ANNALES
POLONICI MATHEMATICI
XXXVIII (1980)

Regular fractional iteration of convex functions

by MAREK KUCZMA (Katowice)*

Abstract. The existence of a unique C^1 solution φ of equation (1) is proved under the condition that $f\colon I \to I$ is convex or concave and of class C^1 in I, $0 < f(x) < x$ in I^*, and $f'(x) > 0$ in I. Here $I = [0, a]$ or $[0, a)$, $0 < a \leqslant \infty$, and $I^* = I \setminus \{0\}$.

1. Let $I = [0, a]$ or $[0, a)$, $0 < a \leqslant \infty$, be a real interval, and write $I^* = I \setminus \{0\}$. Further, let $f\colon I \to I$ be a function of class C^1 in I such that $0 < f(x) < x$ in I^* and $f'(x) > 0$ in I. Put $s = f'(0) \in (0, 1]$.

Several authors (cf. [1], [3], [10], [11], [13], [14], [15]) have studied the C^1 solutions $\varphi\colon I \to I$ of the functional equation

$$\varphi^N(x) = f(x), \tag{1}$$

where $N \geqslant 2$ is a positive integer, and φ^N denotes the N-th iterate of the function φ. The solutions of (1) may be regarded as iterates of the fractional order $1/N$ of the function f.

In [10] one can find the first indication that the C^1 solution of equation (1) might be unique. The uniqueness was then proved, under various additional hypotheses, in [1], [13], [14] and [15]. (Cf. also [3], where the author applies equation (1) to a problem in astronomy.) On the other hand, as has been shown in [11], in the case where $s = 0$, C^1 solutions of equation (1) are, in general, not unique.

The existence of a unique C^1 solution to (1) has been proved in [13] under the additional hypothesis that the function f fulfils the condition

$$f'(x) = s + O(x^\delta), \quad x \to 0^+, \tag{2}$$

if $s \in (0, 1)$, resp.

$$f'(x) = 1 - b(m+1)x^m + O(x^{m+\delta}), \quad x \to 0^+, \tag{3}$$

if $s = 1$, where m, b and δ are positive constants. In [15] M. C. Zdun proved that the unique C^1 solution of (1) exists whenever the function f is convex or concave and $s \in (0, 1)$. However, his proof cannot be adopted to the case where $s = 1$.

---

* This paper has been written during the author's visiting professorship at the University of Marburg, GFR.

Let us note that the convexity condition can be fulfilled although condition (2) resp. (3) is not. This may be seen from the example of the function

$$f(x) = s\int_0^x \left(1+\frac{1}{\log t}\right)dt \tag{4}$$

(cf. [6], [9]), which fulfils all the conditions imposed on f at the beginning of this section provided $a < e^{-1}$, is concave, but does not fulfil (2) resp. (3). Therefore the result of Zdun is of a considerable interest.

In the present paper we give another, simpler proof of Zdun's result, which can be also applied to the case $s = 1$, not covered in [15].

2. In the present section we assume that the function f has the following properties.

(H) $f\colon I \to I$ is of class C^1 and convex or concave in I, $0 < f(x) < x$ in I^*, and $f'(x) > 0$ in I.

Fix an $x_0 \in I^*$. If $s \in (0, 1)$, then for every $x \in I$ there exists the limit

$$\sigma(x) = \lim_{n\to\infty} f^n(x)/f^n(x_0), \tag{5}$$

and the function $\sigma\colon I \to \mathbf{R}$ is convex or concave (just like f), and satisfies the Schröder equation

$$\sigma[f(x)] = s\sigma(x) \tag{6}$$

for all $x \in I$ (cf. [7] and [9], Theorem 6.8). Similarly, if $s = 1$, then for every $x \in I^*$ there exists the limit

$$a(x) = \lim_{n\to\infty} \frac{f^n(x)-f^n(x_0)}{f^{n+1}(x_0)-f^n(x_0)}, \tag{7}$$

and the function $a\colon I^* \to \mathbf{R}$ is convex and satisfies the Abel equation

$$a[f(x)] = a(x)+1 \tag{8}$$

for all $x \in I^*$ (cf. [8] and [9], Theorem 7.5).

The relevant properties of the functions σ and a are described in the following

LEMMA. *Let the function f fulfil hypothesis* (H). *If $s \in (0,1)$, then the function σ given by* (5) *is of class C^1 in I^*, and $\sigma'(x) > 0$ in I^*. If $s = 1$, then the function a given by* (7) *is of class C^1 in I^*, and $a'(x) < 0$ in I^*.*

Proof. If $s \in (0,1)$, then the function σ exists and is convex or concave in I. Thus at every point $x \in (0, a)$ there exist the right derivative $\sigma'_+(x)$ and the left derivative $\sigma'_-(x)$, and the functions $\sigma'_+\colon (0, a) \to \mathbf{R}$

and $\sigma'_-: (0, a) \to \boldsymbol{R}$ are monotonic and both satisfy the functional equation

$$\sigma'[f(x)] = \frac{s}{f'(x)}\,\sigma'(x) \tag{9}$$

in $(0, a)$. Moreover, σ'_+ and σ'_- may differ at most at denumerably many points.

Suppose that $\sigma'_+(x^*) = 0$ for an $x^* \in (0, a)$. Then, by (9), $\sigma'_+[f(x^*)] = 0$, and by induction $\sigma'_+[f^n(x^*)] = 0$ for $n = 1, 2, \ldots$ Since the sequence $f^n(x^*)$ decreases to zero ([9], Theorem 0.4) and the function σ'_+ is monotonic, this implies that σ'_+ vanishes and σ is constant in $(0, x^*)$. But this is impossible in view of (6). Consequently,

$$\sigma'_+(x) \neq 0 \quad \text{ for } x \in (0, a). \tag{10}$$

Since $\lim\limits_{x\to 0+} s/f'(x) = 1$, monotonic solutions of equation (9) are determined uniquely up to a multiplicative constant ([2] and [9], Theorem 5.4; cf. also [5], [6]). Consequently, there exists a constant k such that

$$\sigma'_-(x) = k\sigma'_+(x) \quad \text{ for } x \in (0, a).$$

However, σ'_+ and σ'_- coincide at infinitely many points. Thus $k = 1$ and $\sigma'_+(x) = \sigma'_-(x) = \sigma'(x)$ for all $x \in (0, a)$, which implies that σ' is continuous in $(0, a)$. If $a \in I$, then $f(a) < a$ in virtue of (H). Hence, by (9),

$$\lim_{x\to a-} \sigma'(x) = \lim_{x\to a-} \frac{f'(x)}{s}\,\sigma'[f(x)] = \frac{f'(a)}{s}\,\sigma'[f(a)]$$

exists, is finite and different from zero. Since the function σ is continuous at $f(a) \in (0, a)$, and f is continuous at a, it follows from equation (6) that σ is continuous at a. Consequently σ' exists, is continuous and different from zero in I^*.

Since, for every n, the function f^n is increasing and the constant $f^n(x_0)$ is positive, the function σ is non-decreasing in I. Thus, by (10), $\sigma'(x) > 0$ in I^*.

If $s = 1$, then the function f must be concave. The function α exists and is convex in I^*. The right derivative $\alpha'_+: (0, a) \to \boldsymbol{R}$ and the left derivative $\alpha'_-: (0, a) \to \boldsymbol{R}$ exist, are non-decreasing, and satisfy the equation

$$\alpha'[f(x)] = \frac{1}{f'(x)}\,\alpha'(x). \tag{11}$$

Moreover, similarly as in the case of σ, $\alpha'_+(x) \neq 0$ for $x \in (0, a)$. We have $\alpha'_-(x) = k\alpha'_+(x)$ for $x \in (0, a)$, since α'_+ and α'_- are monotonic solutions of (11), which implies that $\alpha'_+(x) = \alpha'_-(x) = \alpha'(x)$ for all $x \in (0, a)$. If $a \in I$, then

$$\lim_{x\to a-} \alpha'(x) = \lim_{x\to a-} f'(x)\alpha'[f(x)] = f'(a)\alpha'[f(a)]$$

exists, is finite and different from zero. Consequently α is of class C^1 in I^* and $\alpha'(x) \neq 0$ in I^*.

Since, for every n, the function f^n is increasing and the constant $f^{n+1}(x_0)-f^n(x_0)$ is negative, the function α is non-increasing in I^*. Thus $\alpha'(x) < 0$ in I^*.

The result of the present paper is contained in the following

THEOREM. *Let the function f fulfil hypothesis* (H). *Then equation* (1) *has a unique C^1 solution φ: $I \to I$. This solution is given by*

$$\varphi(x) = \sigma^{-1}\left(s^{1/N}\sigma(x)\right) \tag{12}$$

if $s \in (0,1)$, or by

$$\varphi(x) = \begin{cases} \alpha^{-1}\left(\alpha(x)+\dfrac{1}{N}\right) & \text{for } x \in I^*, \\ 0 & \text{for } x = 0, \end{cases} \tag{13}$$

if $s = 1$, where the functions σ and α are given by (5) *and* (7), *respectively.*

Proof. It is easily seen that the function φ is well defined in I by (12) or (13), satisfies equation (1) in I, and is of class C^1 in I^*. For the proof of the existence it remains to show that

$$\lim_{x\to 0+} \varphi'(x) = s^{1/N}, \tag{14}$$

and that φ is continuous at $x = 0$.

Let $s \in (0,1)$. The function σ' is monotonic, say, non-decreasing (if σ' is non-increasing, the proof is analogous). By (12) we have

$$f(x) \leqslant \varphi(x) \leqslant x, \tag{15}$$

whence

$$\sigma'[f(x)] \leqslant \sigma'[\varphi(x)] \leqslant \sigma'(x) \tag{16}$$

for $x \in I^*$. Again by (12) we have $\varphi'(x) = s^{1/N}\sigma'(x)/\sigma'[\varphi(x)]$, whence by (16) and (9)

$$s^{1/N} \leqslant \varphi'(x) \leqslant \frac{s^{1/N}f'(x)}{s}$$

for $x \in I^*$, and (14) follows. Relation (15) implies that

$$\lim_{x\to 0+} \varphi(x) = 0, \tag{17}$$

whereas by (5) $\sigma(0) = 0$, whence we get in view of (12) $\varphi(0) = 0$. Thus φ is continuous at zero.

Now let $s = 1$. Then the function α' is non-decreasing and (13) implies (15) for $x \in I^*$, whence

$$\alpha'[f(x)] \leqslant \alpha'[\varphi(x)] \leqslant \alpha'(x) \tag{18}$$

for $x \in I^*$. Since, by (13), $\varphi'(x) = \alpha'(x)/\alpha'[\varphi(x)]$, (18) and (11) imply that

$$f'(x) \leqslant \varphi'(x) \leqslant 1$$

for $x \in I^*$, and (14) follows. Again (15) implies (17), whence the continuity of φ at zero results in view of (13).

The proof of uniqueness is based on the ideas developed in [12] and does not differ from that given by Zdun [15] in the case $s \in (0, 1)$. Therefore we are going to prove here the uniqueness of φ only in the case $s = 1$.

It has been proved in [10] that if $\varphi\colon I \to I$ is a C^1 solution of equation (1), then it must satisfy the differential equation

$$\varphi' = G(x, \varphi), \tag{19}$$

where

$$G(x, y) = \prod_{n=0}^{\infty} \frac{f'[f^n(x)]}{f'[f^n(y)]}.$$

On the other hand, it has been proved in [2] (cf. also [9], Theorem 5.4) that if $\alpha'\colon I^* \to \boldsymbol{R}$ is a monotonic solution of equation (11), then

$$\alpha'(x) = c \prod_{n=0}^{\infty} \frac{f'[f^n(x)]}{f'[f^n(x_0)]}$$

with a suitable real constant c. Hence we get $G(x, y) = \alpha'(x)/\alpha'(y)$ for $x, y \in I^*$, and equation (19) becomes

$$\alpha'[\varphi(x)]\varphi'(x) = \alpha'(x) \tag{20}$$

for $x \in I^*$. Equation (20) can be easily integrated, and yields

$$\alpha[\varphi(x)] = \alpha(x) + C,$$

whence

$$\varphi(x) = \alpha^{-1}[\alpha(x) + C] \tag{21}$$

for $x \in I^*$. Inserting (21) into equation (1) we get $C = 1/N$. Consequently,

$$f(x) < \varphi(x) < x \quad \text{ for } x \in I^*,$$

and $\varphi(0) = 0$ follows by the continuity of φ. Thus necessarily φ must be given by formula (13), which proves the uniqueness and completes the proof of the theorem.

Actually, in the above theorem it is enough to assume that the function f is convex or concave only in a neighbourhood $[0, b) \subset I$ of $x = 0$. Then the above argument yields the existence of a unique C^1 solution $\varphi\colon [0, b) \to [0, b)$ of equation (1), and this solution can be uniquely extended onto the whole interval I to a C^1 solution $\varphi\colon I \to I$ of (1) (cf. [10]).

It is worthwhile to note that the solution given in the above theorem need not be convex or concave (cf. [4], [10]); but if equation (1) has a convex or concave solution φ: $I \to I$, then the latter is unique and is identical with the C^1 solution given by formula (12) or (13) (cf. [12]). Recently Zdun [16] has proved that if the function f (fulfilling hypothesis (H)) is concave and so is its derivative f', then the (unique) C^1 solution φ of (1) is also concave.

References

[1] S. Bratman, *Uniqueness of regular similarity functions*, Ann. Polon. Math. 31 (1976), p. 265–267.

[2] J. Burek and M. Kuczma, *Einige Bemerkungen über monotone und konvexe Lösungen gewisser Funktionalgleichungen*, Math. Nachr. 36 (1968), p. 121–134.

[3] M. Crum, *On two functional equations which occur in the theory of clock-graduation*, Quart. J. Math., Oxford Ser. 10 (1939), p. 155–160.

[4] J. Ger, *On convex solutions of the functional equation* $\varphi^2(x) = g(x)$. Zeszyty Naukowe Uniw. Jagiell. 252, Prace Mat. 15 (1971), p. 61–65.

[5] M. Kuczma, *Sur une équation fonctionnelle*, Mathematica, Cluj 3 (26) (1961), p. 79–87.

[6] – *On the Schröder equation*, Rozprawy Mat. [Diss. Math.] 34 (1963).

[7] – *Note on Schröder's functional equation*, J. Australian Math. Soc. 4 (1964), p. 149–151.

[8] – *On convex solutions of Abel's functional equation*, Bull. Acad. Polon. Sci., Sér. sci. math. astronom. phys. 13 (1965), p. 645–648.

[9] – *Functional equations in a single variable*, Monografie Mat. 46, Polish Scientific Publishers, Warszawa 1968.

[10] – *Fractional iteration of differentiable functions*, Ann. Polon. Math. 22 (1969), p. 229–237.

[11] – *Fractional iteration of differentiable functions with multiplier zero*, Prace Mat. [Comm. Math.] 14 (1970), p. 35–39.

[12] – and A. Smajdor, *Fractional iteration in the class of convex functions*, Bull. Acad. Polon. Sci., Sér. sci. math. astronom. phys. 16 (1968), p. 717–720.

[13] – – *Regular fractional iteration*, ibidem 19 (1971), p. 203–207.

[14] B. A. Reznick, *A uniqueness criterion for fractional iteration*, Ann. Polon. Math. 30 (1975), p. 219–224.

[15] M. C. Zdun, *Differentiable fractional iteration*, Bull. Acad. Polon. Sci., Sér. sci. math. astronom. phys. 25 (1977), p. 643–646.

[16] – *Continuous and differentiable iteration semigroups*, Prace Nauk. Uniw. Śl. 308 (1979).

Reçu par la Rédaction le 8. 12. 1977