$n \in N$ so that

$$e^{-a_n/k} = \gamma_n^{-s_{n3}/k} \leqslant \gamma_n^{-s_{nj}} = d_n^j, \quad n \in N.$$

This shows that (d_n^k) and $e^{-a_n/k}$ define the same Köthe space so the proof is complete.

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CLARKSON COLLEGE OF TECHNOLOGY
POTSDAM, N.Y.

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A nondentable set without the tree property

by

J. BOURGAIN* (Brussel)

Abstract. The existence is shown of a bounded, closed, convex and nondentable subset of l^∞, which does not contain a tree.

Introduction. Let X, $\| \ \|$ be a Banach space with dual X^*. If $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon)$ denotes the open ball with midpoint x and radius ε. For sets $A \subset X$, let $c(A)$ be the convex hull and $\bar{c}(A)$ the closed convex hull of A. We will say that A is *dentable* if for all $\varepsilon > 0$ there exists $x \in A$ satisfying $x \notin \bar{c}(A \setminus B(x, \varepsilon))$. The Banach space X is said to be *dentable* if every nonempty, bounded subset of X is dentable. We say that X has the *Radon–Nikodým property* (RNP) provided for every measure space (Ω, Σ, μ) with $\mu(\Omega) < \infty$ and every μ-continuous measure $F\colon \Sigma \to X$ of finite variation, there exists a Bochner integrable function $f\colon \Omega \to X$ such that $F(E) = \int_E f\,d\mu$ for every $E \in \Sigma$. The RNP of X is equivalent with the fact that any uniformly bounded X-valued martingale on a finite measure space is convergent a.e. (cf. [5], [18]). It is known that X is a dentable Banach space if and only if X has RNP. The reader will find the history of the equivalence between those two properties in the survey paper [6] of J. Diestel and J. J. Uhl.

A set with the Radon–Nikodým property (RNP-set) is a bounded, closed and convex subset of X such that each of its nonempty subsets is dentable. For some remarkable properties of these sets, I refer the reader to [1], [2] and [15].

DEFINITION 1. A *bush* B is a bounded subset of X such that for some $\varepsilon > 0$ the property $x \in c(B \setminus B(x, \varepsilon))$ holds for all $x \in B$.

A *tree* $\mathscr{T}$ is a bounded subset of X such that for some $\varepsilon > 0$ we have that each point $x \in \mathscr{T}$ is the midpoint of 2 points $y \in \mathscr{T}$, $z \in \mathscr{T}$ with $\|y - z\| \geqslant \varepsilon$.

* Aspirant, N. F. W. O., Belgium.

Obviously every tree is a bush. It is also routine to check that every bush (resp. tree) contains a countable bush (resp. tree).

We say that a subset of X possesses the *bush-* (resp. *tree-*) *property* if it contains a bush (resp. tree). The interest of these notions will be clear from the following 2 results:

PROPOSITION 1 (Huff–Morris, see [9]). *All nondentable bounded, closed and convex sets and therefore all Banach spaces failing* RNP *have the bush-property.*

PROPOSITION 2 (Stegall, see [14]). *Every nondentable convex w^*-compact subset of a conjugate space has the tree property. All duals without* RNP *have the tree property.*

This leads to the natural questions:

PROBLEM 1. Is it true that every Banach space failing RNP has the tree-property?

PROBLEM 2. If a bounded, closed and convex set in a Banach space is not dentable, does it necessarily contain a tree?

Clearly an affirmative answer to Problem 2 would solve Problem 1 affirmatively. Unfortunately the answer to the second question is negative as will be shown in this paper and hence Problem 1 remains open.

If A is a subset of X and Y a subspace of X^*, we agree to call Y *A-norming* provided i^*: $X^{**} \to Y^*$ maps A viewed as a subset of X^{**} isometrically on $i^*(A)$, where i: $Y \to X^*$ is the canonical imbedding. The following is related to the problems mentioned above.

PROPOSITION 3. (1) *If C is a nondentable bounded, closed and convex subset of X and Y is a C-norming subspace of X^*, then either C has the tree-property or l^1 imbeds isomorphically in Y.*

(2) *If X is a separable Banach space failing* RNP, *then X has the tree-property provided c_0 is not isomorphic to a quotient-space of X.*

Proof. (1) If i: $Y \to X^*$ is the canonical imbedding, then $i^*(C)$ is a nondentable subset of Y^*. From Theorem 1 of [2], the required result is obtained.

(2) This is an immediate consequence of (1) and [19].

Counterexample to Problem 2. We will first establish a result about the convergence of martingales in finite-dimensional spaces.

If a is a positive real number, let $[a]$ denotes its integer part. For each integer d, $l^2(d)$ will be the d-dimensional euclidean space.

LEMMA 1. *Let $d \in N$ and $(\xi_k, \Sigma_k)_k$ be an $l^2(d)$-valued martingale on a probability space (Ω, Σ, μ), which is uniformly bounded by $M > 0$. Then for every $\varepsilon > 0$ there exists some $k \leqslant [M^2\varepsilon^{-2}]+1$ satisfying $\|\xi_k - \xi_{k+1}\|_1 < \varepsilon$.*

Proof. Since for every coordinate $i = 1, \ldots, d$, $(\xi_k^i)_k$ is a real martingale, we obtain $\int \xi_k^i \xi_{k+1}^i \, d\mu = \|\xi_k^i\|_2^2$ and hence

$$\|\xi_{k+1}^i - \xi_k^i\|_2^2 = \|\xi_{k+1}^i\|_2^2 - \|\xi_k^i\|_2^2.$$

Therefore

$$\|\xi_{k+1} - \xi_k\|_2^2 = \sum_{i=1}^{d} \|\xi_{k+1}^i - \xi_k^i\|_2^2 = \|\xi_{k+1}\|_2^2 - \|\xi_k\|_2^2.$$

Now suppose $n \in N$ such that $\|\xi_{k+1} - \xi_k\|_1 \geqslant \varepsilon$ for all $k = 1, \ldots, n$. Then we find:

$$n\varepsilon^2 \leqslant \sum_{k=1}^{n} \|\xi_{k+1} - \xi_k\|_1^2 \leqslant \sum_{k=1}^{n} \|\xi_{k+1} - \xi_k\|_2^2 = \|\xi_{n+1}\|_2^2 - \|\xi_1\|_2^2 \leqslant M^2$$

and hence $n \leqslant M^2\varepsilon^{-2}$. This completes the proof.

PROPOSITION 4. *Let F be a real d-dimensional Banach space and let $(\xi_k, \Sigma_k)_k$ be an F-valued martingale on a probability space (Ω, Σ, μ) which is uniformly bounded by $M > 0$. Then for every $\varepsilon > 0$ there exists some $k \leqslant [M^2 d^2 \varepsilon^{-2}]+1$ satisfying $\|\xi_{k+1} - \xi_k\|_1 < \varepsilon$.*

Proof. It is known that F admits a biorthogonal sequence $(e_i, e_i^*)_{1\leqslant i\leqslant d}$ with $\|e_i\| = \|e_i^*\| = 1$. If T: $F \to l^2(d)$ is the operator defined by $Tx = (e_1^*(x), \ldots, e_d^*(x))$, then it is easily verified that $\|T\| \leqslant \sqrt{d}$ and $\|T^{-1}\| \leqslant \sqrt{d}$. Since the $l^2(d)$-valued martingale $(T\xi_k, \Sigma_k)_k$ is bounded by $\sqrt{d}\,M$, there is some $k \leqslant [M^2 d^2 \varepsilon^{-2}]+1$ so that $\|T\xi_{k+1} - T\xi_k\|_1 < \varepsilon/\sqrt{d}$ and thus $\|\xi_{k+1} - \xi_k\|_1 < \varepsilon$.

For all $r, s \in N$ with $r \leqslant s$, we let $B_{r,s}$ be the Banach space l^∞. Define $B = \bigoplus_{r,s\geqslant r} B_{r,s}$ as the l^∞-sum of the spaces $B_{r,s}$. If $r \leqslant s$, then there is a natural projection $\pi_{r,s}$: $B \to B_{r,s}$. Take $\|x\|_{r,s} = \|\pi_{r,s}(x)\|$, $\|x\|_r = \sup_{s\geqslant r} \|x\|_{r,s}$ and $\|x\|^r = \sup_{1\leqslant \varrho\leqslant r} \|x\|_\varrho$ for each $x \in B$.

The next section is devoted to the proof of the following result.

THEOREM 1. *Let η: $N\times N \to N$ be any function which increases in both variables.*

Then there exist a subspace X of B, a sequence $(e_I)_I$ in B and for each $s \in N$ an operator φ_s: $B \to B$, such that the following conditions are satisfied:

(1) $X = \overline{\operatorname{span}}(e_I; I)$;

(2) $\|e_I\|_r \leqslant r^{-1}$;

(3) $\{e_I; I\}$ *is a bush in* X;

(4) *The restriction $\varphi_s|X$ of each operator φ_s has finite rank, which is denoted by* rk_s;

(5) $\|\varphi_s\| \leqslant 3$;

(6) $x = \lim_{s\to\infty} \varphi_s(x)$ *for all* $x \in X$;

(7) *If* $r \leqslant s$, $n \leqslant \eta(\mathrm{rk}_s, r)$, $x_1, \ldots, x_n \in \bar{c}(e_I; I)$ *and* $\lambda_1, \ldots, \lambda_n \geqslant 0$, *then the inequality*

$$\Big\| \sum_{m=1}^{n} \lambda_m x_m - \sum_{m=1}^{n} \lambda_m \varphi_s(x_m) \Big\|_{r+1,s+1} \geqslant \frac{1}{r+1} \sum_{m=1}^{n} \lambda_m \|x_m - \varphi_s(x_m)\|^r$$

holds.

Using Theorem 1, a counterexample to Problem 2 will be obtained. More precisely:

LEMMA 2. *Define* $\eta\colon N \times N \to N$ *by* $\eta(q, r) = 2^{9q^2r^2+2}$ *and let* X, $(e_I)_I$ *and* $(\varphi_s)_s$ *be as is Theorem* 1. *Then* $C = \bar{c}(e_I; I)$ *is a nondentable subset of* X *without the tree-property.*

Proof. Remark that $\|x\|_r \leqslant r^{-1}$ if $x \in C$ and in particular C is contained in the unit ball of X. The fact that C is not dentable is an immediate consequence of (3). It remains to show that C does not possess the tree-property. If we assume the converse, then there is an $\varepsilon > 0$ and a system $(x_{k,l})_{k,l=1,\ldots,2^k}$ in C such that $x_{k,l} = \frac{1}{2}x_{k+1,2l-1} + \frac{1}{2}x_{k+1,2l}$ and $\|x_{k+1,2l-1} - x_{k+1,2l}\| > \varepsilon$.

For each k we let Σ_k be the algebra of subsets of $[0, 1[$ generated by the intervals $[(l-1)2^{-k}, l2^{-k}[$ $(l = 1, \ldots, 2^k)$ and $\xi_k = \Sigma_l x_{k,l} 1_{[(l-1)2^{-k}, l2^{-k}[}$.

Then $(\xi_k, \Sigma_k)_k$ is clearly an X-valued martingale on the Lebesgue space $[0, 1[$ and is uniformly bounded by 1. Hence by (4) and (5), for each s, the martingale $(\varphi_s \xi_k, \Sigma_k)_k$ ranges in an rk_s-dimensional subspace of B and is uniformly bounded by 3. Take $r = [4\varepsilon^{-1}]$ and $k_s = 9\,\mathrm{rk}_s^2 r^2 + 1$ for each $s \geqslant r$. Clearly $\|x_{k,l} - x_{k+1,2l-1}\|^r > \varepsilon/2$, $\|x_{k,l} - x_{k+1,2l}\|^r > \varepsilon/2$ and therefore $\|\xi_k(t) - \xi_{k+1}(t)\|^r > \varepsilon/2$ for $t \in [0, 1[$.

If $s \geqslant r$ and $k \leqslant k_s$, then $2^k < 2^{k+1} \leqslant \eta(\mathrm{rk}_s, r)$ and we deduce from (7)

$$\|x_{0,1} - \varphi_s(x_{0,1})\| \geqslant \|\Sigma_l 2^{-k} x_{k,l} - \Sigma_l 2^{-k} \varphi_s(x_{k,l})\|_{r+1,s+1}$$

$$\geqslant \frac{1}{r+1} \Sigma_l 2^{-k} \|x_{k,l} - \varphi_s(x_{k,l})\|^r = \frac{1}{r+1} \int \|\xi_k(t) - \varphi_s \xi_k(t)\|^r\, dt$$

and the same with k replaced by $k+1$. This implies that

$$2\|x_{0,1} - \varphi_s(x_{0,1})\|$$

$$\geqslant \frac{1}{r+1} \int \|\xi_{k+1}(t) - \xi_k(t)\|^r\, dt - \frac{1}{r+1} \int \|\varphi_s \xi_{k+1}(t) - \varphi_s \xi_k(t)\|^r\, dt$$

$$> \frac{\varepsilon}{2(r+1)} - \frac{1}{r+1} \|\varphi_s \xi_{k+1} - \varphi_s \xi_k\|_1.$$

By Proposition 4, we can find for each $s \geqslant r$ some $k \leqslant k_s$ so that $\|\varphi_s \xi_{k+1} - \varphi_s \xi_k\|_1 < 1/r$. It follows that $\|x_{0,1} - \varphi_s(x_{0,1})\| > 1/2r(r+1)$, for all $s \geqslant r$. From (6), we get the required contradiction.

Proof of Theorem 1. Let m denote the Lebesgue measure on $[0, 1]$. By induction we define sequences $(\sigma_s)_s$ and $(\Omega_s)_s$ of positive integers, taking

$$\sigma_1 = \Omega_1 = 1, \qquad \sigma_{s+1} > \eta(\Omega_s, 1) \ldots \eta(\Omega_s, s) \qquad \text{and} \qquad \Omega_{s+1} = \Omega_s \sigma_{s+1}.$$

For each integer s, let $\mathscr{G}_s$ be the algebra of subsets of $[0, 1[$ generated by the intervals $[(i-1)\Omega_s^{-1}, i\Omega_s^{-1}[$ where $i = 1, \ldots, \Omega_s$, which will be called *s-primitive*. We consider the algebra $\mathscr{G} = \bigcup_s \mathscr{G}_s$. We say that an interval is *primitive* if it is s-primitive for some s. Remark that 2 primitive intervals are either disjoint or comparable. For each integer s, we let $\iota_s\colon \mathscr{G} \to \mathscr{G}_s$ be the mapping defined by $\iota_s S = \bigcup_{A \in \mathscr{G}_s, A \subset S} A$. For all $s \geqslant r$ we introduce a subfamily $\mathscr{S}_{r,s}$ of $\mathscr{G}$. We proceed by induction on r:

Let $\mathscr{S}_{1,s}$ $(s \geqslant 1)$ consist of the intervals which are t-primitive for some $t \geqslant s$. Assume now $(\mathscr{S}_{1,s})_{s \geqslant 1}, \ldots, (\mathscr{S}_{r,s})_{s \geqslant r}$ obtained. $\mathscr{S}_{r+1,s}$ $(s \geqslant r+1)$ will contain all sets of the form $\bigcup_{k=1}^{n} (S_k \setminus \iota_{s-1} S_k)$, where $n \leqslant \eta(\Omega_{s-1}, r)$ and

$$S_k \in \bigcup_{t \geqslant 1} \mathscr{S}_{1,t} \cup \ldots \cup \bigcup_{t \geqslant r} \mathscr{S}_{r,t} \quad \text{for each } k = 1, \ldots, n.$$

Take $\mu_{1,s} = \Omega_s^{-1}$ for $s \geqslant 1$ and $\mu_{r,s} = \eta(\Omega_{s-1}, 1) \ldots \eta(\Omega_{s-1}, r-1)\Omega_s^{-1}$ for $s \geqslant r > 1$.

The reader will verify that $\mu_{r,s}$ increases when r increases and decreases when s increases. Moreover, we have

LEMMA 3. *If* $r \in N$, *then* $\lim\limits_{s \geqslant r, s \to \infty} \mu_{r,s} = 0$.

LEMMA 4. *If* $S \in \mathscr{S}_{r,s}$ $(s \geqslant r)$ *and* $t \geqslant s$, *then* $m(S) \leqslant \mu_{r,s}$ *and* $m(S \setminus \iota_t S) \leqslant \mu_{r,t+1}$.

Proof. (By induction on r.)

For $r = 1$, the statement is almost obvious.

Assume now the property true for $\varrho = 1, \ldots, r$ and let $S \in \mathscr{S}_{r+1,s}$ $(s \geqslant r+1)$. Then $S = \bigcup_{k=1}^{n} (S_k \setminus \iota_{s-1} S_k)$ where $n \leqslant \eta(\Omega_{s-1}, r)$ and $S_k \in \bigcup_{u \geqslant 1} \mathscr{S}_{1,u} \cup \ldots \cup \bigcup_{u \geqslant r} \mathscr{S}_{r,u}$ for all $k = 1, \ldots, n$.

We show that if $1 \leqslant \varrho \leqslant r$, $T \in \mathscr{S}_{\varrho,u}$ $(u \geqslant \varrho)$ and $v+1 \geqslant r$, then $m(T \setminus \iota_v T) \leqslant \mu_{r,v+1}$. We use the induction hypothesis and distinguish 2 cases:

(i) $v < u \Rightarrow m(T \setminus \iota_v T) \leqslant m(T) \leqslant \mu_{\varrho,u} \leqslant \mu_{\varrho,v+1} \leqslant \mu_{r,v+1}$;

(ii) $v \geqslant u \Rightarrow m(T \setminus \iota_v T) \leqslant \mu_{\varrho,v+1} \leqslant \mu_{r,v+1}$.

From this it follows that $m(S) \leqslant \eta(\Omega_{s-1}, r)\mu_{r,s} = \mu_{r+1,s}$. Let further $t \geqslant s$. It is clear that

$$S \setminus \iota_t S \subset \bigcup_{k=1}^{n} \big[S_k \setminus [\iota_{s-1} S_k \cup \iota_t(S_k \setminus \iota_{s-1} S_k)]\big].$$

But since $\iota_{s-1}S_k \cup \iota_t(S_k \setminus \iota_{s-1} S_k) = \iota_t S_k$, it follows that $S\setminus \iota_t S \subset \bigcup_{k=1}^{n}(S_k\setminus \iota_t S_k)$ and hence

$$m(S\setminus \iota_t S) \leqslant \eta(\Omega_{s-1}, r)\mu_{r,t+1} \leqslant \eta(\Omega_t, r)\mu_{r,t+1} = \mu_{r+1,t+1}.$$

So the proof is complete.

LEMMA 5. *If $S \in \mathscr{S}_{r,s+1}$ $(s+1 \geqslant r)$, then $\iota_s S = \emptyset$.*

Proof. By Lemma 4, $m(S) \leqslant \mu_{r,s+1} < \Omega_s^{-1}$, the measure of the s-primitive intervals.

From Lemma 3 and Lemma 4, we obtain

LEMMA 6. *If $r \in N$, then $\lim_{s\to\infty} \sup_{S\in\mathscr{S}_{r,s}} m(S) = 0$.*

It is clear that for $s \geqslant r$ the family $\mathscr{S}_{r,s}$ is infinite and therefore can be identified with N. For each primitive interval I, we introduce an element e_I of $B = \bigoplus_{r,s\geqslant r} B_{r,s}$ by taking $e_I^{r,s,S} = \frac{1}{r}\frac{m(I\cap S)}{m(I)}$ for $S \in \mathscr{S}_{r,s}$. Obviously $\|e_I\|_r \leqslant 1/r$.

Let $X = \overline{\operatorname{span}}\,(e_I;\ I \text{ primitive})$.

LEMMA 7. *$(e_I)_I$ is a bush in X.*

Proof. If I is an s-primitive interval, then I is the disjoint union of $\sigma = \sigma_{s+1}$ $(s+1)$-primitive intervals $I_1, \ldots, I_\sigma$. It is easily verified that

$$e_I = \frac{m(I_1)}{m(I)}e_{I_1} + \ldots + \frac{m(I_\sigma)}{m(I)}e_{I_\sigma} = \frac{1}{\sigma}e_{I_1} + \ldots + \frac{1}{\sigma}e_{I_\sigma}.$$

For each $n = 1, \ldots, \sigma$ we have that

$$\|e_I - e_{I_n}\| \geqslant |e_I^{1,s+1,I_n} - e_{I_n}^{1,s+1,I_n}| = \frac{\sigma-1}{\sigma} > \tfrac{1}{2}.$$

For each $t \in N$ we define $\varphi_t\colon B \to B$ by

$$\varphi_t(x)^{r,s,S} = x^{r,s,S} - \frac{r+1}{r}x^{r+1,t+1,S\setminus\iota_t S} \quad \text{if} \quad r \in N,\ s \geqslant r \text{ and } S \in \mathscr{S}_{r,s}.$$

Obviously φ_t is a linear operator and $\|\varphi_t\| \leqslant 3$.

LEMMA 8. *The rank rk_t of the restriction $\varphi_t|X$ is at most Ω_t.*

Proof. In fact $\varphi_t(X) = \operatorname{span}\,(\varphi_t(e_I); I\ t\text{-primitive})$. To see this, let J be a u-primitive interval. If $u < t$, then using Lemma 7, we obtain $e_J \in c(e_I;\ I\ \ t\text{-primitive})$ and hence $\varphi_t(e_J) \in \operatorname{span}(\varphi_t(e_I);\ I\ \ t\text{-primitive})$. If $u \geqslant t$, then J is contained in some t-primitive interval I. For $r \in N$, $s \geqslant r$ and $S \in \mathscr{S}_{r,s}$, we find

$$\varphi_t(e_J)^{r,s,S} = \frac{1}{r}\frac{m(J\cap S)}{m(J)} - \frac{1}{r}\frac{m(J\cap(S\setminus\iota_t S))}{m(J)} = \frac{1}{r}\frac{m(J\cap \iota_t S)}{m(J)}$$
$$= \frac{1}{r}\frac{m(I\cap\iota_t S)}{m(I)} = \varphi_t(e_I)^{r,s,S},$$

showing that $\varphi_t(e_J) = \varphi_t(e_I)$.

LEMMA 9. *$x = \lim_{t\to\infty}\varphi_t(x)$ for each $x \in X$.*

Proof. Of course we can take $x = e_I$ where I is primitive. If $r \in N$, $s \geqslant r$ and $S \in \mathscr{S}_{r,s}$, then

$$(x - \varphi_t x)^{r,s,S} = \frac{1}{r}\frac{m(I\cap(S\setminus\iota_t S))}{m(I)}.$$

Choose $\varepsilon > 0$ and take $r_0 \in N$ with $r_0^{-1} \leqslant \varepsilon$. Take then $t_0 \geqslant r_0$ such that $m(T) \leqslant m(I)\varepsilon$ whenever $r \leqslant r_0$, $t \geqslant t_0$ and $T \in \mathscr{S}_{r,t}$, which is possible by Lemma 6. We claim that $\|x - \varphi_t(x)\| \leqslant \varepsilon$ if $t \geqslant t_0$. Let thus $r \in N$, $s \geqslant r$ and $S \in \mathscr{S}_{r,s}$. If $r \geqslant r_0$, then $|(x-\varphi_t x)^{r,s,S}| \leqslant r^{-1} \leqslant r_0^{-1} \leqslant \varepsilon$. If $r < r_0$, then $r+1 \leqslant r_0$ and $m(S\setminus\iota_t S) \leqslant m(I)\varepsilon$, since $S\setminus\iota_t S \in \mathscr{S}_{r+1,t+1}$. Therefore also $|(x-\varphi_t x)^{r,s,S}| \leqslant \varepsilon$.

It remains to verify condition (7) of the theorem. By Lemma 8, it will be enough to prove

LEMMA 10. *If $r \leqslant s$, $n \leqslant \eta(\Omega_s, r)$, $x_1, \ldots, x_n \in \bar{c}(e_I;\ I$ primitive$)$ and $\lambda_1, \ldots, \lambda_n \geqslant 0$, then*

$$\Big\|\sum_{m=1}^{n}\lambda_m x_m - \sum_{m=1}^{n}\lambda_m\varphi_s(x_m)\Big\|_{r+1,s+1} \geqslant \frac{1}{r+1}\sum_{m=1}^{n}\lambda_m\|x_m - \varphi_s(x_m)\|^r.$$

Proof. Clearly we may assume $x_1, \ldots, x_n \in c(e_I;\ I$ primitive$)$. Let $x_m = \Sigma_I'\beta_{m,I}e_I$ where $\beta_{m,I} \geqslant 0$ and $\Sigma_I'\beta_{m,I} = 1$ for each $m = 1, \ldots, n$.

Take $\delta > 0$. For each $m = 1, \ldots, n$, there is some $\varrho_m = 1, \ldots, r$, some $s_m \geqslant \varrho_m$ and some $S_m \in \mathscr{S}_{\varrho_m,s_m}$ such that

$$\|x_m - \varphi_s x_m\|^r \leqslant |(x_m - \varphi_s x_m)^{\varrho_m,s_m,S_m}| + \delta.$$

Since $(x_m-\varphi_s x_m)^{\varrho_m,s_m,S_m} = \Sigma_I'\beta_{m,I}(e_I-\varphi_s e_I)^{\varrho_m,s_m,S_m} = \Sigma_I'\beta_{m,I}\dfrac{m(I\cap(S_m\setminus\iota_s S_m))}{\varrho_m\cdot m(I)}$,

we obtain

$$\sum_{m=1}^{n}\lambda_m\|x_m - \varphi_s x_m\|^r \leqslant \sum_{m=1}^{n}\lambda_m\Sigma_I'\beta_{m,I}\frac{m(I\cap(S_m\setminus\iota_s S_m))}{m(I)} + \delta\sum_{m=1}^{n}\lambda_m.$$

Now by definition of $\mathscr{S}_{r+1,s+1}$, we have that $S = \bigcup_{m=1}^{n} (S_m \setminus \iota_s S_m) \in \mathscr{S}_{r+1,s+1}$. Since $\iota_s S = \emptyset$ by Lemma 5, it follows

$$\sum_{m=1}^{n} \lambda_m \|x_m - \varphi_s(x_m)\|^r$$

$$\leqslant \sum_{m=1}^{n} \lambda_m \Sigma_I' \beta_{m,I} \frac{m(I \cap S)}{m(I)} + \delta \sum_{m=1}^{n} \lambda_m$$

$$= \sum_{m=1}^{n} \lambda_m \Sigma_I' \beta_{m,I} (r+1)(e_I - \varphi_s e_I)^{r+1,s+1,S} + \delta \sum_{m=1}^{n} \lambda_m$$

$$= (r+1) \sum_{m=1}^{n} \lambda_m (x_m - \varphi_s x_m)^{r+1,s+1,S} + \delta \sum_{m=1}^{n} \lambda_m$$

$$\leqslant (r+1) \Big\| \sum_{m=1}^{n} \lambda_m x_m - \sum_{m=1}^{n} \lambda_m \varphi_s(x_m) \Big\|_{r+1,s+1} + \delta \sum_{m=1}^{n} \lambda_m .$$

Since $\delta > 0$ was arbitrarily choosen, the proof is complete.

Added in proof. Problem 1 is shown to have also negative solution, by some joint work of H. P. Rosenthal and the author: *Martingales valued in certain subspaces of L^1*, to appear in Israel Journal Math.

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VRIJE UNIVERSITEIT BRUSSEL
DEPARTMENT OF MATHEMATICS

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