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Kac functional and Schrödinger equation

by

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1. Introduction. Let $X = \{x(t),\ t \geqslant 0\}$ be a strong Markov process with continuous paths on $\boldsymbol{R} = (-\infty, +\infty)$. Such a process is often called a *diffusion*. For each real b, we define the hitting time τ_b as follows:

$$\tau_b = \inf\{t > 0 \mid x(t) = b\}. \tag{1}$$

Let P_a and E_a denote as usual the basic probability and expectation associated with paths starting from a. It is assumed that for every a and b, we have

$$P_a\{\tau_b < \infty\} = 1. \tag{2}$$

Now let q be a bounded Borel measurable function on $\boldsymbol{R}$, and write for brevity

$$e(t) = \exp \int_0^t q(x(s))\,ds. \tag{3}$$

This is a multiplicative functional introduced by M. Kac in [3]. In this paper we study the quantity

$$u(a, b) = E_a\{e(\tau_b)\}. \tag{4}$$

Since q is bounded below, (2) implies that $u(a, b) > 0$ for every a and b, but it may be equal to $+\infty$. A fundamental property of u is given by

$$u(a, b)u(b, c) = u(a, c), \tag{5}$$

valid for $a < b < c$, or $a > b > c$. This is a consequence of the strong Markov property (SMP).

2. Probabilistic investigation. We begin by defining two abscissas of finiteness, one for each direction.

$$\begin{aligned} \beta &= \inf\{b \in \boldsymbol{R} \mid \exists a < b:\ u(a, b) = \infty\} \\ &= \sup\{b \in \boldsymbol{R} \mid \forall a < b:\ u(a, b) < \infty\}; \\ \alpha &= \sup\{a \in \boldsymbol{R} \mid \exists b > a:\ u(b, a) = \infty\} \\ &= \inf\{a \in \boldsymbol{R} \mid \forall b > a:\ u(b, a) < \infty\}. \end{aligned} \tag{6}$$

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It is possible, e.g., that $\beta = -\infty$ or $+\infty$. The first case occurs when X is the standard Brownian motion, and $q(x) \equiv 1$; for then, $u(a, b) \geqslant E_a(\tau_b) = \infty$, for any $a \neq b$.

LEMMA 1. *We have*

$$\beta = \inf\{b \in \boldsymbol{R} \mid \forall a < b\colon u(a, b) = \infty\}$$
$$= \sup\{b \in \boldsymbol{R} \mid \exists a < b\colon u(a, b) < \infty\};$$
$$a = \sup\{a \in \boldsymbol{R} \mid \forall b > a\colon u(b, a) = \infty\}$$
$$= \inf\{a \in \boldsymbol{R} \mid \exists b > a\colon u(b, a) < \infty\}.$$

Proof. It is sufficient to prove the first equation above for β, because the second is trivially equivalent to it, and the equations for a follow by similar arguments. Suppose $u(a, b) = \infty$; then for $x < a < b$ we have $u(x, b) = \infty$ by (5). For $a < x < b$ we have by SMP,

$$u(x, b) \geqslant E_x\{e(\tau_a);\ \tau_a < \tau_b\} u(a, b) = \infty$$

since $P_x\{\tau_a < \tau_b\} > 0$ in consequence of (2).

The next lemma is a martingale argument. Let $\mathfrak{F}_t$ be the σ-field generated by $\{x_s,\ 0 \leqslant s \leqslant t\}$ and all null sets, so that $\mathfrak{F}_{t+} = \mathfrak{F}_t$ for $t \geqslant 0$; and for any optional τ let $\mathfrak{F}_{\tau+}$, $\mathfrak{F}_\tau$, $\mathfrak{F}_{\tau-}$ have the usual meanings.

LEMMA 2. *If $a < b < \beta$, then*

(7) $$\lim_{a \uparrow b} u(a, b) = 1;$$

(8) $$\lim_{b \downarrow a} u(a, b) = 1.$$

Proof. Let $a < b_n \uparrow b$ and consider

(9) $$E_a\{e(\tau_b) \mid \mathfrak{F}(\tau_{b_n})\}, \quad n \geqslant 1.$$

Since $b < \beta$, $u(a, b) < \infty$ and the sequence in (9) forms a martingale. As $n \uparrow \infty$, $\tau_{b_n} \uparrow \tau_b$ a.s. and $\mathfrak{F}(\tau_{b_n}) \uparrow \mathfrak{F}(\tau_b^-)$. Since $e(\tau_b) \in \mathfrak{F}(\tau_b^-)$, the limit of the martingale is a.s. equal to $e(\tau_b)$. On the other hand, the conditional probability in (9) is also equal to

$$E_a\left\{e(\tau_b)\exp\left(\int_{\tau_{b_n}}^{\tau_b} q(x(s))\, ds\right) \Big| \mathfrak{F}(\tau_{b_n})\right\} = e(\tau_{b_n}) u(b_n, b).$$

As $n \uparrow \infty$, this must then converge to $e(\tau_b)$ a.s.; since $e(\tau_{b_n})$ converges to $e(\tau_b)$ a.s., we conclude that $u(b_n, b) \to 1$. This establishes (7).

Now let $\beta > b > a_n \downarrow a$, and consider

(10) $$E_a\{e(\tau_b) \mid \mathfrak{F}(\tau_{a_n})\}, \quad n \geqslant 1.$$

This is again a martingale. Although $a \to \tau_a$ is a.s. left continuous, not right continuous, for each fixed a we do have $\tau_{a_n} \downarrow \tau_a$ and $\mathfrak{F}(\tau_{a_n}) \downarrow \mathfrak{F}(\tau_a)$.

Hence we obtain as before $u(a_n, b) \to u(a, b)$ and consequently

$$u(a, a_n) = \frac{u(a, b)}{u(a_n, b)} \to 1.$$

This establishes (8).

The next result illustrates the basic probabilistic method.

THEOREM 1. *The following three propositions are equivalent:*

(i) $\beta = +\infty$;

(ii) $a = -\infty$;

(iii) *For every a and b, we have*

(11) $$u(a, b) u(b, a) \leqslant 1.$$

Proof. Suppose $x(0) = b$ and let $a < b < c$. If (i) is true, then $u(b, c) < \infty$ for every $c > b$. Define a sequence of successive hitting times T_n as follows (where θ denotes the usual shift operator):

$$S = \begin{cases} \tau_a & \text{if} \quad \tau_a < \tau_c, \\ \infty & \text{if} \quad \tau_c < \tau_a; \end{cases}$$

(12) $$T_0 = 0, \qquad T_1 = S,$$
$$T_{2n} = T_{2n-1} + \tau_b \circ \theta_{T_{2n-1}}, \qquad T_{2n+1} = T_{2n} + S \circ \theta_{T_{2n}},$$

for $n \geqslant 1$. Define also

(13) $$N = \min\{n \geqslant 0 \mid T_{2n+1} = \infty\}.$$

It follows from $P_b\{\tau_c < \infty\} = 1$ that $0 \leqslant N < \infty$ a.s. For $n \geqslant 0$, we have

(14) $$E_b\{e(\tau_c)\colon N = n\} = E_b\left\{\exp\left(\sum_{k=0}^{2n} \int_{T_k}^{T_{k+1}} q(x(s))\, ds\right)\right\}$$
$$= E_b\{e(\tau_a);\ \tau_a < \tau_c\}^n\, E_a\{e(\tau_b)\}^n E_b\{e(\tau_c);\ \tau_c < \tau_a\}.$$

Since the sum of the first term in (14) over $n \geqslant 0$ is equal to $u(b, c) < \infty$, the sum of the last term in (14) must converge. Thus we have

(15) $$E_b\{e(\tau_a);\ \tau_a < \tau_c\} u(a, b) < 1.$$

Letting $c \to \infty$ we obtain (11). Hence $u(b, a) < \infty$ for every $a < b$ and so (ii) is true. Exactly the same argument shows that (ii) implies (iii) and so also (i).

We are indebted to R. Durrett for ridding the next lemma of a superfluous condition.

LEMMA 3. *Given any $a \in \boldsymbol{R}$ and $Q > 0$, there exists an $\varepsilon = \varepsilon(a, Q)$ such that*

(16) $$E_a\{e^{Q\sigma_\varepsilon}\} < \infty$$

where $\sigma_\varepsilon = \inf\{t > 0 \mid x(t) \notin (a-\varepsilon, a+\varepsilon)\}$.

Proof. Since X is strong Markov and has continuous paths, there is no "stable" point. This implies $P_a\{\sigma_\varepsilon \geqslant 1\} \to 0$ as $\varepsilon \to 0$ and so there exists ε such that

$$P_a\{\sigma_\varepsilon \geqslant 1\} < e^{-(Q+1)}. \tag{17}$$

Now σ_ε is a terminal time, so $x \to P_x\{\sigma_\varepsilon \geqslant 1\}$ is an excessive function for the process X killed at σ_ε. Hence by standard theory it is finely continuous. By for a diffusion under hypothesis (2) it is clear that fine topology coincides with the Euclidean. Thus $x \to P_x\{\sigma_\varepsilon \geqslant 1\}$ is in fact continuous. (1) It now follows that we have, further decreasing ε if necessary:

$$\sup_{|x-a|<\varepsilon} P_x\{\sigma_\varepsilon \geqslant 1\} < e^{-(Q+1)}. \tag{18}$$

A familiar inductive argument then yields for all $n \geqslant 1$

$$P_a\{\sigma_\varepsilon \geqslant n\} < e^{-n(Q+1)} \tag{19}$$

and (16) follows.

Lemma 4. *For any* $a < \beta$ *we have*

$$u(a, \beta) = \infty; \tag{20}$$

for any $b > a$ *we have* $u(b, a) = \infty$.

Proof. We will prove that if $u(a, b) < \infty$, then there exists $c > b$ such that $u(b, c) < \infty$. This implies (20) by Lemma 1, and the second assertion is proved similarly.

Let $Q = \|q\|$. Given b we choose a and b so that $a < b < d$ and

$$E_b\{e^{Q(\tau_a \wedge \tau_d)}\} < \infty. \tag{21}$$

This is possible by Lemma 3. Now let $b < c < d$; then as $c \downarrow b$ we have

$$E_b\{e(\tau_a);\ \tau_a < \tau_c\} \leqslant E_b\{e^{Q(\tau_a \wedge \tau_d)};\ \tau_a < \tau_c\} \to 0 \tag{22}$$

because $P_b\{\tau_a < \tau_c\} \to 0$. Hence there exists c such that

$$E_b\{e(\tau_a);\ \tau_a < \tau_c\} < \frac{1}{u(a, b)}. \tag{23}$$

This is just (15) above, and so reversing the argument there, we conclude that the sum of the first term in (14) over $n \geqslant 0$ must converge. Thus $u(b, c) < \infty$, as was to be shown.

To sum up:

Theorem 2. *The function* $(a, b) \to u(a, b)$ *is continuous in the region* $a \leqslant b < \beta$ *and in the region* $a < b \leqslant a$. *Furthermore, extended continuity*

(1) This fact can also be proved in an elementary way.

holds in $a \leqslant b \leqslant \beta$ *and* $a \leqslant b \leqslant a$, *except at* (β, β) *when* $\beta < \infty$, *and at* (a, a) *when* $a > -\infty$.

Proof. To see that there is continuity in the extended sense at (a, β), where $a < \beta$, let $a < b_n \uparrow \beta$. Then we have by Fatou's lemma

$$\varliminf_{n\to\infty} u(a, b_n) \geqslant E_a\{\varliminf_{n\to\infty} e(\tau_{b_n})\} = E_a\{e(\tau_\beta)\} = u(a, \beta) = \infty.$$

If $\beta < \infty$, then $u(\beta, \beta) = 1$ by definition, but $u(a, \beta) = \infty$ for all $a < \beta$; hence u is not continuous at (β, β). The case for a is similar.

3. The Schrödinger equation. From now on the process X will be the standard Brownian motion on $\boldsymbol{R}$ and q will be bounded and continuous in $\boldsymbol{R}$. The Schrödinger equation

$$\tfrac{1}{2}\varphi''(x) + q(x)\varphi(x) = 0, \quad x \in \boldsymbol{R} \tag{24}$$

will be referred to as "the equation", and any of its solutions "a solution". The fundamental existence and uniqueness theorem for linear differential equations (with a Lipshitz condition) is applicable and guarantees a unique solution for given initial values $\varphi(a)$ and $\varphi'(a)$ for any $a \in \boldsymbol{R}$. A fortiori, it guarantees the unique extension of any solution given in a non-empty interval to a solution in $\boldsymbol{R}$.

Lemma 5. *Let* φ *be a solution in* $[a, b]$ *with* $\varphi(a) = \varphi(b) = 0$; *then* $b - a \geqslant (2Q)^{-1/2}$, *where* $Q = \|q\|$. *Let* $0 < b - a < (2Q)^{-1/2}$, $A > 0$, $B > 0$; *then there is a unique solution* φ *satisfying*

$$\varphi(a) = A, \quad \varphi(b) = B, \tag{25}$$

and $\varphi > 0$ *in* $[a, b]$.

Proof. The first assertion is known as de la Vallée Poussin's theorem. The second assertion then follows from a well-known criterion for the solvability of the equation with given boundary conditions. For both results see [5], pp. 91–92. The solution cannot vanish more than once by the first assertion. Nor can it vanish just once for then it must assume its minimum there and so Φ' must also vanish which is impossible by the uniqueness theorem.

Lemma 6. *Let* Φ *be a solution in* $\boldsymbol{R}$. *For* $-\infty < a < b < \infty$ *define*

$$\tau = \inf\{t \geqslant 0 \mid x(t) \notin [a, b]\} \tag{26}$$

and

$$M(t) = \Phi(x(t)) \exp \int_0^t q(x(s))\,ds. \tag{27}$$

Then $\{M(t \wedge \tau), \mathfrak{F}(t), t \geqslant 0\}$ *is a martingale.*

Proof. Using Ito's formula (see [2]), we have

$$(28)\qquad dM(t) = \exp\int_0^t q(x(s))\,ds \times \times\Big[\Phi'(x(t))\,dx(t) + \big(\tfrac{1}{2}\Phi''(x(t)) + q(x(t))\,\Phi(x(t))\big)\,dt\Big],$$

namely, for $t \geqslant 0$:

$$(29)\qquad M(t) - M(0) = \int_0^t \Big[\exp\int_0^s q(x(r))\,dr\Big]\Phi'(x(s))\,dx(s) + \int_0^t \Big[\exp\int_0^s q(x(r))\,dr\Big][\tfrac{1}{2}\Phi'' + q\Phi](x(s))\,ds.$$

If we substitute $t \wedge \tau$ for t in the above, the second term on the right vanishes because Φ is a solution. The first term is then of the form $\int_0^t f(s)\,dx(s)$ where

$$f(s) = \chi(s)\Big[\exp\int_0^s q(x(r))\,dr\Big]\,\Phi'(x(s)),$$

$$(30)\qquad \chi(s, w) = \begin{cases} 1 & \text{if} \quad s < \tau(w), \\ 0 & \text{if} \quad s \geqslant \tau(w). \end{cases}$$

Clearly, $f(s, w)$ is progressively measurable, being right continuous, and

$$(31)\qquad E_x\Big\{\int_0^t f(x)^2\,ds\Big\} < \infty, \qquad x \in \boldsymbol{R},$$

because φ' as well as φ is bounded in $[a, b]$. Thus the first term on the right side of (29) is an Ito integral, hence a martingale.

THEOREM 3. *Suppose $u(x, b) < \infty$ for some, hence all, $x < b$. Then $u(\cdot, b)$ is a solution in $(-\infty, b)$.*

Proof. Let $x_1 < x < x_2 < b$ where $x_2 - x_1 < (2Q)^{-1/2}$. Then by Lemma 5, there is a solution Φ satisfying

$$(32)\qquad \Phi(x_i) = u(x_i, b), \qquad i = 1, 2.$$

Let

$$\sigma = \inf\{t \geqslant 0 |\ x(t) \notin [x_1, x_2]\}$$

and M be as in (27). Then $M(t \wedge \sigma)$ is a martingale by Lemma 6. Hence for $t \geqslant 0$ we have

$$(33)\qquad \Phi(x) = E_x\{M(0)\} = E_x\{M(t \wedge \sigma)\} = E_x\{M(\sigma);\ \sigma < t\} + E_x\{M(t);\ \sigma \geqslant t\}.$$

By Lemma 5, we can choose $x_2 - x_1$ so small that

$$(34)\qquad E\{e^{Q\sigma}\} < \infty.$$

Then

$$E_x\{M(t);\ \sigma \geqslant t\} \sup_{x_1 \leqslant x \leqslant x_2} |\Phi| \int_{\{\sigma \geqslant t\}} e^{Q\sigma}\,dP_x$$

converges to zero as $t \to \infty$ by (34), since $\sigma < \infty$ almost surely. Therefore, if we let $t \to \infty$ in (33) and use (32) we obtain

$$(35)\qquad \Phi(x) = E_x\{M(\sigma)\} = E_x\{u(x(\sigma), b)\,e(\sigma)\}.$$

On the other hand, by the strong Markov property applied at σ $(< \tau_b)$, we see that

$$(36)\qquad u(x, b) = E_x\{e(\sigma)E_{x(\sigma)}[e(\tau_b)]\} = E_x\{e(\sigma)\,u(x(\sigma), b)\}.$$

Comparison of (35) with (36) yields $u(x, b) = \Phi(x)$. This being true for each $x < b$, Theorem 3 is established. See the Remarks at the end of the paper.

THEOREM 4. *Let Φ be any solution such that $\Phi(x) > 0$ for $x \in (-\infty, b]$. Then $b < \beta$, and we have*

$$(37)\qquad u(x, b) \leqslant \frac{\Phi(x)}{\Phi(b)}, \qquad -\infty < x \leqslant b.$$

In other words, $u(\cdot, b)$ is the minimal positive solution in $(-\infty, b)$ with $\lim_{x \uparrow b} u(x, b) = 1$.

Proof. Consider the M in (27) but write $\tau_{[a,b]}$ for the τ in (26). Then for each $x \in \boldsymbol{R}$ and $t \geqslant 0$, we have the martingale relation

$$(38)\qquad \Phi(x) = E_x\{M(0)\} = E_x\{M(t \wedge \tau_{[a,b]})\}.$$

If we keep b fixed but let $a \to -\infty$, then under P_x for $x \in (-\infty, b)$ we have $\tau_{[a,b]} \uparrow \tau_b$ and $M(t \wedge \tau_{[a,b]}) \to M(t \wedge \tau_b)$ by continuity. Since $M(t \wedge \tau_{[a,b]}) > 0$ because $\Phi > 0$ in $(-\infty, b]$, it follows from (38) by Fatou's lemma that

$$(39)\qquad \Phi(x) \geqslant E_x\{M(t \wedge \tau_b)\}, \qquad -\infty < x < b.$$

Letting $t \to \infty$, we obtain by another application of Fatou's lemma

$$(40)\qquad \Phi(x) \geqslant E_x\{M(\tau_b)\} = \Phi(b)\,u(x, b).$$

Thus (37) is true for $x < b$; for $x = b$ it reduces to a trivial equation.

As a corollary to Theorem 4, we can relate the two abscissas β and a to the solutions $u(\cdot, b)$. Suppose $\beta > -\infty$. For each $b < \beta$, $u(\cdot, b)$ is a solution in $(-\infty, b)$ by Theorem 3. By the fundamental existence and uniqueness theorem for linear differential equations, there is a solution Φ_b in $\boldsymbol{R}$ satisfying

(41) $$\Phi_b(x) = u(x, b), \qquad -\infty < x < b.$$

It follows from (7) that $\Phi_b(b) = 1$. Furthermore, the uniqueness theorem implies that for $b < c < \beta$ we have

(42) $$\Phi_c(x) = \Phi_b(x)u(b, c), \qquad x \in \boldsymbol{R}.$$

Thus the family of solutions $\{\Phi_b,\ b < \beta\}$ are linearly dependent.

Similarly if $\alpha < +\infty$, then there is a family of linearly dependent solutions $\{{}_a\Phi,\ a > \alpha\}$ satisfying

(43) $${}_a\Phi(x) = u(x, a), \qquad a < x < +\infty.$$

COROLLARY TO THEOREM 4. *For each $b < \beta$, β is the smallest root of Φ_b. For each $a > \alpha$, α is the largest root of ${}_a\Phi$.*

Proof. Fix $b < \beta$ and denote the smallest root of Φ_b by z. Since $\Phi_b > 0$ in $(-\infty, z)$, we must have $z \leqslant \beta$ by Theorem 4. On the other hand, for $b < c < \beta$ we have $\Phi_b(x) = \varphi_c(x)/u(b, c) > 0$ for $x \in (-\infty, c)$ by (42).

Hence $\beta \leqslant z$ and so $\beta = z$ as asserted. The assertion about α is proved similarly.

THEOREM 5. *The following propositions are all equivalent:*

(i) *There is a solution which is positive in $\boldsymbol{R}$;*

(ii) $\beta = +\infty$;

(iii) $\alpha = -\infty$;

(iv) *for every pair of real numbers a and b we have*

(44) $$u(a, b)u(b, a) \leqslant 1;$$

(v) *for a pair of real numbers a and b we have* (44).

Proof. The equivalence of (ii), (iii) and (iv) has already been proved for any diffusion in Theorem 1.

Let Φ be a positive solution in $\boldsymbol{R}$. Then Theorem 4 applies for every b in $\boldsymbol{R}$ and yields $\beta = +\infty$. By symmetry $\alpha = -\infty$.

If $\beta = +\infty$, then for any $b \in \boldsymbol{R}$, Φ_b has no root by Corollary to Theorem 4; hence it is a positive solution in $\boldsymbol{R}$. Similarly if $\alpha = -\infty$, then for any $a \in \boldsymbol{R}$, ${}_a\Phi$ is a positive solution in $\boldsymbol{R}$. We have thus proved the equivalence of (i) with (ii), (iii), and (iv).

It remains to prove that (v) implies (i). Let a and b satisfy (44) and $a < b$. Then

(45) $$\alpha < a < b < \beta$$

by the definition of α and β and Lemma 4. Bisecting the interval $[a, b]$

we deduce from (44) and (5) that $u(a_1, b_1)u(b_1, a_1) \leqslant 1$ for some a_1 and b_1 where $a \leqslant a_1 \leqslant b_1 \leqslant b$ and $b_1 - a_1 = (b-a)/2$. Continuing this process in the grand tradition of Bolzano–Weierstrass we see that there exist a_n and b_n such that $a_n \uparrow c$, $b_n \downarrow c$ where $c \in [a, b]$, and $u(a_n, b_n)u(b_n, a_n) \leqslant 1$. Since $a \leqslant a_n \leqslant b_n \leqslant b$ we may use (41), and (43) and (5) to obtain

(46) $$\Phi_b(a_n)\,{}_a\Phi(b_n) \leqslant \Phi_b(b_n)\,{}_a\Phi(a_n).$$

Writing, e.g., $\Phi_b(b_n) = \Phi_b(a_n) + \Phi_b'(c_n)(b_n - a_n)$ where $a_n \leqslant c_n \leqslant b_n$ by the mean value theorem, substituting into (46) and letting $n \to \infty$, we obtain

(47) $$W(c) = {}_a\Phi(c)\Phi_b'(c) - \Phi_b(c)\,{}_a\Phi'(c) \geqslant 0,$$

where W is the Wronskian of ${}_a\Phi$ and Φ_b. Since ${}_a\Phi$ and Φ_b are both solutions of the equation, it is an elementary fact that W is a constant. Now suppose for the sake of contradiction that Φ_b has a root; then by Corollary to Theorem 4 its smallest root is β. Since $\Phi_b > 0$ in $(-\infty, \beta)$, it is obvious that $\Phi_b'(\beta) < 0$; since $\beta > \alpha$, we have ${}_a\Phi(\beta) > 0$ by the definition of ${}_a\Phi$. Thus

(48) $$W(\beta) = {}_a\Phi(\beta)\Phi_b'(\beta) < 0,$$

which contradicts (47). Therefore, φ_b is a positive solution in $\boldsymbol{R}$ and (i) is proved.

Acknowledgement. Under the assumption (49) below, van Moerbeke proved that condition (v) implies that the solution w_1 in (50) has no zero. The proof above is modelled after his.

4. A particular case. In the analytical study of the Schrödinger equation (24), the following condition on q is often assumed:

(49) $$\int_{-\infty}^{\infty} |xq(x)|\,dx < \infty.$$

It is known (see e.g. [1], p. 284) that there are then two solutions w_1 and w_2 such that

(50) $$\lim_{x\to-\infty} w_1(x) = 1, \qquad \lim_{x\to+\infty} w_2(x) = 1.$$

Any solution v which tends to a finite limit as $x \to -\infty$ $[+\infty]$ must be a constant multiple of w_1 $[w_2]$. For v' must tend to zero at $-\infty$ $[+\infty]$, so that Wronskian of v and w_1 $[w_2]$ must vanish.

The probabilistic counterpart of the result above is given below.

THEOREM 6. *Under the assumption* (49) *we have for any $b < \beta$,*

(51) $$\lim_{x\to-\infty} u(x, b) < \infty$$

and for any $a > \alpha$,

$$\lim_{x\to+\infty} u(x, a) < \infty. \tag{52}$$

Proof. We use the trivial inequality, for $x \leqslant b$,

$$|u(x,b)-1| \leqslant \sum_{n=1}^{\infty} \frac{1}{n!} E_x\Big\{\Big[\int_0^{\tau_b} |q|(x(s))\,ds\Big]^n\Big\} = \sum_{n=1}^{\infty} M^{(n)}(x, b), \tag{53}$$

where $(t_0 = 0)$

$$M^{(n)}(x, b) = E_x\Big\{\prod_{j=1}^{n} \int_{t_{j-1}}^{\tau_b} |q|(x(t_j))\,dt_j\Big\}. \tag{54}$$

Put also

$$M^{(n)}(b) = \sup_{x\leqslant b} M^{(n)}(x, b),$$

then $M^{(n)}(b)$ is nondecreasing in b. Using the Markov property of x at t_{n-1} in (54), we obtain

$$M^{(n)}(x, b) \leqslant M^{(n-1)}(x, b)\, M^{(1)}(b) \leqslant [M^{(1)}(b)]^n. \tag{55}$$

Let τ be as in (26). Then it is part of the standard theory of Brownian motion that for $x \in [a, b]$ we have

$$E_x\Big\{\int_0^{\tau} |q|(x(t))\,dt\Big\} = \int_a^b G(x, y)|q|(y)\,dy, \tag{56}$$

where G is the Green's function for $[a, b]$, specifically (see, e.g. [2]),

$$G(x, y) = \begin{cases} \dfrac{(x-a)(b-y)}{b-a} & \text{if} \quad a \leqslant x \leqslant y \leqslant b; \\[2mm] \dfrac{(y-a)(b-x)}{b-a} & \text{if} \quad a \leqslant y \leqslant x \leqslant b. \end{cases}$$

Letting $a \to -\infty$ and using (49), we have by dominated convergence

$$M^{(1)}(x, b) = E_x\Big\{\int_0^{\tau_b} |q|(x(t))\,dt\Big\} = \int_{-\infty}^{b} [(b-x)\wedge(b-y)]\,|q|(y)\,dy. \tag{57}$$

Hence

$$M^{(1)}(b) \leqslant \int_{-\infty}^{b} (b-y)\,|q|(y)\,dy \tag{58}$$

which tends to zero as $b \to -\infty$ by (49). Choose b_0 so that

$$M^{(1)}(b_0) = \eta < 1. \tag{59}$$

Then we have by (53), (55) and (59) for $b < b_0$:

$$\sup_{x\leqslant b} |u(x, b)-1| \leqslant \frac{M^{(1)}(b)}{1-\eta},$$

and consequently the left member above tends to zero as $b \to -\infty$. This and the fundamental relation (5) for u implies (51) for $b = b_0$, hence also for all $b < \beta$ by the meaning of β (Lemma 1). This proves (51), and (52) is entirely similar.

Corollary. *We have the following bounds for β and α:*

$$\begin{aligned} \beta &\geqslant \sup\Big\{b \in \boldsymbol{R} \,\Big|\, \int_{-\infty}^{b} (b-y)|q|(y)\,dy \leqslant 1\Big\}, \\ \alpha &\leqslant \inf\Big\{a \in \boldsymbol{R} \,\Big|\, \int_{a}^{\infty} (y-a)|q|(y)\,dy \leqslant 1\Big\}. \end{aligned} \tag{60}$$

5. Remarks. In [3], Kac proved that if $p(t;\ a, b)$ denotes the fundamental solution of the partial differential equation:

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} + q(x)\varphi,$$

then

$$p(t;\ a, b)\,db = E_a\{e(t);\ x(t) \in db\},$$

where x is the standard Brownian motion on $\boldsymbol{R}$, and $e(t)$ is defined in (3). The associated semigroup has been called the *Kac semigroup*. It appears that Theorem 3 can be derived by Kac's original method using Laplace transforms (resolvents). This idea is due to Moerbeke, but a difficulty arises because $x \to u(x, b)$ in Theorem 3 is not necessarily bounded, so that a crucial dominated convergence required by the method appears missing. This was pointed out by Murali Rao. Although we have managed to overcome this difficulty by a moderate detour, the present approach to Theorems 3 and 4 throws more light on the "connections between probability theory and differential and integral equations" — Kac's original theme. We are indebted to Moerbeke for some stimulating discussions.

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Added in proof: For results in $\boldsymbol{R}^d$, $d \geqslant 1$ see K. L. Chung and K. M. Rao, *Sur la théorie du potentiel avec la fonctionnelle de Feynman–Kac*, Comp. Rend. Acad. Sci., Paris 290 (31 mars 1980).

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Multilinear singular integrals

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Abstract. This paper uses Fourier Transform and Mellin Transform analysis to obtain L^p estimates for certain multilinear singular integrals. The results obtained here extend estimates by Calderón, Coifman and Meyer on commutators of singular integrals to a wider class of multilinear singular integrals.

§ 1. Introduction. In this paper sharp estimates are obtained for operators of the type:

$$p.v. \int \prod_{j=1}^{n} \left\{ \frac{r_{m_j}(A_j;\, x,\, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y}\, dy \tag{1.1}$$

where

$$r_{m_j}(A_j;\ x,\ y) = A_j(x) - \sum_{k=1}^{m_j-1} \frac{A_j^{(k)}(y)(x-y)^k}{k!}.$$

These operators are related to those introduced by Calderón in [2] and [3] and studied by Coifman and Meyer in [5], [6] and [7]. We will sometimes denote these operators by $D^N H\{\prod_{j=1}^{n} r_{m_j}(A_j;\ x,\ \cdot)f(\cdot)\}$ where $D^N H$ is the Hilbert transform followed by the nth derivative. (The reason for this notation will become apparent in §3.)

The operators studied in this paper arise naturally from the study of higher commutators of differential and pseudo-differential operators. The simplest case is the commutator $[A, D^N H]$ where A is pointwise multiplication by the function $A(x)$. It has been shown by Calderón [3] that this commutator can be written as the sum of pseudo-differential operators of degree less than or equal to $N-1$ plus an operator of the type studied in this paper.

One can show that higher commutators of the form $[A_1, \ldots, [A_n, D^N H] \ldots]$ can be written as the sum of pseudo-differential operators of degree less than or equal to $N-n$ plus the sum of operators of the type considered in this paper. One example is the second commutator $[A, [B, D^N H]]$