By the same method used to handle $\delta < 1$ one arrives at

$$(7)\qquad |I| \leqslant \begin{cases} 4/\pi n^6\delta & \text{if } |N-x\pm\delta| \leqslant 1, \\ 3/\pi^2 n^6\delta\,|N-x\pm\delta| & \text{if } |N-x\pm\delta| \geqslant 1 \text{ when } \delta > 1. \end{cases}$$

By (1) we had

$$|Tf(x)| \leqslant \sum_{n=2}^{\infty} 2n\delta\,|I|.$$

So by (7) we see

$$|Tf(x)| \leqslant \begin{cases} 4/\pi & \text{if } |N-x\pm\delta| \leqslant 1, \\ 6/\pi^2\,|N-x\pm\delta| & \text{if } |N-x\pm\delta| > 1 \text{ when } \delta > 1. \end{cases}$$

Hence we obtain the following bound:

$$(8)\qquad \|Tf\|_p \leqslant 28/\pi \quad \text{if} \quad p \geqslant 3/2 \text{ for } \delta > 1$$

as can be verified by a straightforward computation.

To sum up, for $p \geqslant 3/2$, $\|f\|_p = (2\delta)^{1/p}$, $\|Tf\|_p \leqslant 28/\pi$ for $\delta > 1$ by (8) and $\|Tf\|_p \leqslant 40\delta^{2/3}$ by (6). Hence $\|Tf\|_p \leqslant 40/2^{1/p}\|f\|_p$ whenever $p \geqslant 3/2$. ∎

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Analytic formulae for determinant systems in Banach spaces

by

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Abstract. Formulae are proved for the determinant systems of linear mappings $A = S+T$ where S is a fixed Fredholm mapping from a Banach space X into another one Y, and T is a quasi-nuclear (or nuclear) mapping from X into Y.

The theory of determinants in an arbitrary Banach space X was first created for linear endomorphisms $A = I+T$ where I is the identity mapping in X, and T is a nuclear or quasi-nuclear endomorphism in X. (Grothendieck [2], Leżański [3], Ruston [4], see also Sikorski [5].) The theory yields analytic formulae for the determinant system of A, considered as a function of the quasi-nucleus (or nucleus) F of T. Buraczewski [1] generalized the theory to the case of endomorphsisms of the form $A = S+T$ where S is a Fredholm endomorphism in X, and T is a quasi-nuclear endomorphism in X. He also formulated analytic formulae for the determinant system of $A = S+T$ (when considered as a function of the quasi-nucleus F of T), but only under the additional hypothesis that S is right-hand or left-hand invertible (see Buraczewski [1], Theorem (xiv)). The subject of the present paper is to generalize the formulae to the case of $A = S+T$ where S is any fixed Fredholm mapping of a Banach space X into another one Y, and T is a quasi-nuclear mapping from X into Y. It is not assumed that S is right-hand or left-hand invertible. It is not assumed that $Y = X$, i.e. it is not assumed that A, S and T are endomorphisms. The theory of determinant systems, developped in this paper, is formulated in terms of the category of isomorphically conjugate pairs of Banach spaces. The object of the category are the pairs of Banach spaces just mentioned (for definition, see Section 1). The morphisms are operators defined in Section 2. The main theorem of the paper is Theorem 7.1.

Note that the terminology in our earlier papers on determinant systems differ sometimes from that in the present paper.

1. Pairs of conjugate Banach spaces. In this paper, either all linear spaces are real, or all are complex. If E denotes a normed space, then E^* denotes the Banach space of all continuous linear functionals on E.

A pair (Ξ, X) of Banach spaces is said to be a pair of *conjugate* Banach spaces provided that, to every $\xi \in \Xi$ and $x \in X$, there is assigned a number, denoted by ξx, in such a way that ξx is a continuous bilinear functional on $\Xi \times X$, and the following cancelation laws are satisfied:

(c) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$,

(c′) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

The bilinear functional is called the *scalar product on* $\Xi \times X$. The number ξx is the *scalar product of* ξ *and* x.

It follows from (c) and (c′) and from the continuity of the scalar product that every element $\xi \in \Xi$ can be interpreted as a continuous linear functional on X and, analogously, every element $x \in X$ can be interpreted as a continuous linear functional on Ξ. In symbols,

(1) $$\Xi \subset X^*, \quad X \subset \Xi^*.$$

This causes that every element $\xi \in \Xi$ has two norms: the ordinary norm $|\xi|$ as an element of the Banach space Ξ, and the norm

(2) $$|\xi|^* = \sup_{|x| \leqslant 1} |\xi x|$$

as a functional on X.

Similarly, every element $x \in X$ has two norms: the ordinary norm $|x|$ as an element of the Banach space X, and the norm

(2′) $$|x|^* = \sup_{|\xi| \leqslant 1} |\xi x|$$

as a functional on Ξ. Denoting the scalar product (i.e. the bilinear functional on $\Xi \times X$ in question) by I, and its norm by $|I|$, we have

(3) $$|\xi|^* \leqslant |I| \cdot |\xi| \quad \text{and} \quad |x|^* \leqslant |I| \cdot |x|$$

for all $\xi \in \Xi$ and $x \in X$.

The most important case is when the norms $|\ |$ and $|\ |^*$ are equivalent, for both the spaces Ξ and X. We say then that (Ξ, X) is a pair of *isomorphically conjugate* Banach spaces. A pair (Ξ, X) is a pair of isomorphically conjugate Banach spaces if and only if, in interpretation (1), Ξ is a closed subspace of X^*, and X is a closed subspace of Ξ^*; or equivalently, if and only if there exists a positive number c such that

(4) $$|\xi| \leqslant c \cdot |\xi|^*, \quad |x| \leqslant c \cdot |x|^*$$

for all $\xi \in \Xi$ and $x \in X$.

2. Operators. If A is a bilinear functional defined on the product of a pair of Banach spaces, say Ω and X, then the value of A at a point $(\omega, x) \in \Omega \times X$ will be denoted by $\omega A x$.

Let (Ξ, X) and (Ω, Y) be pairs of conjugate Banach spaces. It follows from the cancelation laws for the pairs of the Banach spaces that every continuous bilinear functional A on $\Omega \times X$ can be interpreted as a linear mapping from X into Ω^* and, simultaneously, as a linear mapping from Ω into X^*. The value of those mappings at points x and ω, respectively, will be denoted by Ax and ωA. Namely, Ax and ωA are the only points in Ω^* and X^*, respectively, such that

$$\omega(Ax) = \omega A x \quad \text{for every } \omega \in \Omega$$

and

$$(\omega A)x = \omega A x \quad \text{for every } x \in X.$$

We shall be interested only in the continuous bilinear functionals A on $\Omega \times X$ that satisfy the conditions

(1) $$Ax \in Y \quad \text{for every } x \in X,$$

and

(1′) $$\omega A \in \Xi \quad \text{for every } \omega \in \Omega.$$

In this case we shall say that A is an *operator* (more precisely, that A is a (Ξ, Y)-*weakly continuous operator on* $(\Omega \times X)$). The set of the operators will be denoted by

(2) $$\mathrm{op}(\Omega \to \Xi, X \to Y).$$

Let (Ξ, X), (Ω, Y) and (Λ, Z) be pairs of conjugate Banach spaces. By the *composition* BA of operators

$$A \in \mathrm{op}(\Omega \to \Xi, X \to Y), \quad B \in \mathrm{op}(\Lambda \to \Omega, Y \to Z)$$

we shall mean the bilinear functional on $\Lambda \times X$ whose value $\lambda(BA)x$ at a point $(\lambda, x) \in \Lambda \times X$ is given by the formula

(3) $$\lambda(BA)x = (\lambda B)(Ax),$$

i.e. it is the scalar product of the elements $\lambda B \in \Omega$ and $Ax \in Y$. It follows from the definition that

(4) $$BA \in \mathrm{op}(\Lambda \to \Xi, X \to Z).$$

Let (Ξ, X) and (Ω, Y) be pairs of conjugate Banach spaces. Let A be an operator in (2) and let B be an operator in

(5) $$\mathrm{op}(\Xi \to \Omega, Y \to X).$$

Then $BA \in \mathrm{op}(\Xi \to \Xi, X \to X)$ and ABA is again in (2). The operator B is said to be a *quasi-inverse* of A provided

(6) $$ABA = A.$$

If, moreover,

(6′) $$BAB = B,$$

B is said to be a *reciprocal quasi-inverse* of A.

3. Bi-skew symmetric multi-linear functionals. We shall examine only continuous multi-linear functionals defined on the cartesian product $\Xi^\mu \times Y^m$ of the Cartesian powers Ξ^μ and Y^m of fixed Banach spaces Ξ and Y. The value of such a multi-linear functional D at a point

$$(\xi_1, \ldots, \xi_\mu, y_1, \ldots, y_m) \in \Xi^\mu \times Y^m$$

will be denoted by

$$D\begin{pmatrix}\xi_1, \ldots, \xi_\mu \\ y_1, \ldots, y_m\end{pmatrix} \quad (\mu \geqslant 0,\ m \geqslant 0). \tag{1}$$

The multi-linear functional (1) is said to be *bi-skew symmetric* if it is skew symmetric in upper variables $\xi_1, \ldots, \xi_\mu$ and it is skew symmetric in lower variables $y_1, \ldots, y_m$, that is if, for any permutation $p = (p_1, \ldots, p_\mu)$ of the integers $1, \ldots, \mu$ and for any permutation $q = (q_1, \ldots, q_m)$ of the integers $1, \ldots, m$,

$$D\begin{pmatrix}\xi_{p_1}, \ldots, \xi_{p_\mu} \\ y_{q_1}, \ldots, y_{q_m}\end{pmatrix} = \operatorname{sgn} p \operatorname{sgn} q D\begin{pmatrix}\xi_1, \ldots, \xi_\mu \\ y_1, \ldots, y_m\end{pmatrix},$$

where $\operatorname{sgn} p$ and $\operatorname{sgn} q$ denote the signs of the permutations p and q, respectively, i.e. they are equal to 1 for even permutations, and equal to -1 for odd permutations.

The Banach space of all continuous (in norm) bi-skew symmetric multi-linear functionals (1) will be denoted by

$$\mathrm{bss}_{\mu,m}(\Xi, Y). \tag{2}$$

Let (1) and

$$D'\begin{pmatrix}\xi_1, \ldots, \xi_\nu \\ y_1, \ldots, y_n\end{pmatrix} \quad (\nu \geqslant 0,\ n \geqslant 0) \tag{3}$$

be bi-skew symmetric multi-linear functionals.

The *bi-skew symmetric* product of D and D' is a bi-skew symmetric functional D'' on $\Xi^{\mu+\nu} \times Y^{m+n}$ defined by

$$D''\begin{pmatrix}\xi_1, \ldots, \xi_{\mu+\nu} \\ y_1, \ldots, y_{m+n}\end{pmatrix} = \sum_{p,q} \operatorname{sgn} p \operatorname{sgn} q\, D\begin{pmatrix}\xi_{p_1}, \ldots, \xi_{p_\mu} \\ y_{q_1}, \ldots, y_{q_m}\end{pmatrix} D'\begin{pmatrix}\xi_{p_{\mu+1}}, \ldots, \xi_{p_{\mu+\nu}} \\ y_{p_{m+1}}, \ldots, y_{q_{m+n}}\end{pmatrix},$$

where $\sum\limits_{p,q}$ is extended over all permutations $p = (p_1, \ldots, p_{\mu+\nu})$ of the integers $1, \ldots, \mu+\nu$ and all permutations $q = (q_1, \ldots, q_{m+n})$ of the integers $1, \ldots, m+n$, such that

$$p_1 < p_2 < \ldots < p_\mu, \quad p_{\mu+1} < p_{\mu+2} < \ldots < p_{\mu+\nu},$$
$$q_1 < q_2 < \ldots < q_m, \quad q_{m+1} < q_{m+2} < \ldots < q_{m+n}.$$

The bi-skew symmetric product D'' of D and D' will be denoted by $D \bullet D'$. The multiplication $\bullet$ is associative.

Suppose that Ξ and Y are members of pairs of conjugate Banach spaces (Ξ, X) and (Ω, Y).

Every element $x \in X$ can be interpreted as an element D in $\mathrm{bss}_{1,0}(\Xi, Y)$, viz. $D(\xi) = \xi x$ for $\xi \in \Xi$. Similarly, every element $\omega \in \Omega$ can be interpreted as an element D in $\mathrm{bss}_{0,1}(\Xi, Y)$, viz. $D(y) = \omega y$ for $y \in Y$. More generally, we can form *multi-vectors*

$$x = x_1 \bullet \ldots \bullet x_\mu \quad \text{where} \quad x_1, \ldots, x_\mu \in X, \tag{4}$$

$$\omega = \omega_1 \bullet \ldots \bullet \omega_m \quad \text{where} \quad \omega_1, \ldots, \omega_m \in \Omega. \tag{4'}$$

If B is an operator in $\mathrm{op}(\Xi \to \Omega, Y \to X)$, then $B \in \mathrm{bss}_{1,1}(\Xi, Y)$. More generally, we can form the following modified *powers of* B

$$B^{\bullet n} = \frac{1}{n!} \underbrace{B \bullet B \bullet \ldots \bullet B}_{n\text{-times}} \quad (n = 0, 1, 2, \ldots) \tag{5}$$

which are elements of $\mathrm{bss}_{n,n}(\Xi, Y)$.

We can also form the bi-skew symmetric products

$$B^{\bullet n} \bullet x \bullet \omega. \tag{6}$$

The multi-linear functionals (4), (4′), (5), (6) are the main examples of bi-skew symmetric functionals which appear in the theory of determinant systems.

4. Weak continuity of multi-linear functionals. Let (Ξ, X) and (Ω, Y) be pairs of conjugate Banach spaces. A multi-linear functional

$$D\begin{pmatrix}\xi_1, \ldots, \xi_\mu \\ y_1, \ldots, y_m\end{pmatrix} \tag{1}$$

defined on $\Xi^\mu \times Y^m$ ($\mu \geqslant 0$, $m \geqslant 0$) is said to be (Ω, X)-*weakly continuous* if the following two conditions are satisfied:

(w_1) for any fixed elements $\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_\mu \in \Xi$ and $y_1, \ldots, y_m \in Y$ there exists an element $x \in X$ such that

$$\xi x = D\begin{pmatrix}\xi_1, \ldots, \xi_{i-1}, \xi, \xi_{i+1}, \ldots, \xi_\mu \\ y_1, \ldots\ldots\ldots, y_m\end{pmatrix}$$

for every $\xi \in \Xi$, $i = 1, \ldots, \mu$;

(w_2) for any fixed elements $\xi_1, \ldots, \xi_\mu \in \Xi$ and $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m \in Y$ there exists and element $\omega \in \Omega$ such that

$$\omega y = D\begin{pmatrix}\xi_1, \ldots\ldots\ldots, \xi_\mu \\ y_1, \ldots, y_{j-1}, y, y_{j+1}, \ldots, y_m\end{pmatrix}$$

for every $y \in Y$, $j = 1, \ldots, m$.

It follows directly from the definition that every (Ω, X)-weakly continuous functional (1) is continuous in each of the variables $\xi_1, \ldots, \xi_\mu$, $y_1, \ldots, y_m$ separately. Thus it is continuous with respect to the ordinary norm

$$(2)\qquad |D| = \sup\left\{\left|D\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix}\right| : |\xi_i| \leqslant 1 \text{ for } i = 1, \ldots, \mu \text{ and } |y_j| \leqslant 1 \text{ for } j = 1, \ldots, m\right\}.$$

4.1. If (Ξ, X) *and* (Ω, Y) *are pairs of isomorphically conjugate Banach spaces, then the set of all* $(\mu+m)$*-linear* (Ω, X)*-weakly continuous functionals* (1) *on* $\Xi^\mu \times Y^m$ *is a linear closed subset of the Banach space of all* $(\mu+m)$*-linear continuous (in the norm* (2)*) functionals on* $\Xi^\mu \times Y^m$*. Therefore that set with the norm* (2) *is a Banach space. Consequently the set of all* $(\mu+m)$*-linear* (Ω, X)*-weakly continuous bi-skew symmetric functionals on* $\Xi^\mu \times Y^m$ *is a Banach space with the norm* (2).

We have to prove that if $D_n \to D$ in the norm (2) and all the functionals D_n are (Ω, X)-weakly continuous, then the limit D is also (Ω, X)-weakly continuous. For this purpose, consider the expressions

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix}, \qquad D\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix}$$

as functions of one variable, say y_1, the remaining variables being fixed. By the (Ω, X)-weak continuity of D_n, there exists an $\omega_n \in \Omega$ such that

$$\omega_n y_1 = D_n\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix} \quad \text{for every } y_1 \in Y.$$

Since (see Section 1, (2))

$$\begin{aligned}|\omega_n - \omega_k|^* &= \sup_{|y_1|\leqslant 1} |(\omega_n - \omega_k) y_1|\\ &\leqslant |D_n - D_k|\,|\xi_1| \ldots |\xi_\mu|\,|y_2| \ldots |y_m|,\end{aligned}$$

the sequence $\omega_1, \omega_2, \ldots$ satisfies the Cauchy condition in the Banach space Ω with the norm $|\ |^*$. Since the norm $|\ |^*$ is equivalent to the norm $|\ |$, the sequence satisfies the Cauchy condition with respect to the norm $|\ |$. Thus the sequence converges, with respect of the norm $|\ |$, to an element $\omega \in \Omega$. Since the scalar multiplication is continuous, we have

$$\begin{aligned}\omega y_1 &= (\lim_{n\to\infty} \omega_n) y_1 = \lim_{n\to\infty} (\omega_n y_1)\\ &= \lim_{n\to\infty} D_n\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix} = D\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix}\end{aligned}$$

for every $y_1 \in Y$.

4.2. *If* (Ξ, X) *and* (Ω, Y) *are pairs of isomorphically conjugate Banach spaces,* $D_0, D_1, D_2, \ldots$ *are* $(\mu+m)$*-linear* (Ω, X)*-weakly continuous functionals defined on* $\Xi^\mu \times Y^m$*, and*

$$(3)\qquad \sum_{k=0}^{\infty} |D_k| < \infty,$$

then the series

$$(4)\qquad D\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix} = \sum_{k=0}^{\infty} D_k\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix}$$

converges in norm and its sum also is a $(\mu+m)$*-linear* (Ω, X)*-weakly continuous functional defined on* $\Xi^\mu \times Y^m$.

This is a direct consequence of 4.1.

4.3. *If* (Ξ, X) *and* (Ω, Y) *are pairs of isomorphically conjugate Banach spaces, then the sets of operators*

$$\mathrm{op}(\Omega \to \Xi, X \to Y) \quad \textit{and} \quad \mathrm{op}(\Xi \to \Omega, Y \to X)$$

are Banach spaces with respect to the ordinary norms

$$|A| = \sup_{\substack{|\omega|\leqslant 1\\ |x|\leqslant 1}} |\omega A x| \quad \textit{and} \quad |A| = \sup_{\substack{|\xi|\leqslant 1\\ |y|\leqslant 1}} |\xi A y|$$

respectively.

This follows from 4.1 (the case where $\mu = 1 = m$) because a bilinear functional A on $\Xi \times Y$ (or $\Omega \times X$) is an operator if and only if it is (Ω, X)-weakly continuous ((Ξ, Y)-weakly continuous).

5. Continuous nuclei. Let (Ξ, X) and (Ω, Y) be pairs of conjugate Banach spaces. If $x \in X$ and $\omega \in \Omega$, then the symbol $x \cdot \omega$ denotes an operator in

$$(1)\qquad \mathrm{op}(\Xi \to \Omega, Y \to X)$$

whose value at a point $(\xi, y) \in \Xi \times Y$ is equal to the product of the numbers ξx and ωy, in symbols

$$\xi(x \cdot \omega) y = \xi x \cdot \omega y.$$

Let F be any continuous linear functional defined on the space (1), i.e. an element in the Banach space

$$(2)\qquad \mathrm{op}(\Xi \to \Omega, Y \to X)^*.$$

The formula

$$(3)\qquad \omega \bar{F} x = F(x \cdot \omega)$$

defines a bilinear functional $\bar{F}$ on $\Omega \times X$. We shall be interested only in the case where $\overline{F}$ is an operator, more precisely, where $\overline{F}$ is an element of the space

(4) $$\mathrm{op}(\Omega\to\Xi, X\to Y).$$

The space of all F in (2) such that $\overline{F}$ is in (4) will be denoted by

(5) $$\mathrm{cn}(\Omega\to\Xi, X\to Y).$$

The elements of (5) will be called *continuous nuclei* (or simply *nuclei*). More precisely, if $T=\overline{F}$, we say that the functional F is a (*continuous*) *nucleus of the operator* T in (4). Let Φ be the mapping from (2) into the Banach space $\mathrm{bss}_{1,1}(\Omega, X)$ of all continuous bilinear functionals on $\Omega\times X$, defined by

(6) $$\Phi(F)=\overline{F}\quad \text{for } F \text{ in } (2).$$

The restriction of Φ to the subspace (5) of (2) will be called the *canonical transformation.*

5.1. *If* (Ξ, X) *and* (Ω, Y) *are pairs of isomorphically conjugate Banach spaces, then the space* (5) *of all continuous nuclei is a closed subset of the Banach space* (2), *and therefore it itself is a Banach space. The canonical transformation is a continuous mapping from* (5) *into* (4).

By a simple calculation (see Section 1, (3)),

$$|\overline{F}|\leqslant |I|\cdot|J|\cdot|F|$$

where $|I|$ and $|J|$ are norms of the scalar products I and J in (Ξ, X) and (Ω, Y), respectively. This proves that Φ is continuous, and consequently so is the canonical transformation. By 4.3, (4) is a closed subspace of $\mathrm{bss}_{1,1}(\Omega, X)$. Consequently, its inverse image by Φ, i.e. the space (5), is a closed subspace of (2).

Let $D\in \mathrm{bss}_{\mu,m}(\Xi, Y)$ be an (Ω, X)-weakly continuous multilinear functional, $\mu>0$, $m>0$, and let F be a nucleus in (2). Fix all the variables $\xi_2,\ldots,\xi_\mu$ and $y_2,\ldots,y_m$ and consider the expression

$$D\begin{pmatrix}\xi_1,\ldots,\xi_\mu\\ y_1,\ldots,y_m\end{pmatrix}$$

as a function of ξ_1 and y_1 only, i.e. as an operator A in (1). Take the value $F(A)$ of F at A. Denote the value by

(7) $$F\square D\begin{pmatrix}\xi_2,\ldots,\xi_\mu\\ y_2,\ldots,y_m\end{pmatrix}.$$

Denote by $F\square D$ the function which assigns to $\xi_2,\ldots,\xi_\mu\in\Xi$ and $y_2,\ldots,y_m \in Y$ the number (7). It is easy to see that

$$F\square D\in \mathrm{bss}_{\mu-1,m-1}(\Xi, Y).$$

If $\mu>1$ and $m>1$, and if $F\square D$ is (Ω, X)-weakly continuous, we can repeat this procedure and define

$$G\square F\square D\in \mathrm{bss}_{\mu-2,m-2}(\Omega, X)$$

for any nucleus G in (2). We can iterate the procedure k-times, where $k=\min(\mu, m)$, and define

$$F_1\square D,\quad F_2\square F_1\square D,\ldots,\quad F_k\square\ldots F_2\square F_1\square D,$$

provided the iteration preserves the (Ω, X)-weak continuity. We shall deal only with the case where the last condition is always satisfied. By definition,

(7′) $$F_i\square F_{i-1}\square\ldots F_2\square F_1\square D\in \mathrm{bss}_{\mu-i,m-i}(\Xi, Y)\qquad (i\leqslant k).$$

If F is a fixed nucleus in (2), then $F\square$ will denote the mapping which assigns $F\square D$ to every $D\in \mathrm{bss}_{\mu,m}(\Xi, Y)$. Clearly,

$$F_i\square\ldots F_1\square$$

is the composition of mappings $F_1\square,\ldots,F_i\square$ determined by nuclei $F_1,\ldots,F_i$ in (2).

We shall denote by $F^{\square i}$ the *modified i-th power* of a nucleus F in (2), that is,

(8) $$F^{\square i}=\frac{1}{i!}\underbrace{F\square\ldots F\square F\square}_{i\text{-times}}.$$

In particular, $F^{\square 1}=F\square$.

6. Estimations. Let (Ξ, X) and (Ω, Y) be pairs of conjugate Banach spaces. Let $|I|$ and $|J|$ denote respectively the norms of the scalar products I, J in those pairs.

We recall (see Section 3, (4)) that if $x_1,\ldots,x_\mu$ are elements in X, then $x=x_1\bullet,\ldots,\bullet x_\mu$ is the following multi-linear functional (multi-vector) in $\mathrm{bss}_{\mu,0}(\Xi, Y)$

(1) $$x(\xi_1,\ldots,\xi_\mu)=\begin{vmatrix}\xi_1x_1\ldots\ \xi_1x_\mu\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ \xi_\mu x_1\ldots\ \xi_\mu x_\mu\end{vmatrix}.$$

Similarly, for any $\omega_1,\ldots,\omega_m$ in Ω the symbol $\omega=\omega_1\bullet,\ldots,\bullet\omega_m$ denotes the following multi-linear functional (multi-vector) in $\mathrm{bss}_{0,m}(\Xi, Y)$

(1′) $$\omega(y_1,\ldots,y_m)=\begin{vmatrix}\omega_1y_1\ \ldots\ \omega_1y_m\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ \omega_my_1\ldots\ \omega_my_m\end{vmatrix}.$$

Under the above notations,

6.1. *The following inequalities hold:*

(2) $$|x| \leqslant \sqrt{\mu^\mu}\cdot |I|^\mu |x_1| \ldots |x_\mu|,$$

(2′) $$|\omega| \leqslant \sqrt{m^m}\, |J|^m |\omega_1| \ldots |\omega_m|.$$

This follows directly from the Hadamard inequality.

Let x and ω be as above and let $B \in \operatorname{op}(\Xi \to \Omega,\ Y \to X)$. Under these hypotheses,

6.2. *The following estimation holds:*

(3) $$|B^{\bullet n}\bullet x \bullet \omega| \leqslant |I|^\mu |J|^m |x_1| \ldots |x_\mu|\, |\omega_1| \ldots |\omega_m|\, |B|^n \cdot \frac{(n+m)!}{n!\, m!} \cdot \frac{(n+\mu)!}{n!\, \mu!}.$$

Denote $B^{\bullet n}\bullet x \bullet \omega$ by D, for brevity. By definition (see p. 88–89)

$$D\begin{pmatrix}\xi_1, \ldots, \xi_{n+\mu}\\ y_1, \ldots, y_{n+m}\end{pmatrix} = \sum_{p,q} a_{p,q}\beta_p\gamma_q$$

where p and q are permutations of the integers $1, \ldots, n+\mu$ and $1, \ldots$ $\ldots, n+m$, respectively, such that

(4) $$p_1 < \ldots < p_n, \qquad p_{n+1} < \ldots < p_{n+\mu},$$

(4′) $$q_1 < \ldots < q_n, \qquad q_{n+1} < \ldots < q_{n+m},$$

and

$$a_{p,q} = \begin{vmatrix}\xi_{p_1}By_{q_1} & \ldots & \xi_{p_1}By_{q_n}\\ \cdot & \cdot & \cdot\\ \xi_{p_n}By_{q_1} & \ldots & \xi_{p_n}By_{q_n}\end{vmatrix},$$

$$\beta_p = \begin{vmatrix}\xi_{p_{n+1}}x_1 & \ldots & \xi_{p_{n+1}}x_\mu\\ \cdot & \cdot & \cdot\\ \xi_{p_{n+\mu}}x_1 & \ldots & \xi_{p_{n+\mu}}x_\mu\end{vmatrix}, \qquad \gamma_q = \begin{vmatrix}\omega_1 y_{q_{n+1}} & \ldots & \omega_1 y_{q_{n+m}}\\ \cdot & \cdot & \cdot\\ \omega_m y_{q_{n+1}} & \ldots & \omega_m y_{q_{n+m}}\end{vmatrix}.$$

Hence

$$\left|D\begin{pmatrix}\xi_1, \ldots, \xi_{n+\mu}\\ y_1, \ldots, y_{n+m}\end{pmatrix}\right| \leqslant \sum_{p,q} |a_{p,q}||\beta_p||\gamma_q|.$$

Suppose that

$$|\xi_i| \leqslant 1 \quad\text{and}\quad |y_j| \leqslant 1 \quad \text{for } i = 1, \ldots, n+\mu \text{ and } j = 1, \ldots, n+m.$$

It follows from 6.1 that

(5) $$|\beta_p| \leqslant \sqrt{\mu^\mu}|I|^\mu |x_1| \ldots |x_\mu|,$$

(5′) $$|\gamma_q| \leqslant \sqrt{m^m}|J|^m\, |\omega_1| \ldots |\omega_m|.$$

Applying the Hadamard inequality to the determinant $a_{p,q}$, we get

(6) $$|a_{p,q}| \leqslant \sqrt{n^n}|B|^n.$$

Inequality (3) follows directly from (5), (5′), (6) and the fact that the numbers of permutations p and q satisfying (4) and (4′), respectively, are equal to

$$\frac{(n+\mu)!}{n!\,\mu!}, \qquad \frac{(n+m)!}{n!\,m!}.$$

Now, let B, x, ω be as above, let

$$D = B^{\bullet n+k}\bullet x \bullet \omega$$

for brevity, let $F \in \operatorname{cn}(\Omega \to \Xi,\ X \to Y)$ be a continuous nucleus, and let

$$D'\begin{pmatrix}\xi_1, \ldots, \xi_{n+\mu}\\ y_1, \ldots, y_{n+m}\end{pmatrix} = \underbrace{F\Box \ldots F\Box}_{k\text{-times}} D\begin{pmatrix}\xi_1', \ldots, \xi_k', & \xi_1, \ldots, \xi_{n+\mu}\\ y_1', \ldots, y_k', & y_1, \ldots, y_{n+m}\end{pmatrix}.$$

Under these hypotheses,

6.3. *The following inequality holds:*

(7) $$|D'| \leqslant |x_1| \ldots |x_\mu|\, |\omega_1| \ldots |\omega_m|\, |I|^\mu |J|^m |B|^n |F|^k \times \times \sqrt{\mu^\mu m^m (n+k)^{n+k}} \cdot \frac{(n+k+\mu)!}{(n+k)!\,\mu!}\, \frac{(n+k+m)!}{(n+k)!\,m!}.$$

We have

$$|D'| = \sup\left\{ D'\begin{pmatrix}\xi_1, \ldots, \xi_{n+\mu}\\ y_1, \ldots, y_{n+m}\end{pmatrix} :\ |\xi_i| \leqslant 1 \text{ for } i = 1, \ldots, n+\mu \text{ and } |y_j| \leqslant 1 \text{ for } j = 1, \ldots, n+m \right\}$$
$$\leqslant |F|^k \cdot \sup\left\{ D\begin{pmatrix}\xi_1', \ldots, \xi_k', \xi_1, \ldots, \xi_{n+\mu}\\ y_1', \ldots, y_k', y_1, \ldots, y_{n+m}\end{pmatrix} :\ |\xi_i'| \leqslant 1 \text{ for } i = 1, \ldots, k,\ |\xi_i| \leqslant 1 \text{ for } i = 1, \ldots, n+\mu,\ |y_j'| \leqslant 1 \text{ for } j = 1, \ldots, k,\ |y_j| \leqslant 1 \text{ for } j = 1, \ldots, n+m \right\}$$
$$= |F|^k |D|.$$

Now apply (3) where n should be replaced by $n+k$.

7. Determinant systems. Let (Ξ, X) and (Ω, Y) be pairs of conjugate Banach spaces.

We shall use the following notation. If D is a multi-linear functional defined on $\Xi^\mu \times Y^{m+1}$, and y is a fixed element in Y, then the symbol Dy

will stand for the multi-linear functional defined on $\Xi^{\mu}\times Y^{m}$ by

$$Dy\begin{pmatrix}\xi_1,\ldots,\xi_\mu\\ y_1,\ldots,y_m\end{pmatrix}=D\begin{pmatrix}\xi_1,\quad\ldots,\xi_\mu\\ y,y_1,\ldots,y_m\end{pmatrix}. \tag{1}$$

Roughly speaking, Dy is obtained from D by the fixation of the first lower variable. If $D\in \mathrm{bss}_{\mu,m+1}(\Xi,Y)$, then $Dy\in\mathrm{bss}_{\mu,m}(\Xi,Y)$. If A is an operator in

$$\mathrm{op}(\Omega\to\Xi,X\to Y), \tag{2}$$

then DA will denote the function which assigns, to every $x\in X$, the functional $D[Ax]$, i.e. the result of the substitution of the element $y=Ax$ for the first lower variable in D.

Analogously, if D is a multi-linear functional defined on $\Xi^{\mu+1}\times Y^m$, and ξ is a fixed element in Ξ, then the symbol ξD will stand for the multi-linear functional defined on $\Xi^{\mu}\times Y^m$ by

$$\xi D\begin{pmatrix}\xi_1,\ldots,\xi_\mu\\ y_1,\ldots,y_m\end{pmatrix}=D\begin{pmatrix}\xi,\xi_1,\ldots,\xi_\mu\\ y_1,\quad\ldots,y_m\end{pmatrix}. \tag{2'}$$

Roughly speaking, ξD is obtained from D by the fixation of the first upper variable. If $D\in\mathrm{bss}_{\mu+1,m}(\Xi,Y)$, then $\xi D\in\mathrm{bss}_{\mu,m}(\Xi,Y)$. If A is an operator in (2), then AD will denote the function which assigns, to every $\omega\in\Omega$, the functional $[\omega A]D$, i.e. the result of the substitution of the element $\xi=\omega A$ for the first upper variable in D.

If $D\in\mathrm{bss}_{\mu,m}(\Xi,Y)$ and $C\in\mathrm{op}(\Xi\to\Xi,X\to X)$, then the symbol $\overleftarrow{C}\bullet D$ will denote the function which assigns, to every $x\in X$, the skew product $Cx\bullet D\in\mathrm{bss}_{\mu+1,m}(\Xi,Y)$ of $Cx\in X\subset\Xi^*$ and D.

Analogously, if $D\in\mathrm{bss}_{\mu,m}(\Xi,Y)$ and $C\in\mathrm{op}(\Omega\to\Omega,Y\to Y)$, then the symbol $\overrightarrow{C}\bullet D$ will denote the function which assigns, to every $\omega\in\Omega$, the skew product $\omega C\bullet D\in\mathrm{bss}_{\mu,m+1}(\Xi,Y)$ of $\omega C\in\Omega\subset Y^*$ and D.

In what follows, the letters I and J will denote the scalar products in (Ξ,X) and (Ω,Y), respectively. The operator I, interpreted as a mapping of Ξ (of X) into itself, is the identity mapping of Ξ (of X) onto itself. Analogously, the operator J interpreted as a mapping of Ω (of Y) into itself is the identity mapping of Ω (of Y) onto itself.

By a *determinant system* for an operator A in (2) we shall mean any infinite sequence of multi-linear functionals

$$D_0,D_1,D_2,\ldots \tag{3}$$

such that

(d_1) all the functionals are bi-skew symmetric, more precisely,

$$D_n\in\mathrm{bss}_{\mu_n m_n}(\Xi,Y) \tag{4}$$

where

$$\mu_n=\mu_0+n,\qquad m_n=m_0+n \tag{5}$$

and

$$\min(\mu_0,m_0)=0; \tag{5'}$$

(d_2) all the multi-linear functionals (3) are (Ω,X)-weakly continuous;
(d_3) there exists a non-negative integer r such that $D_r\neq 0$;
(d_4) the following identities hold for $n=-1,0,1,2,\ldots$

$$D_{n+1}A=\overleftarrow{I}\bullet D_n, \tag{6}$$

$$AD_{n+1}=\overrightarrow{J}\bullet D_n. \tag{6'}$$

The case of $n=-1$ requires an explanation. In that case the right-hand sides of (6) and (6′) have no sense and should be replaced by 0. The left-hand side of (6) is sensible only if $m_0>0$; then it should be equal to zero. Analogously, the left-hand side of (6′) is sensible only if $\mu_0>0$; then it should be equal to zero. By (5′), it is excluded that both μ_0 and m_0 are positive. If $\mu_0=0=m_0$, the left-hand sides of (6) and (6′) have no sense and both (6), (6′) are, therefore, satisfied.

If $n\geqslant 0$, then equations (6) and (6′) are abbreviations of the following identities:

$$D_{n+1}\begin{pmatrix}\xi_0,\xi_1,\ldots,\xi_{\mu_n}\\ Ax,y_1,\ldots,y_{m_n}\end{pmatrix}=\sum_{i=0}^{\mu_n}(-1)^i\xi_i x\cdot D_n\begin{pmatrix}\xi_0,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_{\mu_n}\\ y_1,\ldots\ldots\ldots,y_{m_n}\end{pmatrix}, \tag{7}$$

$$D_{n+1}\begin{pmatrix}\omega A,\xi_1,\ldots,\xi_{\mu_n}\\ y_0,y_1,\ldots,y_{m_n}\end{pmatrix}=\sum_{j=0}^{m_n}(-1)^j\omega y_j\cdot D_n\begin{pmatrix}\xi_1,\ldots\ldots\ldots,\xi_{\mu_n}\\ y_0,\ldots,y_{j-1},y_{j+1},\ldots,y_{m_n}\end{pmatrix}. \tag{7'}$$

7.1. THE MAIN THEOREM. *Let (Ξ,X) and (Ω,Y) be pairs of isomorphically conjugate Banach spaces. For every continuous nucleus F in*

$$\mathrm{cn}(\Omega\to\Xi,X\to Y), \tag{8}$$

let

$$\mathscr{D}_{n,k}(F)=F^{\square k}D_{n+k}\quad\textit{for } n,k=0,1,2,\ldots \tag{9}$$

Then

(i) *the function*

$$\mathscr{D}_{n,k}\colon \mathrm{cn}(\Omega\to\Xi,X\to Y)\to\mathrm{bss}_{\mu_n,m_n}(\Xi,Y) \tag{10}$$

is a homogeneous polynomial, of degree k, which assigns, to every continuous nucleus F, the multi-linear functional $\mathscr{D}_{n,k}(F)$ defined on $\Xi^{\mu_n}\times Y^{m_n}$;

(ii) *there exist positive numbers* $a_{n,k}$ $(n, k = 0, 1, 2, \ldots)$ *that do not depend on* F, *and such that for every fixed* n *the limit* $\lim\limits_{k\to\infty} \frac{a_{n,k+1}}{a_{n,k}}$ *exists and is finite, and the following estimation of the norm of the polynomial* $\mathscr{D}_{n,k}$ *holds*:

(11) $$|\mathscr{D}_{n,k}| \leqslant a_{n,k}/k!,$$

that is,

(11′) $$|\mathscr{D}_{n,k}(F)| \leqslant \frac{a_{n,k}}{k!} |F|^k;$$

(iii) *the series*

(12) $$\sum_{k=0}^{\infty} |\mathscr{D}_{n,k}| \qquad (n = 0, 1, 2, \ldots)$$

converges and, consequently, the series

(12′) $$\sum_{k=0}^{\infty} |\mathscr{D}_{n,k}(F)|, \quad \sum_{k=0}^{\infty} \mathscr{D}_{n,k}, \quad \sum_{k=0}^{\infty} \mathscr{D}_{n,k}(F) \qquad (n = 0, 1, 2, \ldots)$$

converge;

(iv) *the functions*

(13) $$\mathscr{D}_n = \sum_{k=0}^{\infty} \mathscr{D}_{n,k}\colon \operatorname{cn}(\Omega\to\Xi, X\to Y)\to \operatorname{bss}_{\mu_n,m_n}(\Xi, Y) \qquad (n = 0, 1, 2, \ldots),$$

that is,

(13′) $$\mathscr{D}_n(F) = \sum_{k=0}^{\infty} \mathscr{D}_{n,k}(F),$$

are analytic functions (power series) and

(14) $$\frac{d^n}{d\gamma^n} \mathscr{D}_r(\gamma F) = F^{\square n}\mathscr{D}_{n+r}(\gamma F) \qquad (\gamma \text{ a number});$$

(v) *for every fixed* F *in* (8) *the sequence*

(15) $$\mathscr{D}_0(F),\ \mathscr{D}_1(F),\ \mathscr{D}_2(F), \ldots \qquad (\mathscr{D}_n(F) \in \operatorname{bss}_{\mu_n,m_n}(\Xi, Y))$$

is a determinant system for the operator $A + \bar{F}$ *in* (2);

(vi) *moreover, for* $F = 0$ *the determinant system* (15) *coincides with* (3).

To prove the correctness of the definition of the $\mathscr{D}_{n,k}$ and their (Ω, X)-weak continuity we introduce the following terminology. A multilinear functional

(16) $$D\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix}$$

is said to be *simple* if it can be represented in the form

(17) $$D\begin{pmatrix}\xi_1, \ldots, \xi_\mu\\ y_1, \ldots, y_m\end{pmatrix} = c\prod_{j=1}^{n} \xi_{p_j} B_j y_{q_j} \cdot \prod_{j=1}^{m-n} \omega_j y_{q_j+n} \prod_{j=1}^{\mu-n} \xi_{p_j+n} x_j$$

where $B_j \in \operatorname{op}(\Xi\to\Omega, Y\to X)$ for $j = 1, \ldots, n$, $\omega_j \in \Omega$ for $j = 1, \ldots, m-n$, $x_j \in X$ for $j = 1, \ldots, \mu-n$ $(n \leqslant \min(\mu, m))$, c is a number, $p_1, \ldots, p_\mu$ is a permutation of the integers $1, \ldots, \mu$, and $q_1, \ldots, q_m$ is a permutation of the integers $1, \ldots, m$. It may happen that any of the integers n, $\mu-n$, $m-n$ are equal to zero, then the corresponding factor $\prod \ldots$ is supposed to be equal to 1.

A multi-linear functional (16) is said to be *semi-simple* provided it is the sum of a finite number of simple multi-linear functionals.

By a reasoning similar to that in Leżański [3], pp. 248–250, we prove that if (16) is simple, then $F\square D$ is also simple, for any F in (8). This implies that if a multi-linear functional (16) is semi-simple, so is $F\square D$. It follows directly from representation (17) that every simple multilinear functional (16) is (Ω, X)-weakly continuous. On the other hand, it follows from a theorem by Sikorski [6] that if (3) is a determinant system for an operator A in (2), then, for an integer $r \geqslant 0$,

$$D_n = 0 \text{ for } r < n \quad\text{ and }\quad D_n = B^{\bullet n-r}\bullet\omega\bullet x \text{ for } n \geqslant r,$$

where B is a quasi-inverse of A and therefore an operator in

$$\operatorname{op}(\Xi\to\Omega, Y\to X),$$

and x and ω are multivectors, like in formula (6), Section 3. Thus all the multilinear functional (3) are semi-simple, and therefore (Ω, X)-weakly continuous. By the argument just formulated, the multi-linear functionals $F\square D_1, F\square D_2, F\square D_3, \ldots$, are semi-simple and, therefore (Ω, X)-weakly continuous. By the same argument, $F\square F\square D_2, F\square F\square D_3, F\square F\square D_4, \ldots$, are semi-simple and (Ω, X)-weakly continuous, and so on. Thus all the multi-linear functionals

$$\underbrace{F\square \ldots F\square}_{k\text{-times}} D_{n+k}$$

are semi-simple and (Ω, X)-weakly continuous. The functionals $\mathscr{D}_{n,k}$ differ from those functionals by constant factors, thus they are well defined, semi-simple and (Ω, X)-weakly continuous.

(i) is obvious. The estimation quoted in (ii) follows from 6.3 (to get (11) take as $a_{n,k}$ the product of all factors on the right-hand side of 6.3 (7), except the factor $|F|^k$). The convergence of series (12) follows from (11) and from the d'Alembert test applied to the numerical series $\sum_{k=0}^{\infty} a_{n,k}/k!$. All other statements in (iii) are consequences of (12). (iv) is obvious. (vi) follows directly from the defintion of $\mathscr{D}_n$ (see (13), (13′)).

We have to prove (v). Clearly, sequence (15) has property (d_1) with the same μ_n and m_n. Property (d_2) follows from Theorem 4.2 and the (Ω, X)-weak continuity of $\mathscr{D}_{n,k}$ just proved.

Let r be the smallest integer such that $D_r \neq 0$ and let $f(\gamma) = \mathscr{D}_r(\gamma, F)$ for all numbers γ. The function f is analytic. It does not vanish at the point 0 because $f(0) = \mathscr{D}_r(0) = D_r \neq 0$ by (vi). Therefore there exists a non-negative integer n such that $\left(\frac{d^n}{d\gamma^n} f(\gamma)\right)_{\gamma=1} \neq 0$. This implies, by (14), that $\mathscr{D}_{r+n}(F) \neq 0$. Thus for every fixed F sequence (15) has property (d_3).

To prove property (d_4) first observe that the following identities hold:

$$\mathscr{D}_{n+1,k}(F)\begin{pmatrix}\omega A, \xi_1, \ldots, \xi_{\mu_n}\\ y_0, y_1, \ldots, y_{m_n}\end{pmatrix} = \sum_{i=0}^{m_n}(-1)^i\cdot \omega y_i \cdot \mathscr{D}_{n,k}(F)\begin{pmatrix}\xi_1, \ldots\ldots\ldots, \xi_{\mu_n}\\ y_0, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{m_n}\end{pmatrix} - \\ - \mathscr{D}_{n+1,k-1}(F)\begin{pmatrix}\omega \overline{F}, \xi_1, \ldots, \xi_{\mu_n}\\ y_0, y_1, \ldots, y_{m_n}\end{pmatrix}, \tag{18}$$

$$\mathscr{D}_{n+1}(F)\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_{\mu_n}\\ Ax, y_1, \ldots, y_{m_n}\end{pmatrix} = \sum_{i=0}^{\mu_n}(-1)^i\cdot \xi_i x \cdot \mathscr{D}_{n,k}(F)\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{\mu_n}\\ y_1, \ldots\ldots\ldots, y_{m_n}\end{pmatrix} - \\ - \mathscr{D}_{n+1,k-1}(F)\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_{\mu_n}\\ Fx, y_1, \ldots, y_{m_n}\end{pmatrix}. \tag{18'}$$

The proof of (18) and (18′) is similar to that of Leżański [3], pp. 254–256, or Sikorski [5], pp. 37–38. Adding all equations (18) with respect to k we get (7′). Similarly, adding all equations (18′) we get (7).

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