda_\gamma^\infty(x_0)$.

Note that the above theorem can be combined with Theorem 13 to show that $f \in \Lambda_{k,\gamma}^\infty(x_0)$ if and only if all the kth order derivatives of f satisfy (68). (Here $k < \gamma < k+1$ and k is a nonnegative integer.)

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